

Stochastic Targets and Optimal Control with Controlled Loss

B. Bouchard, R. Elie, C. Imbert, N. Touzi

CEREMADE-Univ. Paris-Dauphine, CREST-Ensaes, CMAP-Ecole Polytechnique

Toronto, January 2010

Motivations

Say we have a financial model where Y^ν is the wealth process, X^ν are stock prices/factors, ν is the trading strategy.

Motivations

Say we have a financial model where Y^ν is the wealth process, X^ν are stock prices/factors, ν is the trading strategy.

Quantile hedging / expected loss pricing

Find the minimal condition on the initial wealth such that

$$\mathbb{E}[g(X^\nu(T), Y^\nu(T))] \geq p \text{ for some strategy } \nu.$$

Motivations

Say we have a financial model where Y^ν is the wealth process, X^ν are stock prices/factors, ν is the trading strategy.

Quantile hedging / expected loss pricing

Find the minimal condition on the initial wealth such that

$$\mathbb{E}[g(X^\nu(T), Y^\nu(T))] \geq p \text{ for some strategy } \nu.$$

Optimal management under loss constraint

Optimize $\mathbb{E}[f(X^\nu(T), Y^\nu(T))]$ under the loss constraint

$$\mathbb{E}[g(X^\nu(T), Y^\nu(T))] \geq p$$

Motivations

Say we have a financial model where Y^ν is the wealth process, X^ν are stock prices/factors, ν is the trading strategy.

Quantile hedging / expected loss pricing

Find the minimal condition on the initial wealth such that

$$\mathbb{E}[g(X^\nu(T), Y^\nu(T))] \geq p \text{ for some strategy } \nu.$$

Optimal management under loss constraint

Optimize $\mathbb{E}[f(X^\nu(T), Y^\nu(T))]$ under the loss constraint

$$\mathbb{E}[g(X^\nu(T), Y^\nu(T))] \geq p$$

Our aim is to provide a direct approach

Motivations

Say we have a financial model where Y^ν is the wealth process, X^ν are stock prices/factors, ν is the trading strategy.

Quantile hedging / expected loss pricing

Find the minimal condition on the initial wealth such that

$$\mathbb{E}[g(X^\nu(T), Y^\nu(T))] \geq p \text{ for some strategy } \nu.$$

Optimal management under loss constraint

Optimize $\mathbb{E}[f(X^\nu(T), Y^\nu(T))]$ under the loss constraint

$$\mathbb{E}[g(X^\nu(T), Y^\nu(T))] \geq p$$

Our aim is to provide a direct approach

This requires a preliminary work on stochastic target problems

Outline

Stochastic target problems

Outline

Stochastic target problems

Stochastic target with controlled loss

Outline

Stochastic target problems

Stochastic target with controlled loss

Optimal control under target constraints

Outline

Stochastic target problems

Stochastic target with controlled loss

Optimal control under target constraints

Problem Formulation

Dynamics :

$Z_{t,x,y}^\nu := (X_{t,x}^\nu, Y_{t,x,y}^\nu) \in \mathbb{R}^d \times \mathbb{R}$ solution of

$$X^\nu(s) = x + \int_t^s \mu_X(X^\nu(r), \nu_r) dr + \int_t^s \sigma_X(X^\nu(r), \nu_r) dW_r$$
$$Y^\nu(s) = y + \int_t^s \mu_Y(Z^\nu(r), \nu_r) dr + \int_t^s \sigma_Y(Z^\nu(r), \nu_r) dW_r.$$

Controls

$\nu \in \mathcal{U}$, square integrable, prog. meas., valued in $U \subset \mathbb{R}^d$ (may be unbounded).

Notation

$$z = (x, y)$$

Problem Formulation

Target

$$G := \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : g(x, y) \geq 0\}, \text{ with } g \nearrow y.$$

Problem Formulation

Target

$$G := \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : g(x, y) \geq 0\}, \text{ with } g \nearrow y.$$

Viability set

$$D := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } g(Z_{t,x,y}^\nu(T)) \geq 0 \text{ } \mathbb{P} - \text{a.s.}\}.$$

Problem Formulation

Target

$$G := \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : g(x, y) \geq 0\}, \text{ with } g \nearrow y.$$

Viability set

$$D := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } g(Z_{t,x,y}^\nu(T)) \geq 0 \text{ } \mathbb{P} - \text{a.s.}\}.$$

Value function

$$w(t, x) := \inf\{y \in \mathbb{R} : (t, x, y) \in D\}.$$

Problem Formulation

Target

$$G := \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : g(x, y) \geq 0\}, \text{ with } g \nearrow y.$$

Viability set

$$D := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } g(Z_{t,x,y}^\nu(T)) \geq 0 \text{ } \mathbb{P} - \text{a.s.}\}.$$

Value function

$$w(t, x) := \inf\{y \in \mathbb{R} : (t, x, y) \in D\}.$$

Interpretation

$$D \text{ ``=`` } \{y \geq w(t, x)\}$$

Example : super-hedging in finance

Typical application in finance

Stocks/Factors : X^ν . Wealth : Y^ν . Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ

Example : super-hedging in finance

Typical application in finance

Stocks/Factors : X^ν . Wealth : Y^ν . Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ

Super-hedging price

$Y^\nu(T) \geq \psi(X^\nu(T)) \Leftrightarrow g(Z^\nu(T)) \geq 0$ with $g(x, y) := y - \psi(x)$.

Example : super-hedging in finance

Typical application in finance

Stocks/Factors : X^ν . Wealth : Y^ν . Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ

Super-hedging price

$Y^\nu(T) \geq \psi(X^\nu(T)) \Leftrightarrow g(Z^\nu(T)) \geq 0$ with $g(x, y) := y - \psi(x)$.

$w(t, x) := \inf\{y \in \mathbb{R} : \exists \nu \in \mathcal{U} \text{ s.t. } g(Z_{t,x,y}^\nu(T)) \geq 0 \text{ } \mathbb{P} - \text{a.s.}\}$.

Geometric dynamic programming principle

Recall

$$D := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } g(Z_{t,z}^\nu(T)) \geq 0 \text{ } \mathbb{P} - \text{a.s.}\}.$$
$$w(t, x) := \inf\{y \in \mathbb{R} : (t, x, y) \in D\}$$

Geometric dynamic programming principle

Recall

$$D := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } g(Z_{t,z}^\nu(T)) \geq 0 \text{ } \mathbb{P} - \text{a.s.}\}.$$

$$w(t, x) := \inf\{y \in \mathbb{R} : (t, x, y) \in D\}$$

Theorem [Soner and Touzi]

For any stopping time $\theta \in [t, T] \mathbb{P} - \text{a.s.}$

$$y > w(t, x) \Rightarrow \exists \nu \text{ s.t. } Y_{t,x,y}^\nu(\theta) \geq w(\theta, X_{t,x}^\nu(\theta)) \text{ } \mathbb{P} - \text{a.s.}$$

Geometric dynamic programming principle

Recall

$$D := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } g(Z_{t,z}^\nu(T)) \geq 0 \text{ } \mathbb{P} - \text{a.s.}\}.$$

$$w(t, x) := \inf\{y \in \mathbb{R} : (t, x, y) \in D\}$$

Theorem [Soner and Touzi]

For any stopping time $\theta \in [t, T] \mathbb{P} - \text{a.s.}$

$$y > w(t, x) \Rightarrow \exists \nu \text{ s.t. } Y_{t,x,y}^\nu(\theta) \geq w(\theta, X_{t,x}^\nu(\theta)) \text{ } \mathbb{P} - \text{a.s.}$$

$$y < w(t, x) \Rightarrow \not\exists \nu \text{ s.t. } Y_{t,x,y}^\nu(\theta) > w(\theta, X_{t,x}^\nu(\theta)) \text{ } \mathbb{P} - \text{a.s.}$$

Super-solution property

For " $y = w(t, x)$ ", $\exists \nu \in \mathcal{U}$ s.t. $Y_{t,x,y}^\nu(t+) \geq w(t+, X_{t,x}^\nu(t+))$.

Super-solution property

For " $y = w(t, x)$ ", $\exists \nu \in \mathcal{U}$ s.t. $Y_{t,x,y}^\nu(t+) \geq w(t+, X_{t,x}^\nu(t+))$.
Thus,

$$\begin{aligned} dY_{t,x,y}^\nu(t) &= \mu_Y(x, y, \nu_t)dt + \sigma_Y(x, y, \nu_t)dW_t \\ &\geq dw(t, X_{t,x}^\nu(t)) \\ &= \mathcal{L}_X^{\nu_t}w(t, x)dt + Dw(t, x)\sigma_X(x, \nu_t)dW_t \end{aligned}$$

Super-solution property

For " $y = w(t, x)$ ", $\exists \nu \in \mathcal{U}$ s.t. $Y_{t,x,y}^\nu(t+) \geq w(t+, X_{t,x}^\nu(t+))$.
Thus,

$$\begin{aligned} dY_{t,x,y}^\nu(t) &= \mu_Y(x, y, \nu_t) dt + \sigma_Y(x, y, \nu_t) dW_t \\ &\geq dw(t, X_{t,x}^\nu(t)) \\ &= \mathcal{L}_X^{\nu_t} w(t, x) dt + Dw(t, x) \sigma_X(x, \nu_t) dW_t \end{aligned}$$

Super-solution property

For " $y = w(t, x)$ ", $\exists \nu \in \mathcal{U}$ s.t. $Y_{t,x,y}^\nu(t+) \geq w(t+, X_{t,x}^\nu(t+))$.
Thus,

$$\begin{aligned} dY_{t,x,y}^\nu(t) &= \mu_Y(x, y, \nu_t) dt + \sigma_Y(x, y, \nu_t) dW_t \\ &\geq dw(t, X_{t,x}^\nu(t)) \\ &= \mathcal{L}_X^{\nu_t} w(t, x) dt + Dw(t, x) \sigma_X(x, \nu_t) dW_t \end{aligned}$$

Super-solution property

For " $y = w(t, x)$ ", $\exists \nu \in \mathcal{U}$ s.t. $Y_{t,x,y}^\nu(t+) \geq w(t+, X_{t,x}^\nu(t+))$.
Thus,

$$\begin{aligned} dY_{t,x,y}^\nu(t) &= \mu_Y(x, y, \nu_t) dt + \sigma_Y(x, y, \nu_t) dW_t \\ &\geq dw(t, X_{t,x}^\nu(t)) \\ &= \mathcal{L}_X^{\nu_t} w(t, x) dt + Dw(t, x) \sigma_X(x, \nu_t) dW_t \end{aligned}$$

This leads to

$$\sup_{u \in \mathcal{N}(t, x, w)} \mu_Y(x, w(t, x), u) - \mathcal{L}_X^u w(t, x) \geq 0$$

where $\mathcal{N}(t, x, w) := \{u \in U : \sigma_Y(x, w(t, x), u) = Dw(t, x) \sigma_X(x, u)\}$

Sub-solution property

If

$$\sup_{u \in \mathcal{N}(t,x,w)} \mu_Y(x, w(t,x), u) - \mathcal{L}_X^u w(t,x) > 0$$

where $\mathcal{N}(t,x,w) := \{u \in U : \sigma_Y(x, w(t,x), u) = Dw(t,x)\sigma_X(x, u)\}$,

Sub-solution property

If

$$\sup_{u \in \mathcal{N}(t, x, w)} \mu_Y(x, w(t, x), u) - \mathcal{L}_X^u w(t, x) > 0$$

where $\mathcal{N}(t, x, w) := \{u \in U : \sigma_Y(x, w(t, x), u) = Dw(t, x)\sigma_X(x, u)\}$,

then “we can find” \hat{u} s.t. $\hat{u}(t', x', w + \delta') \in \mathcal{N}(t', x', w + \delta')$ for (t', x') close, δ' small such that

$$\mu_Y(x', w(t', x') + \delta', \hat{u}(\cdot)) - \mathcal{L}_X^{\hat{u}(\cdot)} w(t', x') > 0$$

Sub-solution property

If

$$\sup_{u \in \mathcal{N}(t, x, w)} \mu_Y(x, w(t, x), u) - \mathcal{L}_X^u w(t, x) > 0$$

where $\mathcal{N}(t, x, w) := \{u \in U : \sigma_Y(x, w(t, x), u) = Dw(t, x)\sigma_X(x, u)\}$,

then “we can find” \hat{u} s.t. $\hat{u}(t', x', w + \delta') \in \mathcal{N}(t', x', w + \delta')$ for (t', x') close, δ' small such that

$$\mu_Y(x', w(t', x') + \delta', \hat{u}(\cdot)) - \mathcal{L}_X^{\hat{u}(\cdot)} w(t', x') > 0$$

For $\nu := \hat{u}(\cdot, X_{t,x}^\nu(\cdot), Y_{t,x,y}^\nu(\cdot))$ and $y := w(t, x) - \varepsilon$, we have

$$dY_{t,x,y}^\nu > dw(\cdot, X_{t,x}^\nu(\cdot))$$

Sub-solution property

If

$$\sup_{u \in \mathcal{N}(t, x, w)} \mu_Y(x, w(t, x), u) - \mathcal{L}_X^u w(t, x) > 0$$

where $\mathcal{N}(t, x, w) := \{u \in U : \sigma_Y(x, w(t, x), u) = Dw(t, x)\sigma_X(x, u)\}$,

then “we can find” \hat{u} s.t. $\hat{u}(t', x', w + \delta') \in \mathcal{N}(t', x', w + \delta')$ for (t', x') close, δ' small such that

$$\mu_Y(x', w(t', x') + \delta', \hat{u}(\cdot)) - \mathcal{L}_X^{\hat{u}(\cdot)} w(t', x') > 0$$

For $\nu := \hat{u}(\cdot, X_{t,x}^\nu(\cdot), Y_{t,x,y}^\nu(\cdot))$ and $y := w(t, x) - \varepsilon$, we have

$$dY_{t,x,y}^\nu > dw(\cdot, X_{t,x}^\nu(\cdot))$$

and therefore $Y_{t,x,y}^\nu(\theta) > w(\theta, X_{t,x}^\nu(\theta))$ for θ well chosen.

Sub-solution property

If

$$\sup_{u \in \mathcal{N}(t,x,w)} \mu_Y(x, w(t,x), u) - \mathcal{L}_X^u w(t,x) > 0$$

where $\mathcal{N}(t,x,w) := \{u \in U : \sigma_Y(x, w(t,x), u) = Dw(t,x)\sigma_X(x, u)\}$,

then “we can find” \hat{u} s.t. $\hat{u}(t',x',w + \delta') \in \mathcal{N}(t',x',w + \delta')$ for (t',x') close, δ' small such that

$$\mu_Y(x', w(t',x') + \delta', \hat{u}(\cdot)) - \mathcal{L}_X^{\hat{u}(\cdot)} w(t',x') > 0$$

For $\nu := \hat{u}(\cdot, X_{t,x}^\nu(\cdot), Y_{t,x,y}^\nu(\cdot))$ and $y := w(t,x) - \varepsilon$, we have

$$dY_{t,x,y}^\nu > dw(\cdot, X_{t,x}^\nu(\cdot))$$

and therefore $Y_{t,x,y}^\nu(\theta) > w(\theta, X_{t,x}^\nu(\theta))$ for θ well chosen.

This leads to

$$\sup_{u \in \mathcal{N}(t,x,w)} \mu_Y(x, w(t,x), u) - \mathcal{L}_X^u w(t,x) \leq 0 .$$

PDE Characterization

Theorem (Soner and Touzi ; B., Elie and Touzi)

w is a viscosity solution (in the discontinuous sense) of

$$\sup_{u \in \mathcal{N}_0(t,x,w)} \mu_Y(x, w(t,x), u) - \mathcal{L}_X^u w(t,x) = 0 \quad (t,x) \in [0, T) \times \mathbb{R}^d$$

where

$$\mathcal{N}_\varepsilon(t, x, w)$$

:=

$$\{u \in U : |\sigma_Y(x, w(t,x), u) - Dw(t,x)\sigma_X(x, u)| \leq \varepsilon\}$$

A powerful tool

Robust approach

- Dynamic programming principle is robust.

A powerful tool

Robust approach

- Dynamic programming principle is robust.
- Allows for non-linear dynamics (e.g. impact functions, etc...).
No need of dual formulation.

A powerful tool

Robust approach

- Dynamic programming principle is robust.
- Allows for non-linear dynamics (e.g. impact functions, etc...).
No need of dual formulation.
- If X and Y are fully coupled, can do the same with
 $w(t, z) := \mathbf{1}_{(t,z) \in D}.$

A powerful tool

Robust approach

- Dynamic programming principle is robust.
- Allows for non-linear dynamics (e.g. impact functions, etc...).
No need of dual formulation.
- If X and Y are fully coupled, can do the same with
 $w(t, z) := \mathbf{1}_{(t,z) \in D}$.

Can be extended to many types of control problems

- SDEs with jumps (B., L. Moreau)
- American type constraints (B. and T. N. Vu)
- State constraints (B., M. N. Dang, C.-A. Lehalle)
- Singular control, etc....

Outline

Stochastic target problems

Stochastic target with controlled loss

Optimal control under target constraints

Problem Formulation

Viability set

$$D(p) := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}[g(Z_{t,z}^\nu(T))] \geq p\}, p \in \mathbb{R}.$$

Problem Formulation

Viability set

$$D(p) := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}[g(Z_{t,z}^\nu(T))] \geq p\}, \quad p \in \mathbb{R}.$$

Value function

$$w(t, x; p) := \inf\{y \in \mathbb{R} : (t, x, y) \in D(p)\}.$$

Example

Interpretation

Stocks/Factors : X^ν . Wealth : Y^ν . Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ;

Example

Interpretation

Stocks/Factors : X^ν . Wealth : Y^ν . Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ; Define : $g(x, y) := \mathbf{1}_{\{y \geq \psi(x)\}}$

Example

Interpretation

Stocks/Factors : X^ν . Wealth : Y^ν . Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ; Define : $g(x, y) := \mathbf{1}_{\{y \geq \psi(x)\}}$

Quantile-hedging costs for $0 \leq p \leq 1$

Example

Interpretation

Stocks/Factors : X^ν . Wealth : Y^ν . Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ; Define : $g(x, y) := \mathbf{1}_{\{y \geq \psi(x)\}}$

Quantile-hedging costs for $0 \leq p \leq 1$

$$\mathbb{E} [g(Z_{t,z}^\nu(T))] \geq p \Leftrightarrow \mathbb{P} [Y_{t,x,y}^\nu(T) \geq \psi(X_{t,x}^\nu(T))] \geq p$$

Example

Interpretation

Stocks/Factors : X^ν . Wealth : Y^ν . Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ;

Example

Interpretation

Stocks/Factors : X^ν . Wealth : Y^ν . Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ; Define : $g(x, y) := -\ell([y - \psi(x)]^-)$, $\ell \nearrow$

Example

Interpretation

Stocks/Factors : X^ν . Wealth : Y^ν . Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ; Define : $g(x, y) := -\ell([y - \psi(x)]^-)$, $\ell \nearrow$

Loss function price

Example

Interpretation

Stocks/Factors : X^ν . Wealth : Y^ν . Portfolio strategy : $\nu \in \mathcal{U}$.

Option payoff : ψ ; Define : $g(x, y) := -\ell([y - \psi(x)]^-)$, $\ell \nearrow$

Loss function price

$$\mathbb{E} [g(Z_{t,z}^\nu(T))] \geq p$$

\Leftrightarrow

$$\mathbb{E} [\ell ([Y_{t,x,y}^\nu(T) - \psi(X_{t,x}^\nu(T))]^-)] \leq -p$$

Dynamic Programming for the Quantile Hedging Problem

Dynamic Programming for the Quantile Hedging Problem

Consider the typical example of quantile constraints, i.e.

$$D(p) := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{P}[Y_{t,z}^\nu(T) \geq \psi(X_{t,x}^\nu(T))] \geq p\}.$$

$$\text{and } w(t, x; p) := \inf\{y \in \mathbb{R} : (t, x, y) \in D(p)\}$$

Dynamic Programming for the Quantile Hedging Problem

Consider the typical example of quantile constraints, i.e.

$$D(p) := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{P}[Y_{t,z}^\nu(T) \geq \psi(X_{t,x}^\nu(T))] \geq p\}.$$

$$\text{and } w(t, x; p) := \inf\{y \in \mathbb{R} : (t, x, y) \in D(p)\}$$

Geometric Dynamic Programming ?

Dynamic Programming for the Quantile Hedging Problem

Consider the typical example of quantile constraints, i.e.

$$D(p) := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{P}[Y_{t,z}^\nu(T) \geq \psi(X_{t,x}^\nu(T))] \geq p\}.$$

$$\text{and } w(t, x; p) := \inf\{y \in \mathbb{R} : (t, x, y) \in D(p)\}$$

Geometric Dynamic Programming ?

For a stopping time $\theta \in [t, T] \mathbb{P} - \text{a.s.}$

$$D(p) \stackrel{?}{=} \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } (\theta, Z_{t,z}^\nu(\theta)) \in D(p) \mathbb{P} - \text{a.s.}\} .$$

Dynamic Programming for the Quantile Hedging Problem

Consider the typical example of quantile constraints, i.e.

$$D(p) := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{P}[Y_{t,z}^\nu(T) \geq \psi(X_{t,x}^\nu(T))] \geq p\}.$$

$$\text{and } w(t, x; p) := \inf\{y \in \mathbb{R} : (t, x, y) \in D(p)\}$$

Geometric Dynamic Programming

For a stopping time $\theta \in [t, T] \mathbb{P} - \text{a.s.}$

$$D(p) \neq \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } (\theta, Z_{t,z}^\nu(\theta)) \in D(p) \mathbb{P} - \text{a.s.}\}.$$

Dynamic Programming for the Quantile Hedging Problem

Consider the typical example of quantile constraints, i.e.

$$D(p) := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{P}[Y_{t,z}^\nu(T) \geq \psi(X_{t,x}^\nu(T))] \geq p\}.$$

and $w(t, x; p) := \inf\{y \in \mathbb{R} : (t, x, y) \in D(p)\}$

Geometric Dynamic Programming

For a stopping time $\theta \in [t, T] \mathbb{P} - \text{a.s.}$

$$D(p) = \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } (\theta, Z_{t,z}^\nu(\theta)) \in D(P_\theta) \mathbb{P} - \text{a.s.}\}.$$

with $P_\theta := \mathbb{P}[Y_{t,z}^\nu(T) \geq \psi(X_{t,x}^\nu(T)) \mid \mathcal{F}_\theta]$

Dynamic Programming for the Quantile Hedging Problem

Consider the typical example of quantile constraints, i.e.

$$D(p) := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{P}[Y_{t,z}^\nu(T) \geq \psi(X_{t,x}^\nu(T))] \geq p\}.$$

and $w(t, x; p) := \inf\{y \in \mathbb{R} : (t, x, y) \in D(p)\}$

Geometric Dynamic Programming

For a stopping time $\theta \in [t, T] \mathbb{P} - \text{a.s.}$

$$D(p) = \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } (\theta, Z_{t,z}^\nu(\theta)) \in D(P_\theta) \mathbb{P} - \text{a.s.}\}.$$

with $P_\theta := \mathbb{P}[Y_{t,z}^\nu(T) \geq \psi(X_{t,x}^\nu(T)) \mid \mathcal{F}_\theta] = p + \int_t^\theta \alpha_s dW_s$

Dynamic Programming for the Quantile Hedging Problem

Consider the typical example of quantile constraints, i.e.

$$D(p) := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{P}[Y_{t,z}^\nu(T) \geq \psi(X_{t,x}^\nu(T))] \geq p\}.$$

and $w(t, x; p) := \inf\{y \in \mathbb{R} : (t, x, y) \in D(p)\}$

Theorem (B., Elie and Touzi)

$(t, z) \in D(p)$ iff there exists (ν, α) s.t., for any stopping time
 $\theta \in [t, T] \mathbb{P} - a.s.$,

$$(\theta, Z_{t,z}^\nu(\theta)) \in D(P_{t,p}^\alpha(\theta)) \mathbb{P} - a.s.$$

where

$$P_{t,p}^\alpha = p + \int_t^\cdot \alpha_s dW_s = " \mathbb{P}[Y_{t,z}^\nu(T) \geq \psi(X_{t,x}^\nu(T)) | \mathcal{F}] "$$

Dynamic Programming for the general case

Back to general framework

$$D(p) := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}[g(Z_{t,z}^\nu(T))] \geq p\}.$$

Dynamic Programming for the general case

Back to general framework

$$D(p) := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}[g(Z_{t,z}^\nu(T))] \geq p\}.$$

Theorem (B., Elie and Touzi)

$$\exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}[g(Z_{t,z}^\nu(T))] \geq p$$

\Leftrightarrow

$$\exists \nu \in \mathcal{U} \text{ and } \alpha \text{ s.t. } g(Z_{t,z}^\nu(T)) \geq P_{t,p}^\alpha(T) \mathbb{P} - a.s.$$

with $P_{t,p}^\alpha = p + \int_t^\cdot \alpha_s dW_s$ " $= \mathbb{E}[g(Z_{t,z}^\nu(T)) | \mathcal{F}_\cdot]$

Dynamic Programming for the general case

Back to general framework

$$D(p) := \{(t, z) : \exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}[g(Z_{t,z}^\nu(T))] \geq p\}.$$

Theorem (B., Elie and Touzi)

$$\exists \nu \in \mathcal{U} \text{ s.t. } \mathbb{E}[g(Z_{t,z}^\nu(T))] \geq p$$

\Leftrightarrow

$$\exists \nu \in \mathcal{U} \text{ and } \alpha \text{ s.t. } g(Z_{t,z}^\nu(T)) \geq P_{t,p}^\alpha(T) \mathbb{P} - a.s.$$

with $P_{t,p}^\alpha = p + \int_t^\cdot \alpha_s dW_s$ " = " $\mathbb{E}[g(Z_{t,z}^\nu(T)) | \mathcal{F}_\cdot]$

Back to a.s. stochastic target problems

Apply the previous approach to the new controlled process $(Z_{t,z}^\nu, P_{t,p}^\alpha)$ and controls (ν, α) .

PDE Derivation

Theorem (B., Elie and Touzi)

w is a viscosity solution (in the discontinuous sense) of

$$\sup_{(u,\alpha) \in \mathcal{N}_0(t,x,p,w)} \mu_Y(x, w(t,x,p), u) - \mathcal{L}_{X,P}^{u,\alpha} w(t,x,p) = 0$$

where

$$\mathcal{N}_\varepsilon(\cdot, w)$$

:=

$$\{(u, \alpha) \in U \times \mathbb{R}^d : |\sigma_Y(\cdot, w, u) - D_x w \sigma_X(\cdot, u) - D_p w \alpha| \leq \varepsilon\}.$$

Comments

- No general comparison theorem, only examples.

Comments

- No general comparison theorem, only examples.
- Explicit resolution in simple case (e.g. quantile hedging in B.-S. type models with local volatility)

Comparison with other approaches

- Follmer and Leukert approach : needs a dual formulation.

Comparison with other approaches

- Follmer and Leukert approach : needs a dual formulation.
- Maximize $\mathbb{E}[g(X^\nu(T), Y^\nu(T))]$ for y fixed and invert.

Comparison with other approaches

- Follmer and Leukert approach : needs a dual formulation.
- Maximize $\mathbb{E}[g(X^\nu(T), Y^\nu(T))]$ for y fixed and invert.
- If g is invertible in y , $g(X^\nu(T), Y^\nu(T)) = P^\alpha(T)$ reduces to $Y^\nu(T) = g^{-1}(X^\nu(T), P^\alpha(T))$. In finance, one could consider

$$\inf_{\alpha} \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[g^{-1}(X^\nu(T), P^\alpha(T))] .$$

Comparison with other approaches

- Follmer and Leukert approach : needs a dual formulation.
- Maximize $\mathbb{E}[g(X^\nu(T), Y^\nu(T))]$ for y fixed and invert.
- If g is invertible in y , $g(X^\nu(T), Y^\nu(T)) = P^\alpha(T)$ reduces to $Y^\nu(T) = g^{-1}(X^\nu(T), P^\alpha(T))$. In finance, one could consider

$$\inf_{\alpha} \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[g^{-1}(X^\nu(T), P^\alpha(T))].$$

- Consider the family of BSDEs (Y^α, Z^α) such that

$$Y_t^\alpha = g^{-1}(X(T), P^\alpha(T)) + \int_t^T f(Y_s^\alpha, Z_s^\alpha) ds - \int_t^T Z_s^\alpha dW_s$$

and consider $\bar{Y}_t^\alpha := \text{essinf}\{Y_t^{\alpha'}, \alpha' = \alpha \text{ on } [0, t]\}$.

Outline

Stochastic target problems

Stochastic target with controlled loss

Optimal control under target constraints

Problem formulation

Stochastic target constraints problem (\mathbb{P} – a.s. sense)

$$V(t, z) := \sup_{\nu \in \mathcal{U}_{t,z}} \mathbb{E} [f(Z_{t,z}^{\nu}(T))]$$

$$\text{with } \mathcal{U}_{t,z} := \left\{ \nu \in \mathcal{U} \text{ s.t. } g(Z_{t,z}^{\nu}(T)) \geq 0 \text{ } \mathbb{P} - \text{a.s.} \right\}$$

Problem formulation

Stochastic target constraints problem (\mathbb{P} – a.s. sense)

$$V(t, z) := \sup_{\nu \in \mathcal{U}_{t,z}} \mathbb{E} [f(Z_{t,z}^\nu(T))]$$

$$\text{with } \mathcal{U}_{t,z} := \{ \nu \in \mathcal{U} \text{ s.t. } g(Z_{t,z}^\nu(T)) \geq 0 \text{ } \mathbb{P}-\text{a.s.} \}$$

Target constraints in expectation/probability

$$V(t, z, p) := \sup_{\nu \in \mathcal{U}_{t,z,p}} \mathbb{E} [f(Z_{t,z}^\nu(T))]$$

$$\text{with } \mathcal{U}_{t,z,p} := \{ \nu \in \mathcal{U} \text{ s.t. } \mathbb{E} [g(Z_{t,z}^\nu(T))] \geq p \}$$

Problem formulation

Stochastic target constraints problem (\mathbb{P} – a.s. sense)

$$V(t, z) := \sup_{\nu \in \mathcal{U}_{t,z}} \mathbb{E} [f(Z_{t,z}^\nu(T))]$$

$$\text{with } \mathcal{U}_{t,z} := \{\nu \in \mathcal{U} \text{ s.t. } g(Z_{t,z}^\nu(T)) \geq 0 \text{ } \mathbb{P}-\text{a.s.}\}$$

Example : Super-hedging constraint/Index tracking

$$\mathcal{U}_{t,x,y} := \{\nu \in \mathcal{U} \text{ s.t. } Y_{t,x,y}^\nu(T) \geq \psi(X_{t,x}^\nu(T)) \text{ } \mathbb{P}-\text{a.s.}\}$$

Problem formulation

Stochastic target constraints problem (\mathbb{P} – a.s. sense)

$$V(t, z) := \sup_{\nu \in \mathcal{U}_{t,z}} \mathbb{E} [f(Z_{t,z}^\nu(T))]$$

$$\text{with } \mathcal{U}_{t,z} := \{\nu \in \mathcal{U} \text{ s.t. } g(Z_{t,z}^\nu(T)) \geq 0 \text{ } \mathbb{P}-\text{a.s.}\}$$

Example : Quantile-hedging constraint/Index tracking

$$\mathcal{U}_{t,x,y,p} := \{\nu \in \mathcal{U} \text{ s.t. } \mathbb{P} [Y_{t,x,y}^\nu(T) \geq \psi(X_{t,x}^\nu(T))] \geq p\}$$

Problem formulation

Stochastic target constraints problem (\mathbb{P} – a.s. sense)

$$V(t, z) := \sup_{\nu \in \mathcal{U}_{t,z}} \mathbb{E} [f(Z_{t,z}^{\nu}(T))]$$

$$\text{with } \mathcal{U}_{t,z} := \left\{ \nu \in \mathcal{U} \text{ s.t. } g(Z_{t,z}^{\nu}(T)) \geq 0 \text{ } \mathbb{P}-\text{a.s.} \right\}$$

Example : Loss constraint

$$\mathcal{U}_{t,x,y,p} := \left\{ \nu \in \mathcal{U} \text{ s.t. } \mathbb{E} \left[\ell \left([Y_{t,x,y}^{\nu}(T) - \psi(X_{t,x}^{\nu}(T))]^- \right) \right] \leq -p \right\}$$

Problem re-formulation

Stochastic target constraints problem (\mathbb{P} – a.s. sense)

$$V(t, z) := \sup_{\nu \in \mathcal{U}_{t,z}} \mathbb{E} [f(Z_{t,z}^{\nu}(T))]$$

$$\text{with } \mathcal{U}_{t,z} := \left\{ \nu \in \mathcal{U} \text{ s.t. } g(Z_{t,z}^{\nu}(T)) \geq 0 \text{ } \mathbb{P} - \text{a.s.} \right\} .$$

Problem re-formulation

Stochastic target constraints problem (\mathbb{P} – a.s. sense)

$$V(t, z) := \sup_{\nu \in \mathcal{U}_{t,z}} \mathbb{E} [f(Z_{t,z}^{\nu}(T))]$$

$$\text{with } \mathcal{U}_{t,z} := \left\{ \nu \in \mathcal{U} \text{ s.t. } g(Z_{t,z}^{\nu}(T)) \geq 0 \text{ } \mathbb{P} - \text{a.s.} \right\} .$$

State constraint problem formulation

$$\mathcal{U}_{t,z} := \left\{ \nu \in \mathcal{U} \text{ s.t. } (s, Z_{t,z}^{\nu}(s)) \in D \text{ } \mathbb{P} - \text{a.s. } \forall s \in [t, T] \right\} , \text{ where}$$

$$D = \{(t, z) : \mathcal{U}_{t,z} \neq \emptyset\} .$$

Problem re-formulation

Stochastic target constraints problem (\mathbb{P} – a.s. sense)

$$V(t, z) := \sup_{\nu \in \mathcal{U}_{t,z}} \mathbb{E} [f(Z_{t,z}^{\nu}(T))]$$

$$\text{with } \mathcal{U}_{t,z} := \left\{ \nu \in \mathcal{U} \text{ s.t. } g(Z_{t,z}^{\nu}(T)) \geq 0 \text{ } \mathbb{P} - \text{a.s.} \right\} .$$

State constraint problem formulation

$$\begin{aligned} \mathcal{U}_{t,z} &:= \left\{ \nu \in \mathcal{U} \text{ s.t. } (s, Z_{t,z}^{\nu}(s)) \in D \text{ } \mathbb{P} - \text{a.s. } \forall s \in [t, T] \right\} , \text{ where} \\ D &= \{(t, z) : \mathcal{U}_{t,z} \neq \emptyset\} . \end{aligned}$$

Important point

D is given by “the” viscosity solution w of a PDE. Not a-priori.

More complex but implies reflexion on the boundary automatically.

PDE formulation

Assumption

The value function w of the target problem is continuous in the domain, with a continuous extension at T

PDE formulation

Assumption

The value function w of the target problem is continuous in the domain, with a continuous extension at T

Decomposition of the domain

$$\text{int}D := \{(t, x, y) \in [0, T] \times \mathbb{R}^{d+1} : y > w(t, x)\}$$

$$\partial_Z D := \{(t, x, y) \in [0, T] \times \mathbb{R}^{d+1} : y = w(t, x)\}$$

$$\partial_T D := \{(t, x, y) \in [0, T] \times \mathbb{R}^{d+1} : y \geq w(t, x), t = T\}$$

PDE formulation

On $\text{int}D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y > w(t, x)\}$

PDE formulation

On $\text{int}D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y > w(t, x)\}$

- $\forall \nu, \exists \theta > t \mathbb{P} - \text{a.s. s.t. } Y_{t,x,y}^\nu(\theta) > w(\theta, X_{t,x}^\nu(\theta)) \mathbb{P} - \text{a.s.}$

PDE formulation

On $\text{int}D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y > w(t, x)\}$

- $\forall \nu, \exists \theta > t \mathbb{P} - \text{a.s. s.t. } Y_{t,x,y}^\nu(\theta) > w(\theta, X_{t,x}^\nu(\theta)) \mathbb{P} - \text{a.s.}$
- The state constraint does not play any role.

PDE formulation

On $\text{int}D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y > w(t, x)\}$

- $\forall \nu, \exists \theta > t \mathbb{P} - \text{a.s. s.t. } Y_{t,x,y}^\nu(\theta) > w(\theta, X_{t,x}^\nu(\theta)) \mathbb{P} - \text{a.s.}$
- The state constraint does not play any role.
- Usual HJB equation

$$\inf_u -\mathcal{L}_{X,Y}^u V(t, x, y) = 0 .$$

PDE formulation

On $\text{int}D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y > w(t, x)\}$

- $\forall \nu, \exists \theta > t \mathbb{P} - \text{a.s. s.t. } Y_{t,x,y}^\nu(\theta) > w(\theta, X_{t,x}^\nu(\theta)) \mathbb{P} - \text{a.s.}$
- The state constraint does not play any role.
- Usual HJB equation

$$\inf_u -\mathcal{L}_{X,Y}^u V(t, x, y) = 0 .$$

On $\partial_T D := \{(t, x, y) \in [0, T] \times \mathbb{R}^{d+1} : y \geq w(t, x), t = T\}$

Standard boundary condition $V(T-, x, y) = f(x, y).$

PDE formulation

On $\partial_Z D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y = w(t, x)\}$

PDE formulation

On $\partial_Z D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y = w(t, x)\}$

- Must choose ν s.t. $dY_{t,x,y}^\nu(t) \geq dw(t, X_{t,x}^\nu(t))$

PDE formulation

On $\partial_Z D := \{(t, x, y) \in [0, T] \times \mathbb{R}^{d+1} : y = w(t, x)\}$

- Must choose ν s.t. $dY_{t,x,y}^\nu(t) \geq dw(t, X_{t,x}^\nu(t))$
- This implies
 - $\sigma_Y(x, y, u) = Dw(t, x)\sigma_X(x, u)$

PDE formulation

On $\partial_Z D := \{(t, x, y) \in [0, T] \times \mathbb{R}^{d+1} : y = w(t, x)\}$

- Must choose ν s.t. $dY_{t,x,y}^\nu(t) \geq dw(t, X_{t,x}^\nu(t))$
- This implies
 - $\sigma_Y(x, y, u) = Dw(t, x)\sigma_X(x, u)$
 - $\mu_Y(x, y, u) - \mathcal{L}_X^u w(t, x) \geq 0$

PDE formulation

On $\partial_Z D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y = w(t, x)\}$

- Must choose ν s.t. $dY_{t,x,y}^\nu(t) \geq dw(t, X_{t,x}^\nu(t))$
 - This implies
 - $\sigma_Y(x, y, u) = Dw(t, x)\sigma_X(x, u)$
 - $\mu_Y(x, y, u) - \mathcal{L}_X^u w(t, x) \geq 0$
- ⇒ Defines a set $U(t, x, y, w)$.

PDE formulation

On $\partial_Z D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y = w(t, x)\}$

- Must choose ν s.t. $dY_{t,x,y}^\nu(t) \geq dw(t, X_{t,x}^\nu(t))$
- This implies
 - $\sigma_Y(x, y, u) = Dw(t, x)\sigma_X(x, u)$
 - $\mu_Y(x, y, u) - \mathcal{L}_X^u w(t, x) \geq 0$
- ⇒ Defines a set $U(t, x, y, w)$.
- Constrained HJB equation

$$\inf_{u \in U(t, x, y, w)} -\mathcal{L}_{X,Y}^u V(t, x, y) = 0 .$$

PDE formulation

Precise formulation on

$$\partial_Z D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y = w(t, x)\}$$

PDE formulation

Precise formulation on

$$\partial_Z D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y = w(t, x)\}$$

- V_* is a super-solution of

$$\inf_{\varphi \in \mathcal{T}^*(t, x)} \inf_{u \in U(t, x, y, \varphi)} -\mathcal{L}_{X, Y}^u V_*(t, x, y) \geq 0 .$$

$$\mathcal{T}^*(t, x) := \{\varphi \in C^{1,2} \text{ s.t. } 0 = \max(w - \varphi) = (w - \varphi)(t, x)\}.$$

PDE formulation

Precise formulation on

$$\partial_Z D := \{(t, x, y) \in [0, T) \times \mathbb{R}^{d+1} : y = w(t, x)\}$$

- V_* is a super-solution of

$$\inf_{\varphi \in \mathcal{T}^*(t, x)} \inf_{u \in U(t, x, y, \varphi)} -\mathcal{L}_{X, Y}^u V_*(t, x, y) \geq 0 .$$

$$\mathcal{T}^*(t, x) := \{\varphi \in C^{1,2} \text{ s.t. } 0 = \max(w - \varphi) = (w - \varphi)(t, x)\}.$$

- V^* is a sub-solution of

$$\sup_{\varphi \in \mathcal{T}_*(t, x)} \inf_{u \in U(t, x, y, \varphi)} -\mathcal{L}_{X, Y}^u V^*(t, x, y) \leq 0 .$$

$$\mathcal{T}_*(t, x) := \{\varphi \in C^{1,2} \text{ s.t. } 0 = \min(w - \varphi) = (w - \varphi)(t, x)\}.$$

PDE formulation in the smooth case

Assumption

- $w \in C^{1,2}([0, T) \times \mathbb{R}^d)$ with a continuous extension at T .

PDE formulation in the smooth case

Assumption

- $w \in C^{1,2}([0, T) \times \mathbb{R}^d)$ with a continuous extension at T .
- There exists a locally Lipschitz map $\check{u} : \mathbb{R}^{d+1} \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\mathcal{N}_0(z, p) \neq \emptyset \Rightarrow \mathcal{N}_0(z, p) = \{\check{u}(z, p)\}.$$

PDE formulation in the smooth case

Assumption

- $w \in C^{1,2}([0, T) \times \mathbb{R}^d)$ with a continuous extension at T .
- There exists a locally Lipschitz map $\check{u} : \mathbb{R}^{d+1} \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\mathcal{N}_0(z, p) \neq \emptyset \Rightarrow \mathcal{N}_0(z, p) = \{\check{u}(z, p)\}.$$

In this case, w is a solution of

$$0 = \min \left\{ \mu_Y(\cdot, w, \hat{u}) - \mathcal{L}_X^{\hat{u}} w, \mathbf{1}_{\hat{u} \in \text{int}(U)} \right\} \text{ with } \hat{u} := \check{u}(\cdot, w, Dw)$$

PDE formulation in the smooth case

Dirichlet boundary condition

PDE formulation in the smooth case

Dirichlet boundary condition

Set $\mathcal{V}_*(t, x) := V_*(t, x, w(t, x))$ and $\mathcal{V}^*(t, x) := V^*(t, x, w(t, x))$

PDE formulation in the smooth case

Dirichlet boundary condition

Set $\mathcal{V}_*(t, x) := V_*(t, x, w(t, x))$ and $\mathcal{V}^*(t, x) := V^*(t, x, w(t, x))$

Theorem

\mathcal{V}_* and \mathcal{V}^* are viscosity super- and sub- solution of

$$\begin{aligned}-\mathcal{L}_X^{\hat{u}} \varphi \mathbf{1}_{\hat{u} \in \text{int}(U)} &= 0 \quad \text{with } \hat{u} := \check{u}(\cdot, w, Dw) \\ \varphi(T-, \cdot) &= f(\cdot, w(T-, \cdot))\end{aligned}$$

PDE formulation in the smooth case

Dirichlet boundary condition

Set $\mathcal{V}_*(t, x) := V_*(t, x, w(t, x))$ and $\mathcal{V}^*(t, x) := V^*(t, x, w(t, x))$

Theorem

\mathcal{V}_* and \mathcal{V}^* are viscosity super- and sub- solution of

$$\begin{aligned}-\mathcal{L}_X^{\hat{u}} \varphi \mathbf{1}_{\hat{u} \in \text{int}(U)} &= 0 \quad \text{with } \hat{u} := \check{u}(\cdot, w, Dw) \\ \varphi(T-, \cdot) &= f(\cdot, w(T-, \cdot))\end{aligned}$$

Do the following

PDE formulation in the smooth case

Dirichlet boundary condition

Set $\mathcal{V}_*(t, x) := V_*(t, x, w(t, x))$ and $\mathcal{V}^*(t, x) := V^*(t, x, w(t, x))$

Theorem

\mathcal{V}_* and \mathcal{V}^* are viscosity super- and sub- solution of

$$\begin{aligned}-\mathcal{L}_X^{\hat{u}} \varphi \mathbf{1}_{\hat{u} \in \text{int}(U)} &= 0 \quad \text{with } \hat{u} := \check{u}(\cdot, w, Dw) \\ \varphi(T-, \cdot) &= f(\cdot, w(T-, \cdot))\end{aligned}$$

Do the following

- Solve the PDE for $w \Rightarrow \partial_Z D$

PDE formulation in the smooth case

Dirichlet boundary condition

Set $\mathcal{V}_*(t, x) := V_*(t, x, w(t, x))$ and $\mathcal{V}^*(t, x) := V^*(t, x, w(t, x))$

Theorem

\mathcal{V}_* and \mathcal{V}^* are viscosity super- and sub- solution of

$$\begin{aligned}-\mathcal{L}_X^{\hat{u}} \varphi \mathbf{1}_{\hat{u} \in \text{int}(U)} &= 0 \quad \text{with } \hat{u} := \check{u}(\cdot, w, Dw) \\ \varphi(T-, \cdot) &= f(\cdot, w(T-, \cdot))\end{aligned}$$

Do the following

- Solve the PDE for $w \Rightarrow \partial_Z D$
- Solve the PDE for $\mathcal{V} : (t, x) \mapsto V(t, x, w(t, x))$.

PDE formulation in the smooth case

Dirichlet boundary condition

Set $\mathcal{V}_*(t, x) := V_*(t, x, w(t, x))$ and $\mathcal{V}^*(t, x) := V^*(t, x, w(t, x))$

Theorem

\mathcal{V}_* and \mathcal{V}^* are viscosity super- and sub- solution of

$$\begin{aligned}-\mathcal{L}_X^{\hat{u}} \varphi \mathbf{1}_{\hat{u} \in \text{int}(U)} &= 0 \quad \text{with } \hat{u} := \check{u}(\cdot, w, Dw) \\ \varphi(T-, \cdot) &= f(\cdot, w(T-, \cdot))\end{aligned}$$

Do the following

- Solve the PDE for $w \Rightarrow \partial_Z D$
- Solve the PDE for $\mathcal{V} : (t, x) \mapsto V(t, x, w(t, x))$.
- Solve the HJB PDE for V in $\text{int}D$ with
 $V(t, x, y) = \mathcal{V}(t, x)$ on $\partial_Z D$ and $V(T-, \cdot) = f$ on $\partial_T D$.

Possible extensions

Could be extended to

- Jump diffusion processes (in progress by L. Moreau).
- American type constraints (Dynamic programming by B. and T. N. Vu)
- Multiple constraints (in progress with T. N. Vu)
- etc...

References

- B., R. Elie and C. Imbert, Optimal Control under Stochastic Target Constraints, to appear in *SIAM Journal on Control and Optimization*.
- B., R. Elie and N. Touzi, Stochastic target problems with controlled loss, *SIAM Journal on Control and Optimization*, 2009.
- B. and T. N. Vu, The obstacle version of the Geometric Dynamic Programming Principle : Application to the pricing of American options under constraints, to appear in *Applied Mathematics and Optimization*.