# Dynamic Markov Bridges and Kyle-Back Models of Insider Trading

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# Back's model of insider trading

Inspired by Kyle (1985), Back (1992) studies a market for a bond and a risky asset with three types of participants:

1. *Noise traders:* The noise traders have no information about the future value of the risky asset, their cumulative demand is modeled by a standard BM $B$.
2. *Informed trader:* The insider knows the value $V \sim N(0, 1)$, $V \perp B$, of the risky asset at time 1. Being risk-neutral, her objective is to maximize her expected profit.
3. *Market maker:* The market maker observes the total order, sets the price of the risky asset and clears the market.

# The pricing mechanism of the market

- The market maker decides the price looking at the total order $X^\theta$ given by
$$X_t^\theta = B_t + \theta_t,$$
where $\theta_t$ is the position of the insider in the risky asset at time $t$.
- Thus, the filtration of the market maker is $\mathcal{F}^X$. Note that $\theta$ is not necessarily adapted to $\mathcal{F}^X$, i.e. the insider's trade is not observed directly by the market maker.
- The market maker has a *pricing rule*, $H : [0,1] \times \mathbb{R} \mapsto \mathbb{R}$, to assign the price in the following form:
$$S_t = H(t, X_t),$$
where $S_t$ is the market price of the risky asset at time $t$.

# Equilibrium

## Definition 1

A pair $(H^*, \theta^*)$ is said to form an equilibrium if $H^*$ is a pricing rule, $\theta^* \in \mathcal{A}$, and the following conditions are satisfied:

1. *Market efficiency condition:* Given $\theta^*$, $H^*$ is a rational pricing rule, i.e.

$$H^*(t, X_t^*) = \mathbb{E}[V|\mathcal{F}_t^{X^*}], \quad t \in [0, 1],$$

   where $X_t^* = B_t + \theta_t^*$.

2. *The optimality condition:* Given $H^*$, $\theta^*$ maximizes the expected profit of the insider.

# Equilibrium: Back's solution.

- In the equilibrium $X^*$, the equilibrium level of the total order, satisfies

$$dX_t^* = dB_t + \frac{V - X_t^*}{1-t}dt,$$

so that $X^*$ is a Brownian bridge. The price is given by $S_t = X_t^*$.

- $X^*$ is a BM in its own filtration: the insider cannot be detected (so-called "*Inconspicuous trade theorem*").

# An equilibrium model for a defaultable bond I

A company issues a bond that pays €1 at time 1 unless it defaults before that time. Default time is given by

$$\tau := \inf\{t > 0 : Z_t = -1\},$$

where $Z$ is a BM starting at 0 and $Z \perp B$. C. & Çetin (2008) study a similar problem where insider knows $\tau$ from the beginning. In the equilibrium total order solves

$$dX_t^* = dB_t + \left\{ \frac{1}{1 + X_t^*} - \frac{1 + X_t^*}{\tau - t} \right\} dt$$

and the price of the defaultable bond is given by $H^*(t, X_t^*)$ where on the set $\{\tau > t\}$

$$H^*(t,x) := \int_{1-t}^{\infty} \frac{x+1}{\sqrt{2\pi y^3}} e^{-\frac{(x+1)^2}{2y}} dy = \mathbb{P}(\tau > 1 | Z_t = x).$$

# An equilibrium model for a defaultable bond II

- Note that, this time, $1 + X^*$ is a 3-dimensional Bessel bridge of length $\tau$ in insider's view. Moreover, $\tau$ is an $\mathcal{F}^{X^*}$-stopping time. Indeed,

$$\tau = \inf\{t > 0 : X_t^* = -1\}.$$

- $X^*$ is a BM in its own filtration: the insider cannot be detected (*Inconspicuous trade theorem*).
- Related literature: Wu (1999), Föllmer-Wu-Yor (1999), Cho (2003), Lasserre (2004).

# One common mathematical characteristic

- In the models above the insider's optimal strategy, $\theta^*$ satisfies
$$d\theta^*_t = \frac{\partial}{\partial x}\log\rho(t, X^*_t, \text{ signal})dt$$
where
$$\rho(t,x,z)\,dz = P(\text{signal} \in dz|X^*_t = x).$$
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- Indeed, by standard filtering theory,
$$dX_t^* \quad = \quad dB_t^{X^*} + \mathbb{E}\left[\frac{\partial}{\partial x}\log\rho(t, X_t^*,\ \text{signal})\middle|\mathcal{F}_t^{X^*}\right] dt$$

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# This is not a coincidence!

- Consider a more general Markov setting in which the insider knows the price at time 1, $Z_1$, where

$$Z_t = \int_0^t a(Z_s) dB_s^Z \quad (B^Z \perp B)$$

- Then in the equilibrium the market maker uses the following process for the pricing purposes:

$$dX_t = a(X_t)\,(dB_t + d\theta_t)\,.$$

- It can be shown along the similar lines that it is necessary in the equilibrium that $X$ is a $\mathcal{F}^X$-martingale and $Z_1 = X_1$.

- It is well-known, at least since Fitzsimmons, Pitman & Yor (1993) (see also Baudoin (2002)), that the solution $X$ of

$$dX_t = a(X_t)dB_t + a^2(X_t)\frac{G_x(1-t, X_t, z)}{G(1-t, X_t, z)}dt,$$

is a Markov process converging to $z$ as $t \to 1$, where $G$ is the transition density of

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- If $Z_1$, independent of $B$, has a density given by $G(1,0,\cdot)$, then defining

$$dX_t = a(X_t)dB_t + a^2(X_t)\frac{G_x(1-t, X_t, Z_1)}{G(1-t, X_t, Z_1)}dt,$$

gives the process we want: $\mathcal{F}^X$-martingale with $X_1 = Z_1$.

# Partial resumé so far

In the models presented so far

- there is a private signal $Z_1$ of the insider giving the true price at the end of the trading horizon;
- the cumulative demand does not change its law, i.e. it stays as a Brownian motion if the insider trades optimally;
- $\lim_{t \to 1} S_t = Z_1$, where $S$ is the market price of the asset.

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**Question:** What about dynamic private information? Do we have the same probabilistic structure?

# Dynamic information asymmetry

- Back and Pedersen (1998) analyze the same problem when the insider receives a continuous signal

$$Z_t = Z_0 + \int_0^t \sigma(u) dB_u^Z$$

where $Z_0$ is a $N(0,1)$ r.v., $B^Z$ is a BM independent of $B$, the noise demand, and $Var(Z_0) + \int_0^1 \sigma^2(s)ds = 1$.

- The asset value at time 1 is given by $Z_1$. The equilibrium demand in this case is given by

$$dX_t^* = dB_t + \frac{Z_t - X_t^*}{V(t) - t} dt,$$

where $V(t) = Var(Z_0) + \int_0^t \sigma^2(s)ds$. $S_t = X_t^*$ and, moreover, $\lim_{t \to 1} S_t = Z_1$.

- Similar problems in varying generality are discussed in Wu (1999), Föllmer, Wu and Yor (1999) and Danilova (2008).

# Extension to a general diffusion setting

Goal: Given

$$Z_t = Z_0 + \int_0^t \sigma(s) a(Z_s) dB_s^Z$$

with $a(z)$ satisfying regularity conditions, construct a process $X$ with $X_0 = 0$ and adapted to $\mathcal{F}_t^{Z,B}$ (recall that $B^Z \perp B$), such that:

**C1** $(X, Z)$ is Markov.

**C2** $X_1 = Z_1$, $Q^z$-a.s., where $Q^z$ is the law of $(X, Z)$ with $Z_0 = z$ and $X_0 = 0$.

**C3** $X$ is a local martingale in its own filtration and $[X, X]_t = \int_0^t a^2(X_s) ds$.

# Model assumptions

Föllmer, Wu and Yor (1999) showed that such a construction is impossible when $\sigma \equiv 1$.

## Assumption 1

*Fix a real number $c \in [0,1]$. $\sigma : [0,1] \mapsto \mathbb{R}_+$ and $a : \mathbb{R} \mapsto \mathbb{R}_+$ are two measurable functions such that:*

1. $V(t) := c + \int_0^t \sigma^2(u)du > t$ *for every* $t \in [0,1)$, *and* $V(1) = 1$.
2. $\sigma^2(\cdot)$ *is bounded on* $[0,1]$.
3. $a(\cdot)$ *is bounded away from zero.*
4. $a(\cdot)$ *is twice continuously differentiable, such that* $Z$ *is well-defined as unique strong solution.*

## General solution

- **Conjecture:** the solution to our problem is $(X, Z)$ such that $Z_0 \sim G(c, 0, z)$, $X$ solves

$$dX_t = a(X_t)dB_t + a^2(X_t)\frac{\rho_x(t, X_t, Z_t)}{\rho(t, X_t, Z_t)}dt, \quad t < 1$$

and

$$\rho(t, x, z) = G(V(t) - t, x, z),$$

where

  - $G(t, x, z)$ is the transition probability of $d\xi_t = a(\xi_t)d\beta_t$ and
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  - $V(t) = c + \int_0^t \sigma^2(u)du$.
- We need to prove that
  - $X$ is a $\mathcal{F}^X$-martingale $\Leftarrow$ $\rho$ is the conditional density of $Z_t$ given $\mathcal{F}_t^X$
  - $\lim_{t\to 1} X_t = Z_1$

# Where does our guess for $\rho$ come from? Some heuristics

- We expect $\rho(t,x,z)$ to be the signal conditional density given $X_t = x$, to have that $dX_t = a(X_t)dB_t^X$ in its own filtration, where $B^X$ is standard BM under $\mathcal{F}^X$
- Compare with $dZ_t = \sigma(t)a(Z_t)dB_t^Z$
- Recall that $V(t) = c + \int_0^t \sigma^2(u)du$, it suggests to use $X_{V(t)}$ as a proxy for $Z_t$
- Moreover $G(V(t)-t,x,z)$ is the transition density of $X_{V(t)}$ given $X_t = z$, so that it's natural to conjecture that $\rho(t,x,z) = G(V(t)-t,x,z)$
- We check our guess using a slight generalization of Kurtz-Ocone (1988)

# Existence of $G$

We need assumptions to get existence of the transition probabilities $G(t, x, z)$ of $d\xi_t = a(\xi_t) d\beta_t$. Let

$$A(x) := \int_0^x \frac{dy}{a(y)},$$

and $\zeta_t = A(\xi_t)$. Itô's formula yields

$$d\zeta_t = d\beta_t + b(\zeta_t) dt, \quad \text{where } b(y) := -\frac{1}{2} a_z(A^{-1}(y)).$$

**Assumption 2**

*$b$ and $b_y$ are bounded and $b_y$ is Hölder.*

Under all our assumptions, there exists a fund. solution, $G$, to $u_t = (1/2)(a^2(z)u)_{zz}$.
Moreover, $G(t-s, y, x) = \Gamma(t-s, A(y), A(x)) \frac{1}{a(x)}$, where $\Gamma$ is transition density of $\zeta_t$.

# Is $\rho$ indeed the conditional density?

- We have seen that $\rho(t, x, z) = G(V(t) - t, x, z)$ is a good candidate for the conditional density of $Z$ given $\mathcal{F}_t^X$. Let's verify our guess.

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- Let $U_t = A(Z_t)$ and $R_t = A(X_t)$ so that

$$\begin{aligned} dU_t &= \sigma(t)d\beta_t + \sigma^2(t)b(U_t)dt \\ dR_t &= dB_t + \left\{ \frac{p_x(t,R_t,U_t)}{p(t,R_t,U_t)} + b(R_t) \right\} dt, \end{aligned} \tag{2}$$

where $p(t,x,z) := a(A^{-1}(z))\rho(t, A^{-1}(x), A^{-1}(V(z)))$.

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  where $p(t,x,z) := a(A^{-1}(z))\rho(t,A^{-1}(x),A^{-1}(V(z)))$.
- *Then, $p(t,R_t,\cdot)$ is the $\mathcal{F}_t^R$-conditional density of $U_t$ if and only if $\rho(t,X_t,\cdot)$ is the conditional density of $Z_t$ given $\mathcal{F}_t^X$.*

# Is $\rho$ indeed the conditional density?

How to check that $p(t, R_t, \cdot)$ is the $\mathcal{F}_t^R$-conditional density of $U_t$? Our approach is based on the following steps: Let $\mathcal{P}$ the set of all probability measures on $\mathcal{B}(\mathbb{R})$

- the $\mathcal{P}$-valued process $\pi_t(\omega, dx)$ is well-defined by $\pi_t f = \mathbb{E}[f(U_t)|\mathcal{F}_t^R]$, $f$ measurable bounded
- consider the operator

  $$\mathcal{A}_0 := \partial_t + \frac{1}{2}\sigma^2(t)\partial_{xx}^2 + \sigma^2(t)b(t,x)\partial_x$$

  the corresponding martingale problem is well-posed and has a unique solution $(t, U_t)$ so that ...
- ... we can apply arguments from Kurtz-Ocone (1988) implying that the Kushner-Stratonovich equation satisfied by $U_t$'s conditional density has a unique solution under our assumptions
- since $p(t, R_t, \cdot)$ satisfies that equation, thus it equals the $\mathcal{F}_t^R$-conditional density of $U_t$.

# Convergence : Gaussian case I

- When $a \equiv 1$, $dZ_t = \sigma(t)dB^Z_t$, $B^Z$ standard BM. It's well-known that $G(t-s, y, x) = \frac{1}{\sqrt{2\pi(t-s)}}\exp(-\frac{(x-y)^2}{2(t-s)})$. In this case
$$dX_t = dB_t + \frac{Z_t - X_t}{V(t) - t}dt.$$
- This is the equilibrium demand obtained by Back and Pedersen (1998).
- Back and Pedersen (1998) and Wu (1999) only prove the convergence
$$\lim_{t \to 1} X_t = Z_1$$
in $L^2(\mathbb{P})$ where $\mathbb{P}$ is the market maker's probability given by
$$\mathbb{P}(E) = \int Q^z(E)P(Z_0 \in dz), \qquad \text{for } E \in \mathcal{F}.$$

# Convergence of $X_t$: Gaussian case II

We shall now give a proof of the convergence with respect to the insider's probability given $Z_0 = z$, i.e.

$$\lim_{t \to 1} X_t = Z_1, \quad Q^z - a.s.$$

Here are the main steps:

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Here are the main steps:

- Find a cont. function $\varphi(t, x, z)$ such that $(\varphi(t, X_t, Z_t))_{t \in [0,1)}$ is a positive $Q^z$-supermartingale and, under some mild conditions on $\sigma$,

$$\lim_{t \to 1} \varphi(t, x, z) = +\infty, \quad x \neq z$$

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- Let $M_t := \varphi(t, X_t, Z_t)$. Supermartingale conv theorem gives that $\lim_{t \to 1} M_t = M_1$, $Q^z$-a.s. By Fatou's lemma we have

$$M_0 \geq \liminf_{t \to 1} E^z[M_t] \geq E^z\left[\lim_{t \to 1} \varphi(t, X_t, Z_t)\right]$$

This yields $Q^z(\lim_{t \to 1} X_t \neq Z_1) = 0$.

# Convergence of $X_t$: The general case.

- Let $U_t = A(Z_t)$ and $R_t = A(X_t)$, where $A(x) = \int_0^x a(y)^{-1} dy$. Recall that

$$\begin{aligned} dU_t &= \sigma(t)d\beta_t + \sigma^2(t)b(U_t)dt \\ dR_t &= dB_t + \left\{ \frac{p_x(t, R_t, U_t)}{p(t, R_t, U_t)} + b(R_t) \right\} dt, \end{aligned}$$

  with $p(t, x, z) := a(A^{-1}(z))\rho(t, A^{-1}(x), A^{-1}(z))$.
- Note that $X_t \to Z_1 \iff R_t \to U_1$.

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  with $p(t, x, z) := a(A^{-1}(z)) \rho(t, A^{-1}(x), A^{-1}(z))$.
- Note that $X_t \to Z_1 \iff R_t \to U_1$.
- It is easy to show that $p(t, x, z) = \Gamma(V(t) - t, x, z)$ where $\Gamma$ is the transition density of

$$d\zeta_t = d\beta_t + b(\zeta_t) dt,$$

- As the law of $\zeta$ is equivalent to the Wiener measure, we can write

$$\Gamma(t, x, z) = h(t, x, z) q(t, x, z)$$

  where $q$ is the transition density of a standard BM.

# Sketch of proof for convergence of $R_t$

Consider a new measure, $P^z$ under which, and with an abuse of notation,

$$\begin{aligned} dU_t &= \sigma(t)d\beta_t \\ dR_t &= dB_t + \frac{p_x(t, R_t, U_t)}{p(t, R_t, U_t)}dt = \\ &= dB_t + \frac{U_t - R_t}{V(t) - t}dt + \frac{h_x(V(t) - t, R_t, U_t)}{h(V(t) - t, R_t, U_t)}dt. \end{aligned} \quad (3)$$

Let

$$r_t = R_t - e^{-\int_0^t \frac{ds}{V(s)-s}} \int_0^t e^{\int_0^s \frac{du}{V(u)-u}} \frac{h_x(V(s) - s, R_s, U_s)}{h(V(s) - s, R_s, U_s)}ds.$$

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This $r_t$ satisfies

$$dr_t = dB_t + \frac{U_t - r_t}{V(t) - t}ds.$$

So as in the Gaussian case $r_t$ converges to $U_1$.

So we need

$$\lim_{t\to 1} e^{-\int_0^t \frac{ds}{V(s)-s}} \int_0^t e^{\int_0^s \frac{du}{V(u)-u}} \frac{h_x(V(s)-s, R_s, U_s)}{h(V(s)-s, R_s, U_s)} ds = 0.$$

So we need

$$\lim_{t\to 1} e^{-\int_0^t \frac{ds}{V(s)-s}} \int_0^t e^{\int_0^s \frac{du}{V(u)-u}} \frac{h_x(V(s)-s, R_s, U_s)}{h(V(s)-s, R_s, U_s)} ds = 0.$$

by de L'Hôpital rule the limit equals

$$\lim_{t\to 1}(V(t)-t)\frac{h_x(V(t)-t, R_t, U_t)}{h(V(t)-t, R_t, U_t)} = 0$$

due to the following: Let $x_n \to x$, $z_n \to z$ and $t_n \to 0$. Then

$$\lim_{n\to\infty} t_n \frac{h_x}{h}(t_n, x_n, z_n) = 0.$$

The result above continues to hold when $x = \pm\infty$ as well.

# A straightforward corollary: signal with drift

Let $Z$ be the unique strong solution to

$$Z_t = Z_0 + \int_0^t \sigma(s) d\beta_s + \int_0^t \sigma^2(s) b(Z_s) ds,$$

where $b \in C_b^2$ with bounded derivatives, $\sigma$ is as before and $P(Z_0 \in dz) = \Gamma(c, 0, z)dz$ for some $c \in (0, 1)$. Define $X$ by

$$dX_t = dB_t + \left\{ b(X_s) + \frac{\rho_x(t, X_t, Z_t)}{\rho(t, X_t, Z_t)} \right\} dt,$$

for $t \in (0, 1)$ with $X_0 = 0$. Here $\rho(t, x, z) := \Gamma(V(t) - t, x, z)$ where $V(t) = c + \int_0^t \sigma^2(u) du$ and $\Gamma(t, x, z)$ is the transition density of $\zeta_t = \beta_t + b(\zeta_t)dt$. Then

# A straightforward corollary: signal with drift

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1. $X_t - \int_0^t b(X_s) ds$ is a standard BM;
2. $X_1 = Z_1$, $Q^z$-a.s. where $Q^z$ is the law of $(X, Z)$ with $Z_0 = z$ and $X_0 = 0$.

# An example

Suppose $Z$ is an Ornstein-Uhlenbeck type process, i.e.

$$dZ_t = \sigma(t)d\beta_t - b\sigma^2(t)Z_t dt,$$

where $b > 0$ is a constant and $Z_0$ has law $G(c, 0, \cdot)$.

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for $t \in (0, 1)$. Then, the previous theorem implies that $X$ is an Ornstein-Uhlenback process in its own filtration and a bridge, i.e. $X_1 = Z_1$, $Q^z$-a.s.