Market indifference prices

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Asset price models

Mathematical Finance:

- price dynamics **exogenous**: semimartingale models
- stochastic analysis

+ mathematically tractable
+ dynamic model: hedging
+ 'easy' to calibrate: volatility

– only suitable for (very) *liquid markets* or *small investors*

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Economics:

- prices **endogeneous**: demand matches supply
- equilibrium theory
- \+ undeniably reasonable explanation for price formation
- \+ excellent qualitative properties
- – difficult to calibrate: preferences, endowments
- – quantitative accuracy?

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Our goal:

Bridge the gap between these price formation principles!

Outline

Basic principle: Stay close to Black-Scholes

- Wealth dynamics induced by 'small' trades should be given by the usual stochastic integrals at least to first order:

$$V_T(\varepsilon Q) = \varepsilon \int_0^T Q_s \, dS_s^0 + o(\varepsilon) \quad \text{for} \quad \varepsilon \to 0$$

- Specify wealth dynamics for 'any' predictable trading strategy
- Asset prices for small exposures should allow for an expansion of the form

$$p(\varepsilon\psi) = \varepsilon \underbrace{\mathbb{E}_{\mathbb{Q}}\psi}_{\text{Black-Scholes price}} + \underbrace{\frac{1}{2}\varepsilon^2 C(\psi)}_{\text{liquidity correction}} + o(\varepsilon^2) \quad \text{for} \quad \varepsilon \to 0$$

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Main idea:

Use dynamic indifference prices!

General setting

Financial model

- beliefs and information flow described by stochastic basis $(\Omega, \mathscr{F}_T, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P})$
- marketed claims: European with payoff profiles $\psi_i \in L^0(\mathscr{F}_T)$ $(i = 1, \dots, I)$ possessing all exponential moments
- utility functions $u_m : \mathbb{R} \to \mathbb{R}$ $(m = 1, \dots, M)$ with bounded absoulte risk aversion:

$$0 < c_* \le -\frac{u_m''(x)}{u_m'(x)} \le c^* < \infty$$

  $\rightsquigarrow$ similar to exponential utilities
- initial endowments $\alpha_0^m \in L^0(\mathscr{F}_T)$ $(m = 1, \dots, M)$ have finite exponential moments and form a Pareto-optimal allocation

Pareto-optimal allocations

Recall:

- $\alpha = (\alpha^m) \in L^0(\mathscr{F}_T, \mathbb{R}^M)$ is **Pareto-optimal** if $\Sigma = \Sigma_m \alpha^m$ cannot be re-distributed to form a better allocation $\tilde{\alpha} = (\tilde{\alpha}^m)$:

$$\mathbb{E}u_m(\tilde{\alpha}^m) \geq \mathbb{E}u_m(\alpha^m) \quad \text{with '>' for some } m \in \{1, \ldots, M\} \quad .$$

- $\alpha = (\alpha^m)$ Pareto-optimal iff same marginal indifference price quotes from all market makers, i.e., we have a universal marginal pricing measure $\mathbb{Q}(\alpha)$ for the market:

$$\frac{d\mathbb{Q}(\alpha)}{d\mathbb{P}} \propto u_m'(\alpha^m) \quad \text{independent of } m$$

- Pareto-optimal allocations realized through trades among market makers $\rightsquigarrow$ complete OTC-market
- 1-1 correspondence to weight vectors $w \in \mathbb{R}_+^M$, $\sum w_m = 1$.

A single transaction

- pre-transaction endowment of market makers: $\alpha = (\alpha^m)$ with total endowment $\Sigma = \sum_m \alpha^m$
- investor submits passes $q = (q^1, \dots, q^I)$ claims on to the market makers along with a cash transfer of size x
- total endowment of market makers after transaction

$$\tilde{\Sigma} = \Sigma + (x + \langle q, \psi \rangle)$$

  is redistributed among the market makers to form a new Pareto optimal allocation of endowments $\tilde{\alpha} = (\tilde{\alpha}^m)$

Obvious question:

How exactly to determine the cash transfer x and the new allocation $\tilde{\alpha}$?

A single transaction

Theorem

There exists a unique cash transfer $x = x(q)$ *and a unique Pareto-optimal allocation* $\tilde{\alpha} = (\tilde{\alpha}^m(q))$ *of the total endowment* $\tilde{\Sigma}(x, q) = \Sigma + (x + \langle q, \psi \rangle)$ *such that each market maker is as well-off after the transaction as he was before:*

$$\mathbb{E} u_m(\tilde{\alpha}^m) = \mathbb{E} u_m(\alpha^m) \quad (m = 1, \ldots, M)\,.$$

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Note:

The cash transfer x can be viewed as the **market's indifference price** for the transaction q: it is the minimal amount for which the market makers can accommodate the investor's order without anyone of them being worse-off.

$\rightsquigarrow$ most friendly market environment for our investor!

Basic questions about market indifference prices

- How does the market indifference price depend on the transaction's size?
- Under what conditions is there a liquidity premium?
- What are its key determinants?
- How does the market's pre-transaction exposure affect the market indifference price?
- How to take into account the market makers' risk aversion and ability to hedge?
- Is there a difference between a model with several market makers and one with a representative market maker?
- ...

Expansions of market indifference prices

Theorem

The indifference price $x = x(q)$ is twice cont. differentiable with

$$x(q+\Delta q) - x(q) = -\mathbb{E}_{\mathbb{Q}}[\langle \Delta q, \psi \rangle] + \frac{1}{2R_0}\mathbb{E}_{\mathbb{R}}[(\langle \Delta q, \psi\rangle - \mathbb{E}_{\mathbb{Q}}\langle \Delta q, \psi\rangle)^2] + \frac{R_0}{2}\mathbb{E}_{\mathbb{R}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{R}}\right)^2 var_{\rho}[Z\Delta q]\right] + o(|\Delta q|^2), \quad \Delta q \to 0,$$

where

- $\mathbb{Q} \sim \mathbb{P}$ *is the equilibrium pricing measure determined by the market makers' Pareto allocation*
- R_0 *is the market's risk tolerance at transaction time*
- $\mathbb{R} \sim \mathbb{Q}$ *is the market's risk tolerance measure*
- ρ *is the vector of the market makers' risk relative tolerances*
- Z *describes the sensitivities of Pareto weights w.r.t.* q

Some observations

$$
\begin{aligned}
x(q+\Delta q) - x(q) &= -\mathbb{E}_{\mathbb{Q}}[\langle \Delta q, \psi \rangle] \\
&+ \frac{1}{2R_0}\mathbb{E}_{\mathbb{R}}[(\langle \Delta q, \psi \rangle - \mathbb{E}_{\mathbb{Q}}\langle \Delta q, \psi \rangle)^2] + \frac{R_0}{2}\mathbb{E}_{\mathbb{R}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{R}}\right)^2 \mathrm{var}_{\rho}[Z\Delta q]\right] \\
&+ o(|\Delta q|^2), \quad \Delta q \to 0,
\end{aligned}
$$

- Up to 1st order, the transaction costs are as in a small investor setting with pricing measure $\mathbb{Q}$.
- The market indifference price is convex in the transaction size.
- The liquidity premium is always nonnegative and vanishes if and only if we have a pure (and pointless) cash transaction: $\langle \Delta q, \psi \rangle \equiv$ const

Some observations

$$x(q+\Delta q) - x(q) = -\mathbb{E}_{\mathbb{Q}}[\langle \Delta q, \psi \rangle]$$
$$+ \frac{1}{2R_0}\mathbb{E}_{\mathbb{R}}[(\langle \Delta q, \psi \rangle - \mathbb{E}_{\mathbb{Q}}\langle \Delta q, \psi \rangle)^2] + \frac{R_0}{2}\mathbb{E}_{\mathbb{R}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{R}}\right)^2 \mathrm{var}_\rho[Z\Delta q]\right]$$
$$+ o(|\Delta q|^2), \quad \Delta q \to 0,$$

- The liquidity premium splits into an aggregate component and one featuring the relative risk tolerances $\rho^m = R^m / \sum_l R^l$.
- Up to 2nd order, there is no difference between our multiple market maker model and a representative market maker model if and only if

  $$\mathbb{E}_{\mathbb{R}^l}\psi = \mathbb{E}_{\mathbb{R}^m}\psi \quad (l, m = 1, \dots, M)$$

  where $\mathbb{R}^m$ is market maker m's risk tolerance measure, i.e., if and only if the extra endowment with any tradable claim has the same 2nd order impact on every market maker's expected utility.

Key tool: Convex duality of saddle functions

Theorem

The representative agent's utility

$$r(v, x, q) = \max_{\alpha \,:\, \sum_m \alpha^m = \Sigma + (x + \langle q, \psi \rangle)} \sum_m v^m \mathbb{E} u_m(\alpha^m)$$

has the dual

$$\tilde{r}(u, y, q) = \sup_v \inf_x \{\langle v, u \rangle + xy - r(v, x, q)\}$$

in the sense that

$$r(v, x, q) = \inf_u \sup_y \{\langle v, u \rangle + xy - \tilde{r}(u, y, q)\}$$

and, for fixed q, (v, x) is a saddle point for $\tilde{r}(u, y, q)$ if and only if (u, y) is a saddle point for $r(v, x, q)$.

Implications of duality

- properties of r translate into properties of $\tilde{r}$
- $r \in C^2$ iff $\tilde{r} \in C^2$
- derivatives of r can be computed in terms of derivatives of $\tilde{r}$
- For conjugate saddle points (v, x) and (u, y):

$$v = \partial_u \tilde{r}(u, y, q),\, x = \partial_y \tilde{r}(u, y, q),$$

and

$$u = \partial_v r(v, x, q),\, y = \partial_x r(v, x, q)\,.$$

$\rightsquigarrow$ explicit construction of cash transfer $x = \tilde{r}(u, 1, q)$ and Pareto weights $w = \partial_u \tilde{r}(u, 1, q)/|\partial_u \tilde{r}(u, 1, q)|_1$ for given utility vector u and transaction q

The wealth dynamics for simple strategies

When our investor follows a simple strategy

$$Q_t = \sum_n q_n 1_{(t_{n-1},t_n]}(t) \quad \text{with} \quad q_n \in L^0(\mathscr{F}_{t_{n-1}})$$

we can proceed inductively to determine the corresponding cash balance process

$$X_t = \sum_n x_n 1_{(t_{n-1},t_n]}(t)$$

and (conditionally) Pareto-optimal allocations

$$A_t = \sum_n \alpha_n 1_{(t_{n-1},t_n]}(t)\,.$$

In particular, we obtain the investor's terminal wealth mapping:

$$Q \mapsto V_T(Q) == \langle Q_T, \psi \rangle = X_T = \sum_m \alpha_T^m - \sum_m \alpha_0^m$$

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Mathematical challenge:

How to consistently pass to general predictable strategies?

The technical key observation

Hence: Sufficient to track the evolution of weight vectors W_t and of the overall endowment Σ_t...

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Hence: Sufficient to track the evolution of weight vectors W_t and of the overall endowment Σ_t... or more simply, given the current cumulatively generated position Q_t, keep track of the amount of cash X_t exchanged so far:

$$\Sigma_t = \Sigma_0 + (X_t + \langle Q_t, \psi \rangle) \, .$$

But: (W_t, X_t) changes whenever Q_t does: 'wild' dynamics!

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Fortunately: Given $q = Q_t$, (W_t, X_t) can be recovered from the vector of the market makers' expected utilities $u = U_t$:

$$W_t = W_t(u, q), \quad X_t = X_t(u, q)$$

— and these utilities evolve as martingales:

- no changes because of transactions: indifference pricing principle
- changes induced by arrival of new information: martingales

An SDE for the utility process

We need to understand the martingale dynamics of expected utilities.

Assumption

- *filtration generated by Brownian motion* B
- *contingent claims* ψ *and total initial endowment* Σ_0 *Malliavin differentiable with bounded Malliavin derivatives*
- *bounded prudence:* $\left|-\frac{u_m'''(x)}{u_m''(x)}\right| \leq K < +\infty$

Notation:

- $A(w, x, q) =$ Pareto allocation of $\Sigma_0 + (x + \langle q, \psi \rangle)$ with weights w
- $U_t(w, x, q) = (\mathbb{E}\left[u_m(A^m(w, x, q)) \,|\, \mathscr{F}_t\right])_{m=1,\ldots,M}$
- $dU_t(w, x, q) = F_t(w, x, q)\, dB_t$

An SDE for the utility process

Theorem

For every simple strategy Q the induced process of expected utilities for our market makers solves the SDE

$$dU_t = G_t(U_t, Q_t)\, dB_t, \quad U_0 = (\mathbb{E} u_m(\alpha_0^m))$$

where

$$G_t(u, q) = F_t(W_t(u, q), X_t(u, q), q)\,.$$

Note:

This SDE makes sense for any predictable (sufficiently integrable) strategy Q!

The rest: Stability theory for SDEs

Corollary

For Q^n such that $\int_0^T (Q_t^n - Q_t)^2\, dt \to 0$ in probability, the corresponding solutions U^n converge uniformly in probability to the solution U corresponding to Q.
In particular, we have a consistent and continuous extension of our terminal wealth mapping $Q \mapsto V_T(Q)$ from simple strategies to predictable, a.s. square-integrable strategies.

No arbitrage

Theorem

There is no arbitrage opportunity for the large investor among **all** *predictable strategies.*

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Sketch of Proof: For the large investor to make a profit, some market makers have to lose in terms of expected utility.

However, utility processes are local martingales and bounded from above — thus submartingales! $\square$

Hedging of contingent claims

Problem

Large investor wishes to hedge against a claim H using the assets ψ available on the market.

- Is it possible at all?
- How much initial capital is needed?
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Solution

Assume that H has all exponential moments and let $\psi = W_T$. Then the initial capital the large investor needs to replicate the option H is given by the market indifference price that would be quoted for H if this claim was traded at time 0. The hedging strategy can be computed in terms of the martingale representations for the utility processes induced by the corresponding Pareto allocation:

$$G_t(U_t, Q_t) = I_t \,.$$

Conclusion

- new model for obtaining endogenous price dynamics of illiquid assets: market indifference pricing
- 2nd order expansions of transaction prices with insights into the structure of liquidity premia
- nonlinear wealth dynamics accounting for liquidity premia
- consistent and continuous extension from simple to general predictable strategies via SDE for utility process
- complete market with simple pricing rule: indifference yet again

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THANK YOU VERY MUCH!