Robust quantification of the exposure to operational risk:
Bringing economic sense to economic capital

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Please, ask questions!

Basel II

Basel Committee
on Banking Supervision

International Convergence
of Capital Measurement
and Capital Standards

A Revised Framework
Comprehensive Version

This document is a compilation of the June 2004 Basel II Framework, the elements of the 1988 Accord that were not revised during the Basel II process, the 1996 Amendment to the Capital Accord to Incorporate Market Risks, and the 2005 paper on the Application of Basel II to Trading Activities and the Treatment of Double Default Effects. No new elements have been introduced in this compilation.

June 2006

BANK FOR INTERNATIONAL SETTLEMENTS

http://www.bis.org/publ/bcbs128.htm

The three pillars approach

- **First pillar**: **Minimum capital requirements** (quantification of risk)

    - Specifies the guiding principles for the estimation of regulatory/economic capital.
    - Operational risk is included as a new type of risk.

- **Second pillar: Supervisory review process.**

- **Third pillar: Market discipline (+ public disclosure)**

Operational Risk: Definition

[source: Basel II]

644.   Operational risk is defined as the risk of loss resulting from **inadequate** or **failed internal processes**, **people** and **systems** or from **external events**.

This definition includes legal risk, but excludes strategic and reputational risk.

Operational Risk: Measurement

Measurement approaches

- ■ Basic Indicator Approach (BIA)

$$K_{\mathrm{BIA}} = \alpha \times \mathrm{EI}, \text{ where } \begin{cases} \alpha = 0.15 \\ \mathrm{EI} = \text{gross income (mean of the last 3 years)} \end{cases}$$

- ■ Standardised Approach (TSA)

$$K_{\mathrm{TSA}} = \sum_{i=1}^{8} \beta_i \times \mathrm{EI}_i, \text{ where } \begin{cases} \beta_i \text{ are defined by the regulator} \\ \mathrm{EI}_i \text{ are the gross income for line i.} \end{cases}$$

- ■ Advanced Measurement Approaches (AMA)
  - ■ Scorecard approach.
  - ■ Loss Distribution approach.

AMA Soundness Standard (Basel II)

667.   Given the continuing evolution of analytical approaches for operational risk, the Committee is not specifying the approach or distributional assumptions used to generate the operational risk measure for regulatory capital purposes. However, a bank must be able to demonstrate that its approach captures potentially severe 'tail' loss events. Whatever approach is used, a bank must demonstrate that its operational risk measure meets a soundness standard comparable to that of the internal ratings-based approach for credit risk, (i.e. comparable to a **one year holding** period and a **99.9th** percentile confidence interval).

Business lines & risk types

Business line ↓	Risk type						
	Internal fraud	External fraud	Employment Practices and Workplace Safety	Clients, Products & Business Practices	Damage to physical assets	Business disruption and system failures	Execution, Delivery & Process Management
Corporate Finance							
Trading & Sales							
Retail Banking							
Commercial Banking							
Payment and Settlement							
Agency Services and Custody							
Asset Management							
Retail Brokerage							

Loss distribution approach

■ Model the distribution of the **aggregate losses** for a given **business line & risk type**

$$Loss_t^{[i,j]} = \sum_{n=1}^{N_t^{[i,j]}} X_{nt}^{[i,j]} \; ;$$

$N_t^{[i,j]}$ is the number of losses in year t

for business line i and risk type j.

■ Calculate **Capital at Risk** (99.9% percentile) of the aggregate loss distribution per **business line & risk type** and **add** them.

$$CaR = \sum_{i=1}^{8} \sum_{j=1}^{7} CaR^{[i,j]}$$
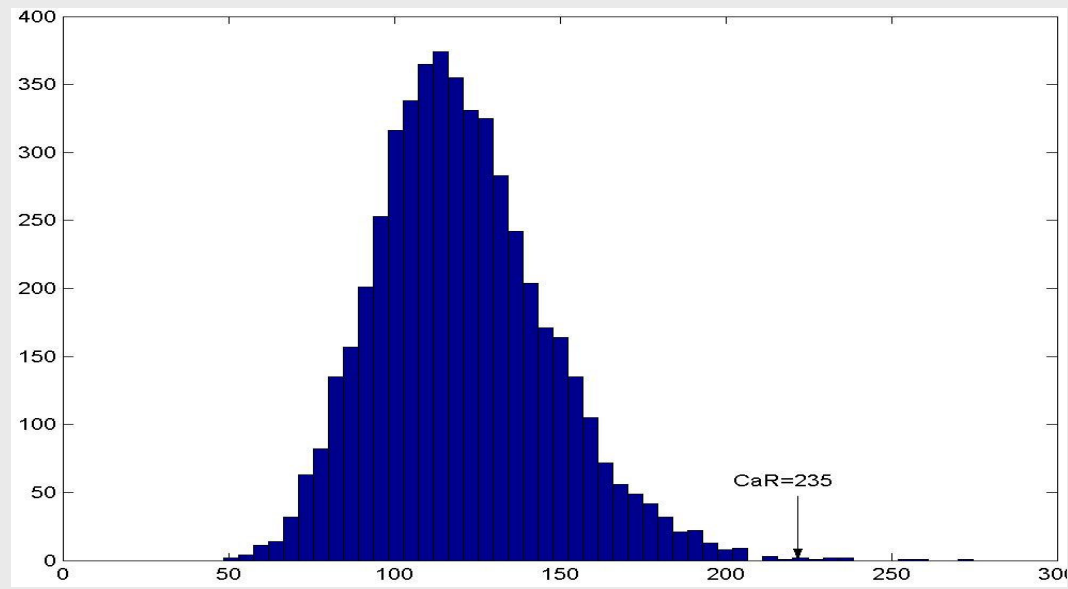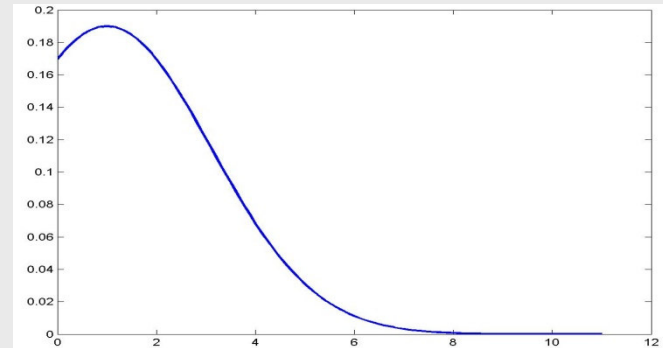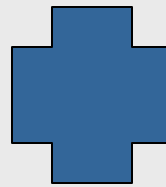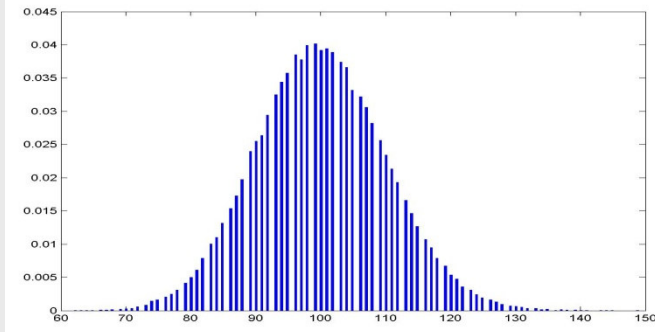
Actuarial models: Frequency + Severity

- Hypothesis
  - **Severities** of losses are **independent**
  - **Severities and frequencies** are **independent**
- Model separately
  - Frequency $\{N_t\}$

    E.g. Poisson, negative binomial, Cox process,…
  - Severity $\{X_{nt}\}$

    E. g. Lognormal, Gaussian inverse, Gamma, Weibull, …
- Obtain the distribution of aggregate losses by combining these distributions. [Panjer, FFT, **MC sim.**]
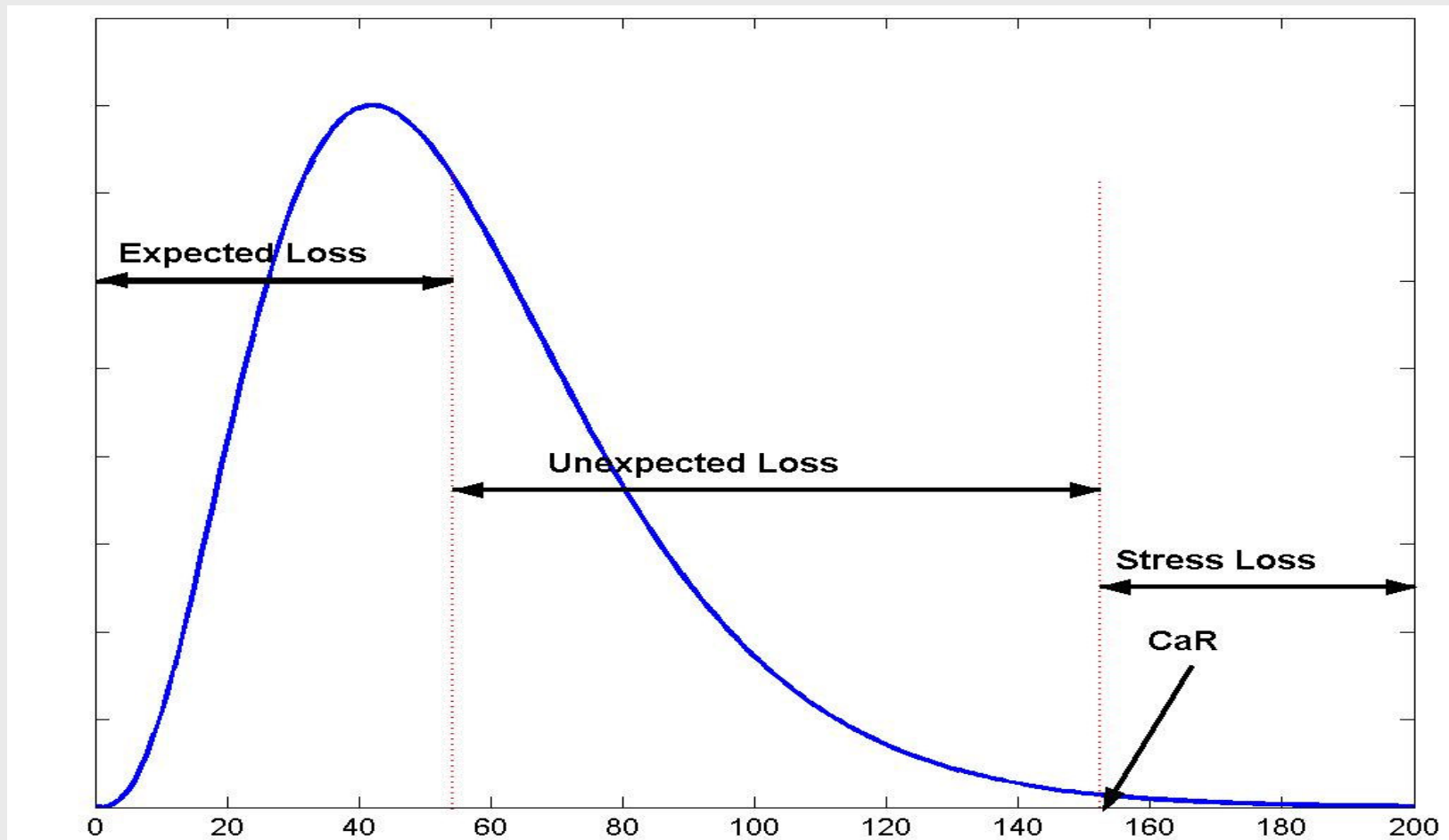
Risk analysis

- **Calculate aggregate yearly loss distribution** from the frequency and severity distributions.

- **Compute risk measures**

  - Expected loss

  - **Capital at Risk (CaR)**

    e.g. 99.9% percentile of the aggregate loss distribution.

  - **Conditional CaR (Expected shortfall)**

    Expected loss, given that the loss is above CaR

Aggregation of frequency and severity dists.

Expected & unexpected loss

Computational issues in risk analysis

Algorithms to compute **risk measures**

- **Deterministic algorithms**

  Discretized approximation to aggregate loss distribution.

  - Panjer
  - Fast Fourier Transform (FFT)

- **Monte Carlo algorithms**

  Empirical compound distribution obtained by simulation.

  Computationally costly.

  - **Use variance reduction techniques**
  - Hardware solutions: Grid computing, GPU's, …

Modeling the frequency of events

- Use **only internal data**
- **Time unit for fit**: 1 day, 1week, 1 month, 1 year (too few data!)
- Model distributions:
  - **Poisson**
  - **Negative binomial**
  - Cox process
  - Empirical

  The differences between the risk measures obtained with different models are generally small.
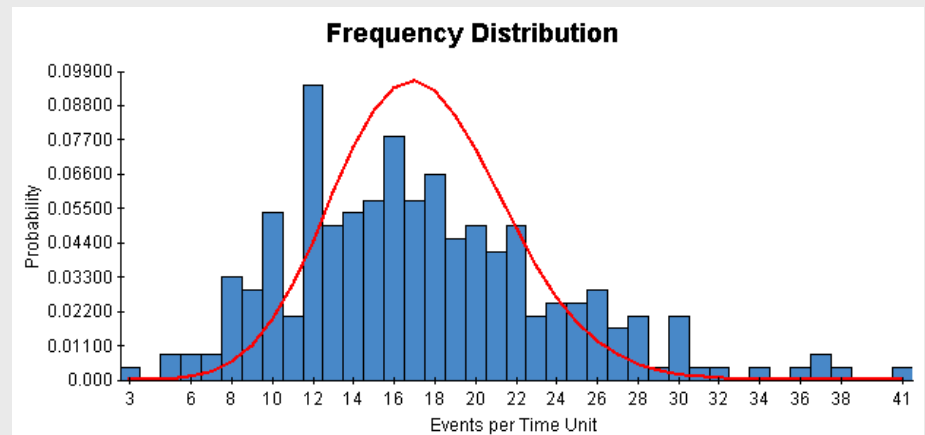- **Correct the distribution parameters** to take into account the **collection threshold**.

Model distributions for frequencies

- **Poisson model**

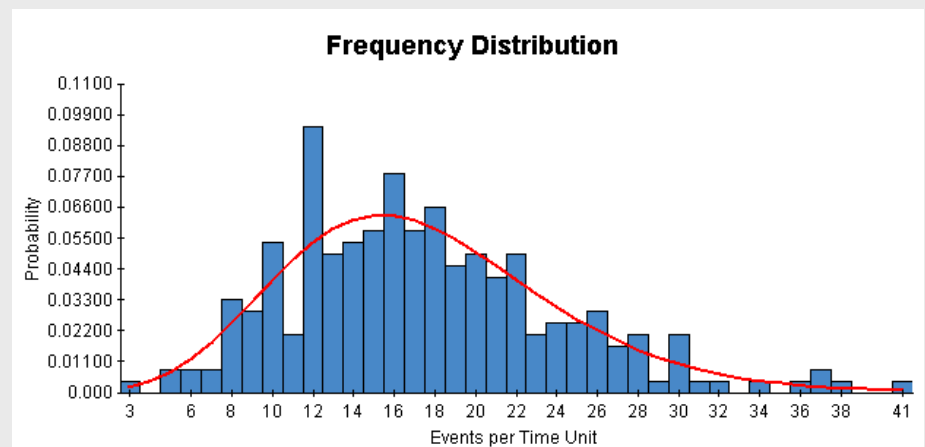 One-parameter model

  average frequency: λ

  mean = variance

- **Negative binomial**

  Two parameters

  mean < variance

Modeling the severity of events

- Use **internal + external + scenarios**
- Take into account the **collection threshold** in the fit (truncated data)
- Model distributions:
  - **Lognormal**
  - **Piecewise models:**
    - Model for the body (e.g. empirical, lognormal)
    - Model for the tail
      - Generalized Pareto
      - g-and-h distribution

The **differences** among the risk measures obtained with **different models** are generally **large**.

Modeling the severity of events

Focus on the tail
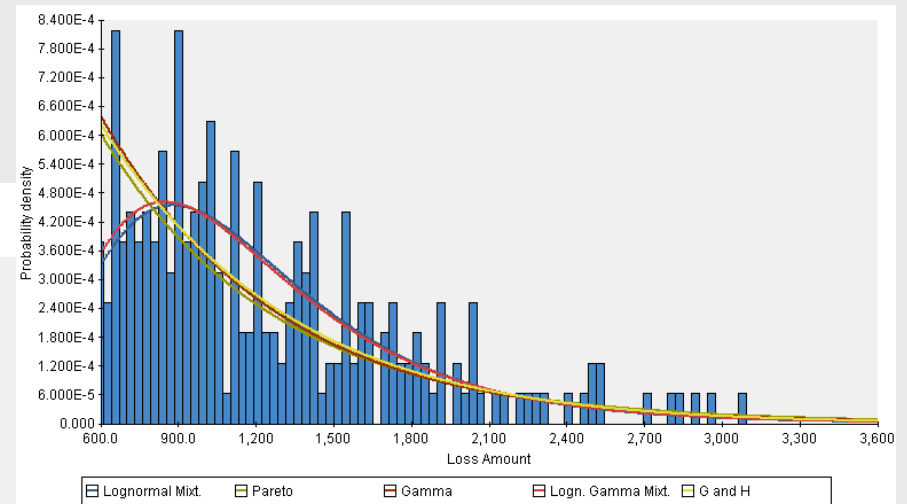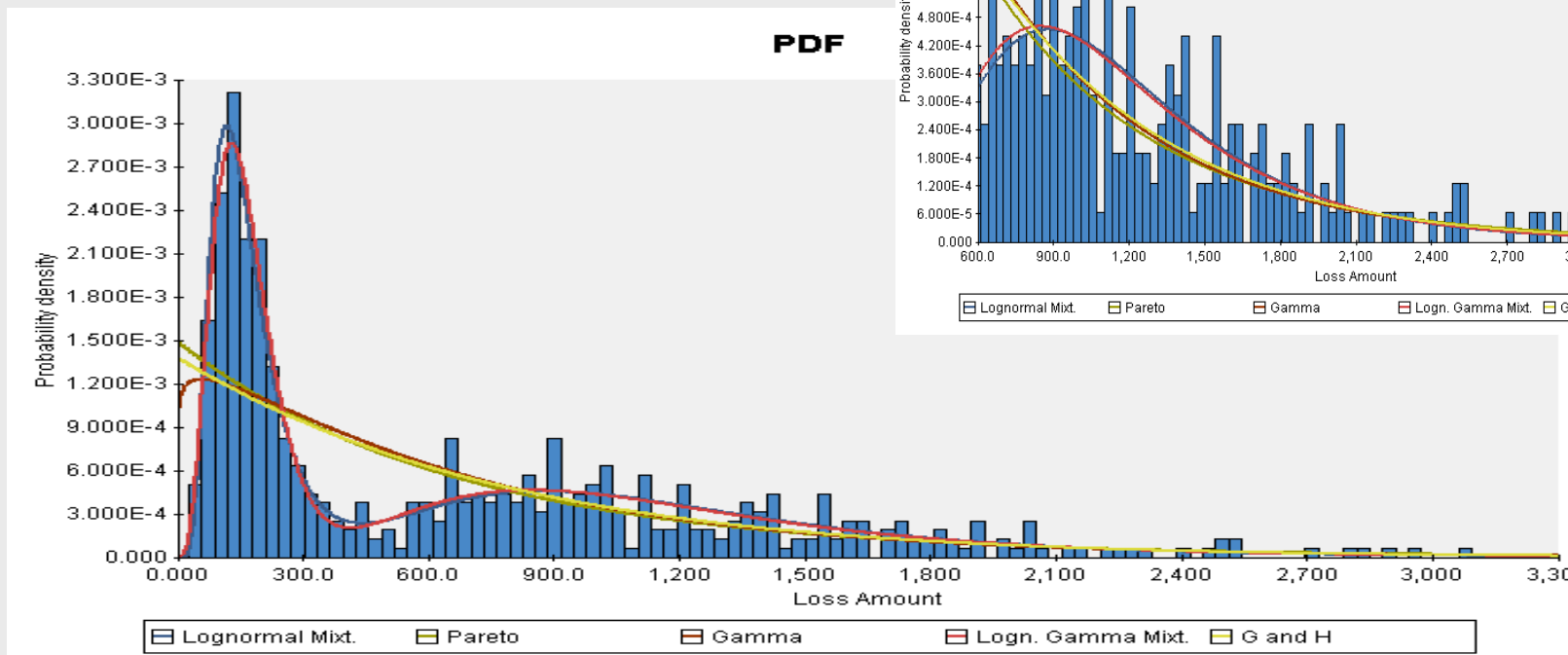
Separate models for the body and tail
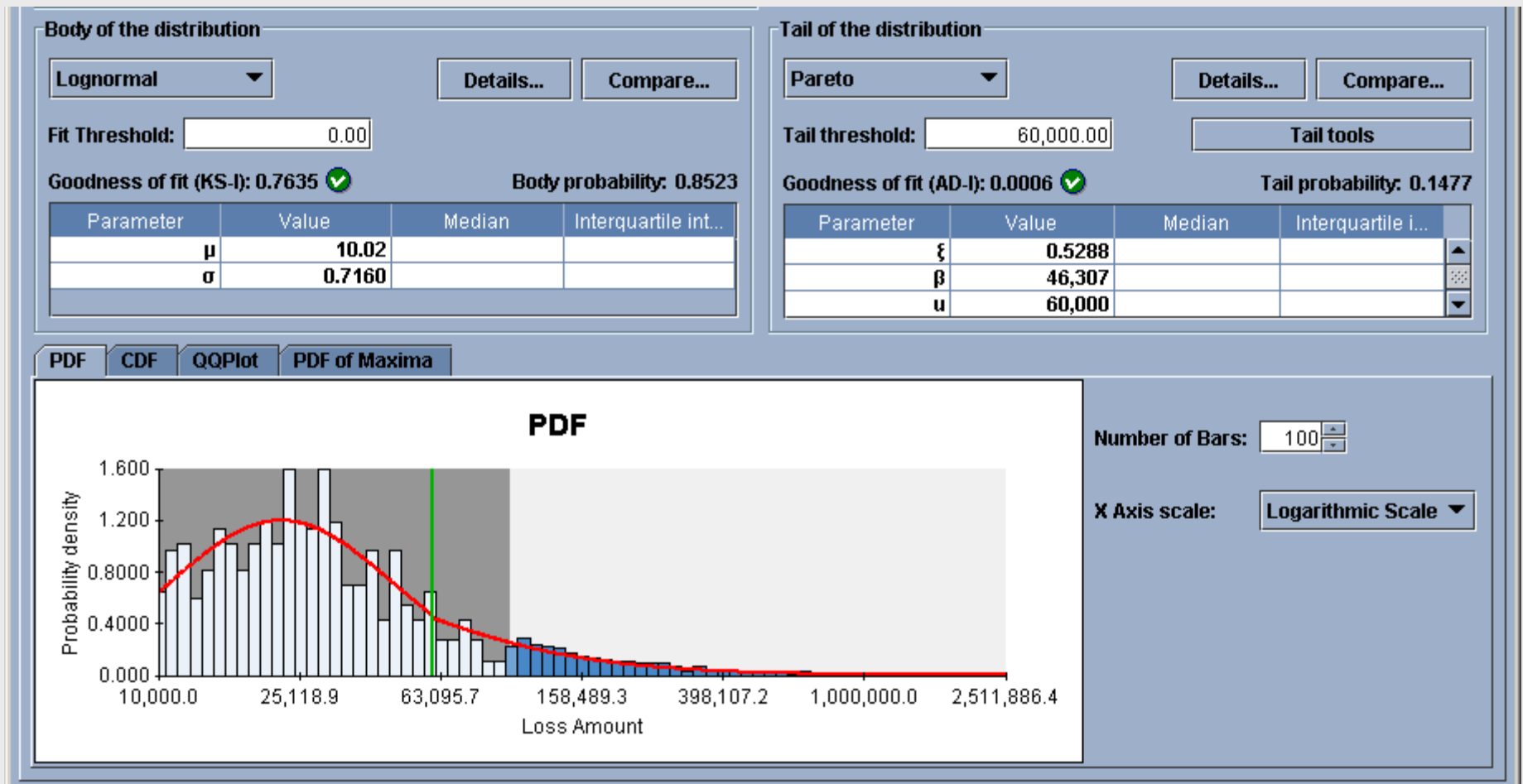
A cautionary tale

Tails are notoriously **difficult to model**
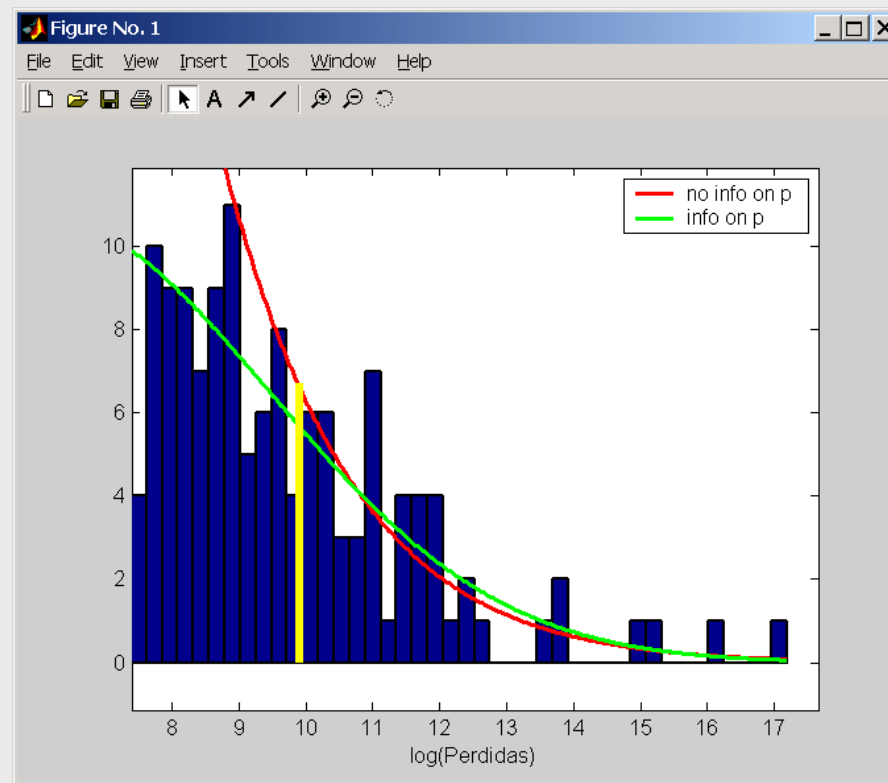
The lognormal distribution

$$LNpdf(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}\,x}\exp\left\{-\frac{1}{2\sigma^2}\left(\log x - \mu\right)^2\right\}$$

$$x > 0$$

$$X \sim \exp(\mu + \sigma Z); \quad Z \sim N(0,1)$$

- $\exp(\mu) \Rightarrow$ scale
- $\sigma \Rightarrow$ tails

LN distribution (μ = 10.00, σ = 2.50)

log(X)
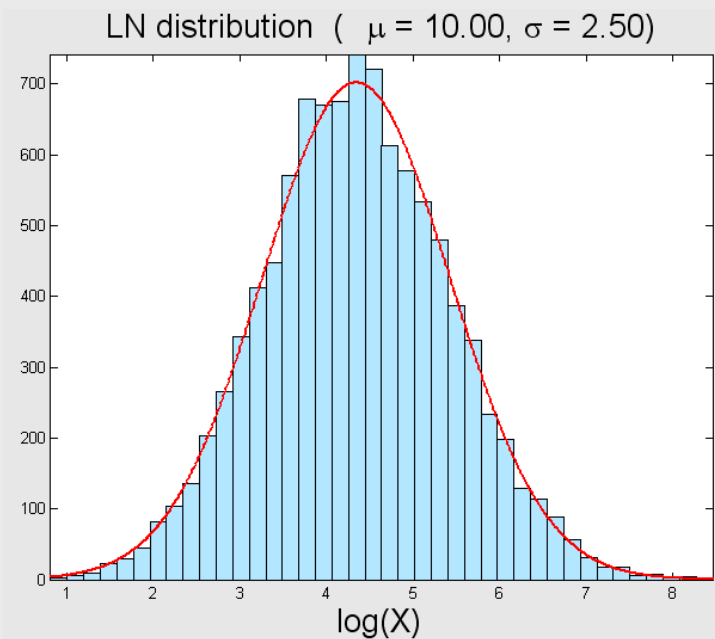
The g-and-h distribution

$$X = a + b \frac{e^{gZ} - 1}{g} \exp\left(\frac{1}{2} h Z^2 \right)$$

$$Z \sim N(0,1)$$

- g $\Rightarrow$ skewness
- h $\Rightarrow$ kurtosis

g-and-h distribution (g = 1.10 h = 0.10)

Advantages

- Flexible, realistic fits (Dutta & Perry, 2007)

Disadvantages

- Slow convergence to asymptotic regime (EVT) (Degen et al., 2007)
- Unstable estimates of parameters
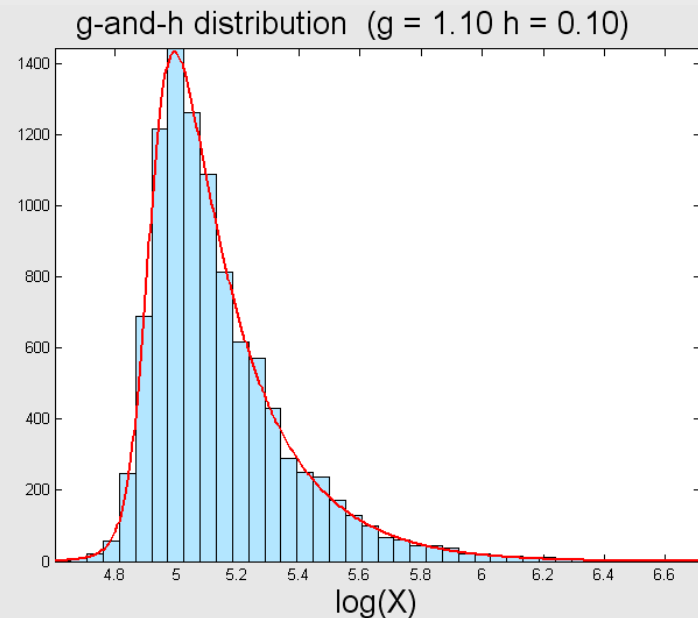
Extreme Value Theory and operational risk

- **Asymptotic regime**:

  CaR is dominated by **single extreme events** from the tail of the severity distribution.

  - Asymptotically, the tail of a distribution is has **Generalized Pareto** form.
  - These extreme events should be
    - Independent.
    - Identically distributed.
    - Constant probability occurrence per unit time.
      $\Rightarrow$ **Poisson** distribution.

- **Model**: **Poisson + Pareto tail**.

The Generalized Pareto distribution

- Probability density function

$$GPpdf(x; u, \beta, \xi)$$

$$= \frac{1}{\beta}\left(1 + \frac{\xi}{\beta}(x-u)_+\right)^{-1-\frac{1}{\xi}}$$

$$\beta > 0$$

$$x \geq u \qquad (\text{if } \xi \geq 0)$$

GP distribution (u = 3.00e+005,   β = 5.00e+005, ξ = 1.00)

log(X)
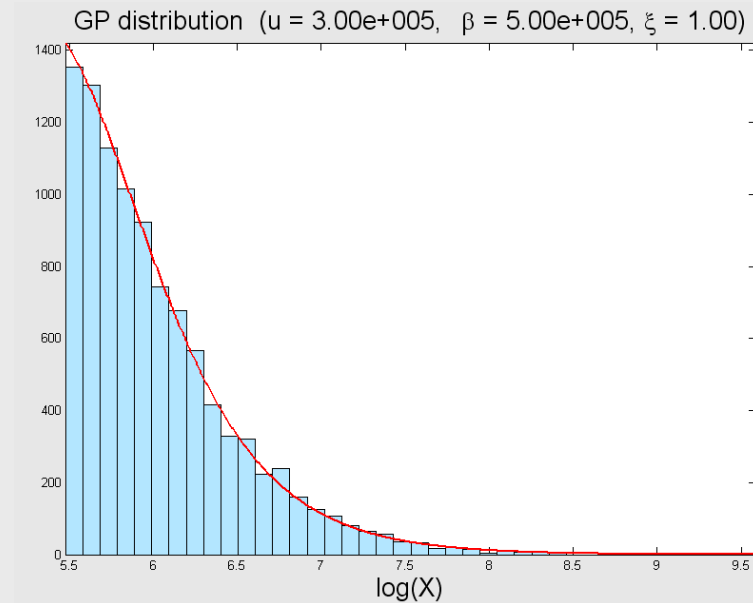
The parameter ξ

- If $\xi \geq 0.5$ the **variance diverges.**
- If $\xi \geq 1$ the **mean diverges.**
  - The **expected loss** is not defined.
  - Empirical estimates of the **unexpected loss** (the difference between a high percentile of the aggregate loss distribution and the expected loss) **can be negative !**
- In Pareto fits to **empirical** operational loss data, values of ξ **close to 1** and even larger can be found.

Pareto fit: Estimates of ξ (N = 10^5)

- **Theoretical value for Pareto data** ξ **= 0.7**
- **Theoretical value for lognormal data** ξ **= 0**
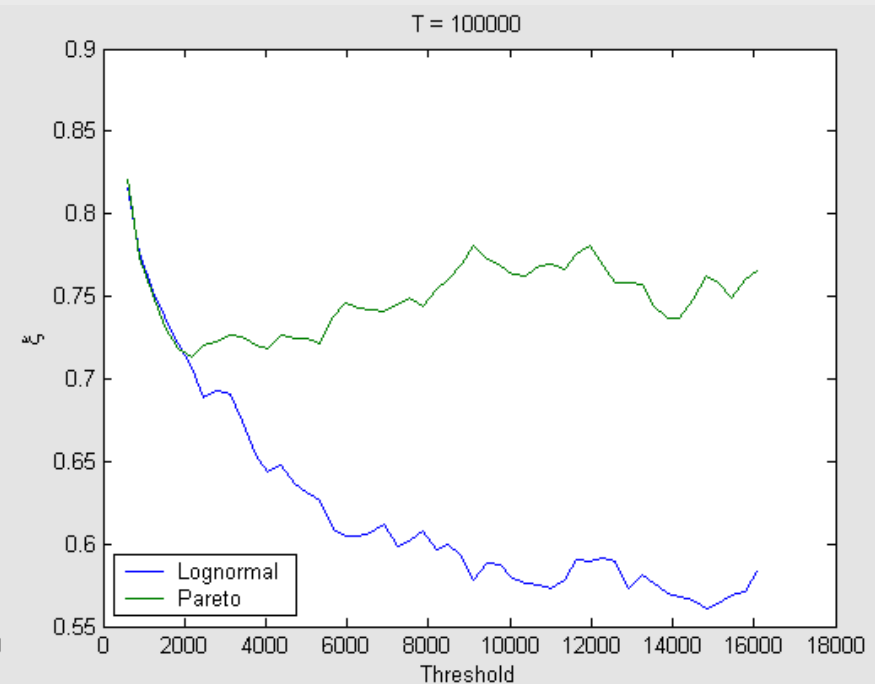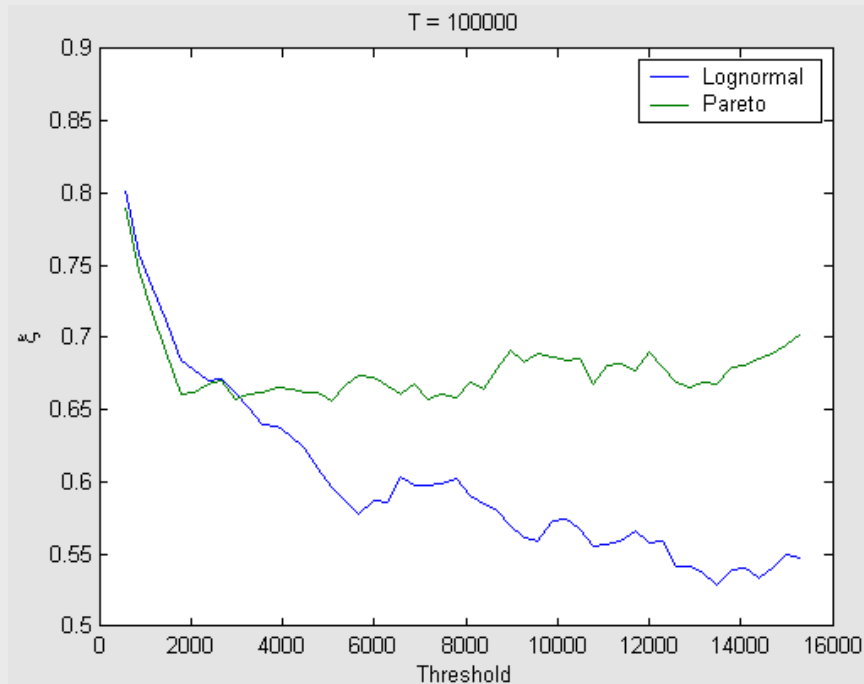
Pareto fit: Estimates of ξ ($N = 10^3$)

- **Theoretical value for Pareto data $\xi = 0.7$**
- **Asymptotic value for lognormal data $\xi = 0$**
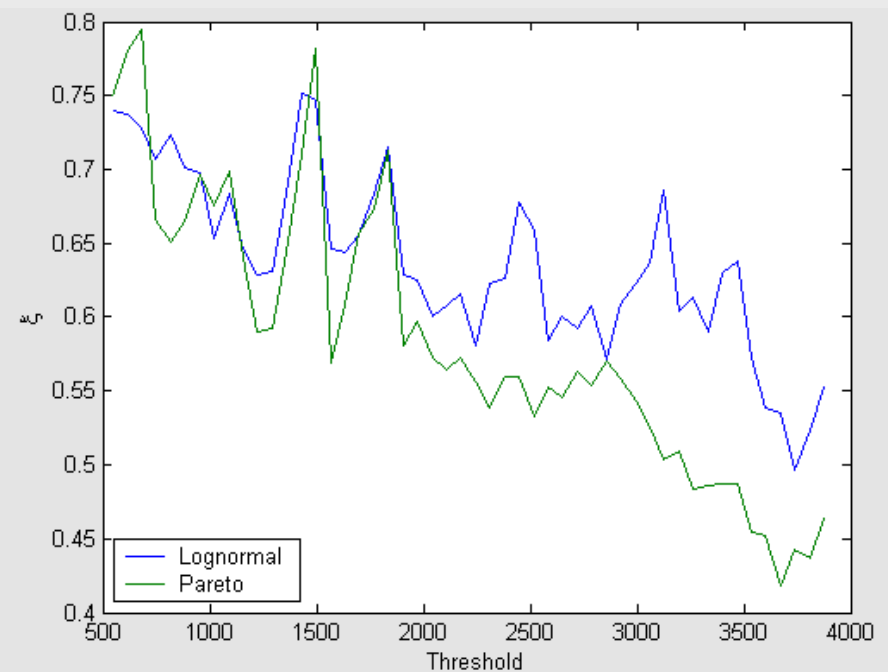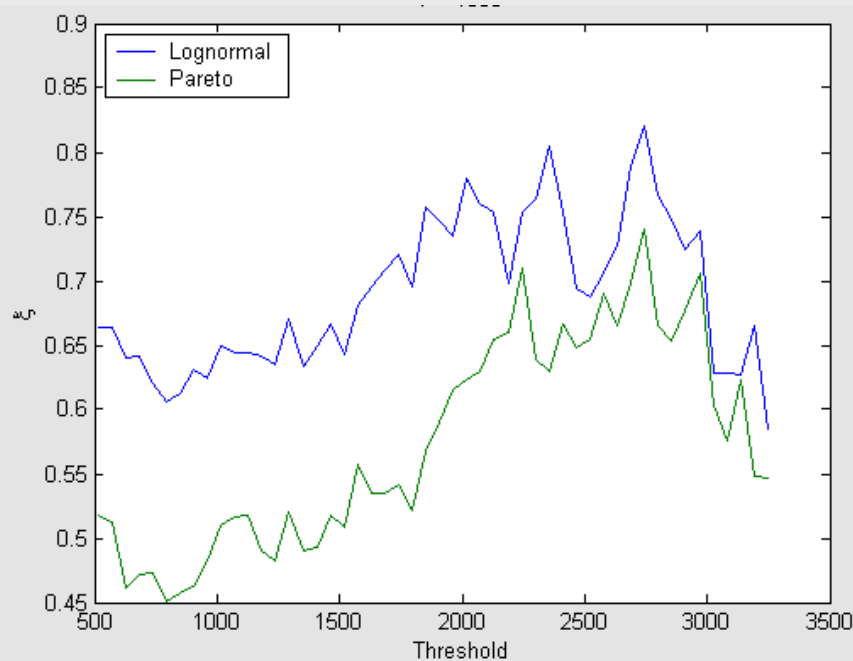
Sensitivity to single events ($N=10^3$, $M=100$)

	u	β	ξ	CaR ($\times 10^{-3}$)
Theoretical	1930	2300	0.7	3604
Maximum excluded	1900 [1876, 1914]	2352 [2086, 2726]	0.55 [0.45, 0.70]	1144 [661, 3167]
Maximum included	1928 [1913, 1.934]	2303 [2022, 2619]	0.66 [0.55, 0.81]	2492 [1261, 7695]
variation	32 [19, 50]	-65 [-107, -39]	0.1 [0.08, 0.13]	1335 [538, 4537]
% variation	1.67 [0.97, 2.69]	-2.78 [-4.48, -1.72]	17.92 [13.03, 26.96]	104.01 [70.36, 145.72]

Model uncertainty

- Losses sampled from a **lognormal distribution** ($\mu = 5, \sigma$)
- **Sample size N = 10,000**

  5 yeas of loss data $\Rightarrow$ Poisson model ($\lambda = 2,000$)
- **Collection threshold: u**

  Best severity fit

	$u = 3,000$			$u = 10,000$		
σ	best fit	CaR	error	best fit	CaR	error
1.00	LN-gamma	8.07E+07	-0.01%	Gamma mixture	8.27E+07	2.37%
1.25	g-and-h	1.15E+08	0.29%	g-and-h	1.15E+08	-0.26%
1.50	g-and-h	1.85E+08	3.04%	Burr	2.31E+08	28.52%
1.75	LN-gamma	2.73E+08	-12.08%	LN-gamma	2.74E+08	-11.70%
2.00	LN	7.17E+08	0.43%	Lognormal	7.18E+08	0.50%
2.25	LN	1.99E+09	6.94%	LN mixture	1.85E+09	-0.74%
2.50	LN mixture	3.25E+09	-30.54%	g-and-h	3.42E+09	-26.92%
2.75	Burr	9.59E+10	462.20%	Burr	4.66E+10	173.29%
3.00	Burr	1.89E+11	201.73%	Burr	2.24E+11	257.60%

Which model for the tail?

- **5 yeas of data** of losses
- Data sampled from a
  Lognormal ($\mu = 5$, $\sigma = 2$)
- **The sample size is N.**
- Model:

  frequency:
  - **Poisson $\lambda = N/5$**

  Severity:
  - **lognormal**
  - **LN body + g-and-h tail**
  - **LN body + Pareto tail**
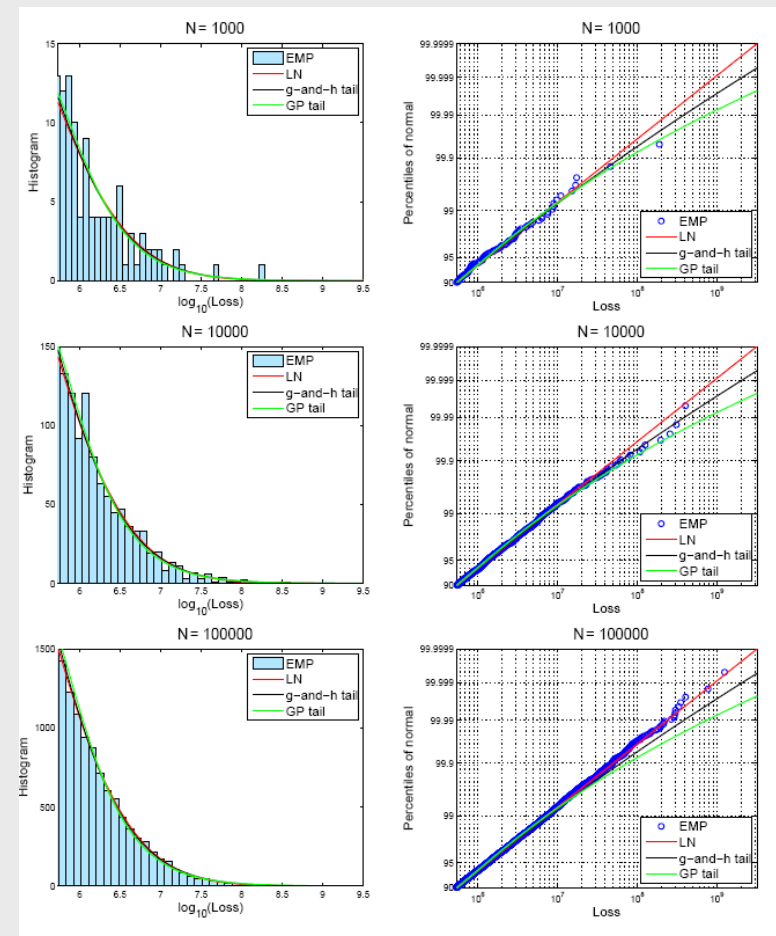
Which model to measure of risk?

λ	Tail model	CaR $\times 10^{-9}$	cCaR $\times 10^{-9}$
200	LN	1.48	2.87
	GP	15.14	∞
	g-and-h	3.49	9.75
2,000	LN	5.55	9.44
	GP	151.93	∞
	g-and-h	16.98	42.79
20,000	LN	23.60	33.76
	GP	1522.28	∞
	g-and-h	80.48	181.61

Statistics for goodness of fit tests

$F_N(x)$: Empirical cdf

$F(x)$: Model distribution (fitted to the data)

- **Kolmogorov-Smirnov** (KS)

$$KS = \arg\max_x \left\| F(x) - F_N(x) \right\|$$

- **Cramer-von Mises** (CvM)

$$CvM = \int_0^\infty dF(x) \left(F(x) - F_N(x) \right)^2$$

- Anderson-Darling (AD) + right-tailed variant (rt-AD)

$$AD = \int_0^\infty dF(x) \frac{\left(F(x) - F_N(x) \right)^2}{F(x)\left(1 - F(x)\right)}$$

$$\text{rt-}AD = \int_0^\infty dF(x) \frac{\left(F(x) - F_N(x) \right)^2}{1 - F(x)};$$

Goodness of fit tests (lognormal sample)

N	Tail model	KS	CvM	AD	rt-AD
1,000	LN	0.457	0.597	0.627	0.705
	GP	0.540	0.657	0.710	0.836
	g-and-h	0.653	0.797	0.828	0.891
10,000	LN	0.618	0.572	0.673	0.682
	GP	0.071	0.104	0.124	0.066
	g-and-h	0.180	0.143	0.183	0.238
100,000	LN	0.769	0.785	0.899	0.838
	GP	0.006	0.004	0.003	0.002
	g-and-h	0.007	0.014	0.019	0.027

Goodness of fit tests (LN body + g-and-h tail)

N	Tail model	KS	CvM	AD	rt-AD
1,000	LN	0.640	0.674	0.695	0.703
	GP	0.875	0.755	0.735	0.839
	g-and-h	0.770	0.752	0.768	0.776
10,000	LN	0.116	0.240	0.163	0.120
	GP	0.440	0.372	0.239	0.205
	g-and-h	0.525	0.684	0.568	0.414
100,000	LN	0.026	0.050	0.045	0.025
	GP	0.022	0.098	0.042	0.010
	g-and-h	0.247	0.233	0.170	0.252

Goodness of fit tests (LN body + GP tail)

N	Tail model	KS	CvM	AD	rt-AD
1,000	LN	0.827	0.681	0.813	0.775
	GP	0.810	0.889	0.949	0.962
	g-and-h	0.694	0.734	0.831	0.859
10,000	LN	0.649	0.659	0.370	0.222
	GP	0.586	0.468	0.532	0.493
	g-and-h	0.245	0.290	0.288	0.304
100,000	LN	0.003	0.006	0.000	0.000
	GP	0.254	0.195	0.177	0.163
	g-and-h	0.007	0.008	0.002	0.002

Lognormal vs. Pareto

- It is extremely **difficult to distinguish** between **lognormal** and **Pareto** tails for **small data samples**.
- If **data** is actually **lognormal**, but we describe it using a **Pareto model**, **CaR** is typically **overestimated**.
- If **data** is actually **Pareto** , but we describe it using a **lognormal model**, **CaR** is typically **underestimated**.
- **Is EVT directly applicable?**
    - We may not be in the asymptotic regime yet.
    - There is an upper bound for the losses an institution can have (use of distributions with finite support?)

Lognormal vs. Pareto (in the Internet)

A. B. Downey (2005) *Computer communications,*
"**Lognormal and Pareto distributions in the Internet**"

- **Insufficient or ambiguous evidence** for long-tailed (Pareto) behavior in many datasets

    Example: Distribution of file sizes

- In many cases **lognormal fit as good as Pareto model**

- Some **evidence for long-tailed distributions** in
    - Interarrival times of TCP packets
    - Distribution of transfer times

If data were Pareto

- Empirical estimates of ξ can be close to 1 for real operational loss data.

- Extremely large unrealistic estimates of CaR (economic interpretation?).

- Very unstable estimates in samples with less than $T = 10^4 - 10^5$ events
  - Difficulties in the choice of threshold for POT fit.
  - Sensitivity to the presence or absence of extreme events.
  - Lack of stability of risk measures with time.

Assuming a cap on the losses

■ Fits become more **robust**

- Models with **finite moments**.
- **Less sensitive to single events**.
- **Risks measures more stable** with time.

■ Loss cap can be used as a single control parameter

- Can be set using **economic arguments.**
- **Less arbitrary** than other modeling choices (in particular, than the parametric form of the severity distribution).
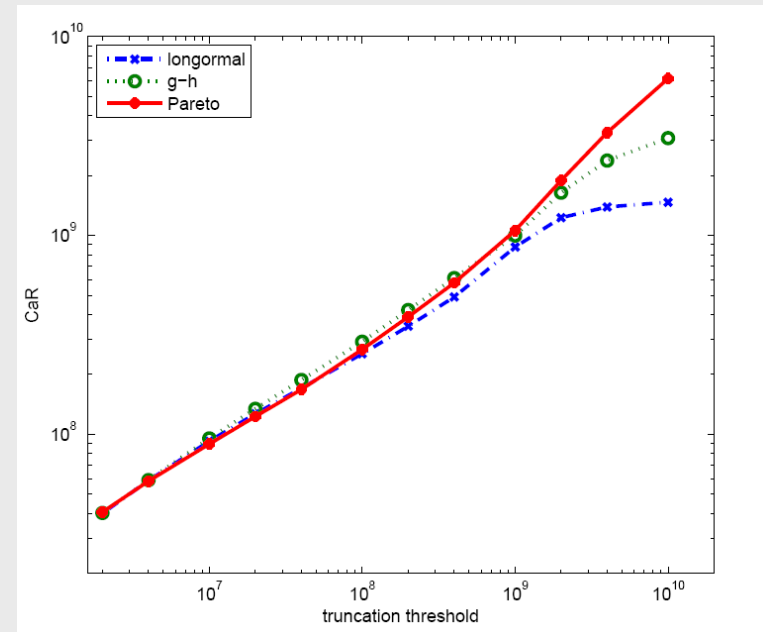
CaR estimates with LN data + loss cap

- **Frequency**: Poisson losses ($\lambda = 200$)
- **Severity**:

  LN distribution

  $\mu = 5$, $\sigma = 2$.

Plots

- without right truncation (top plot)
- truncated at $u_{right} = 10^9$ (middle plot)
- truncated at $u_{right} = 10^{10}$ (bottom plot)
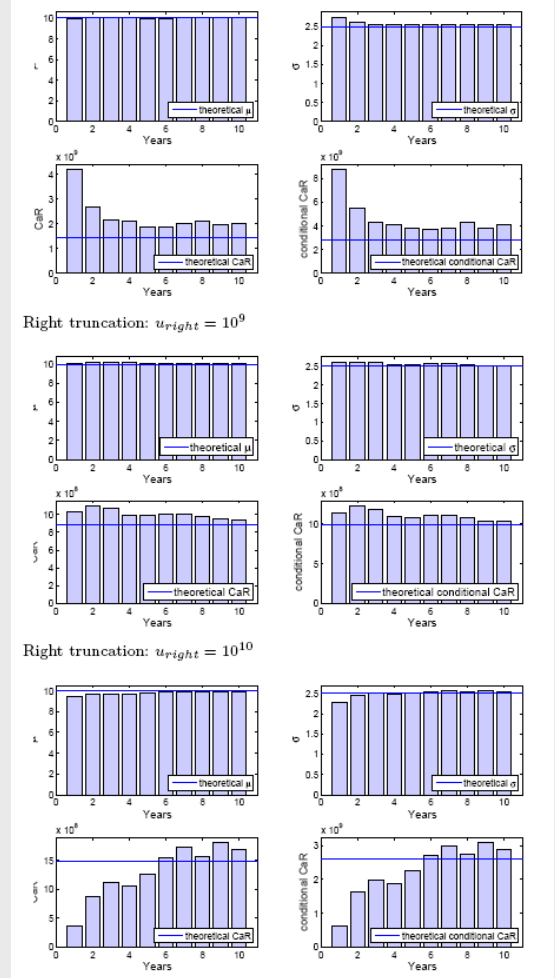
CaR estimates with g-and-h tail + loss cap

- **Frequency**: Poisson losses ($\lambda = 200$)
- **Severity**:

  LN body $\qquad \mu = 5, \sigma = 2.$

  g-and-h tail ($p_{tail} = 0.15$)

  $u = 3 \times 10^5; \; a = 0; \; b = 5 \times 10^4;$

  $g = 2.10; \; h = 0.25,$

Plots

- without right truncation   (top plot)
- truncated at $u_{right} = 10^9$ (middle plot)
- truncated at $u_{right} = 10^{10}$ (bottom plot)
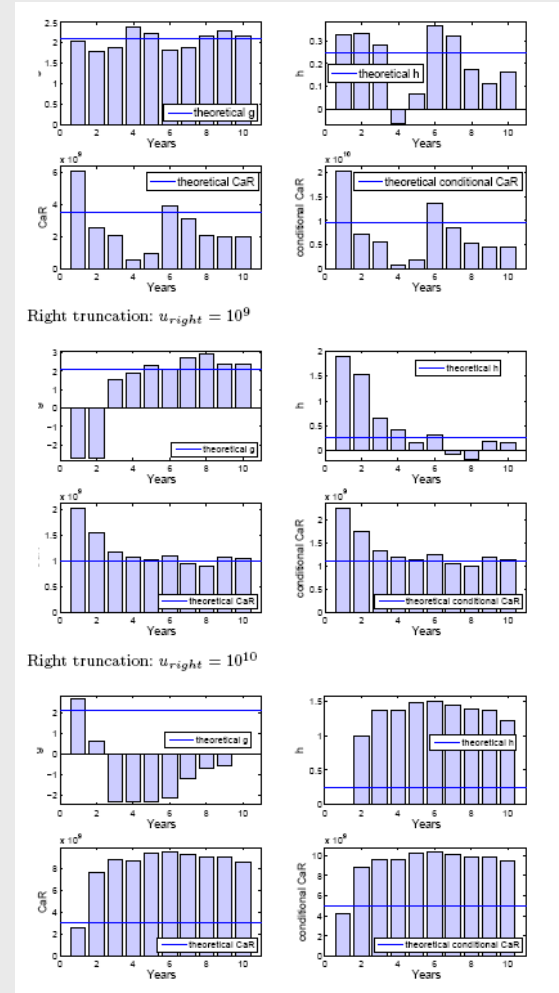
CaR estimates with GP tail + loss cap

- **Frequency**: Poisson losses ($\lambda = 200$)
- **Severity**:

   LN body $\mu = 5$, $\sigma = 2$.

   GP tail ($p_{tail} = 0.15$)

   $u = 3 \times 10^5$; $\beta = 5 \times 10^5$; $\xi = 1$,

Plots

- without right truncation   (top plot)
- truncated at $u_{right} = 10^9$ (middle plot)
- truncated at $u_{right} = 10^{10}$ (bottom plot)
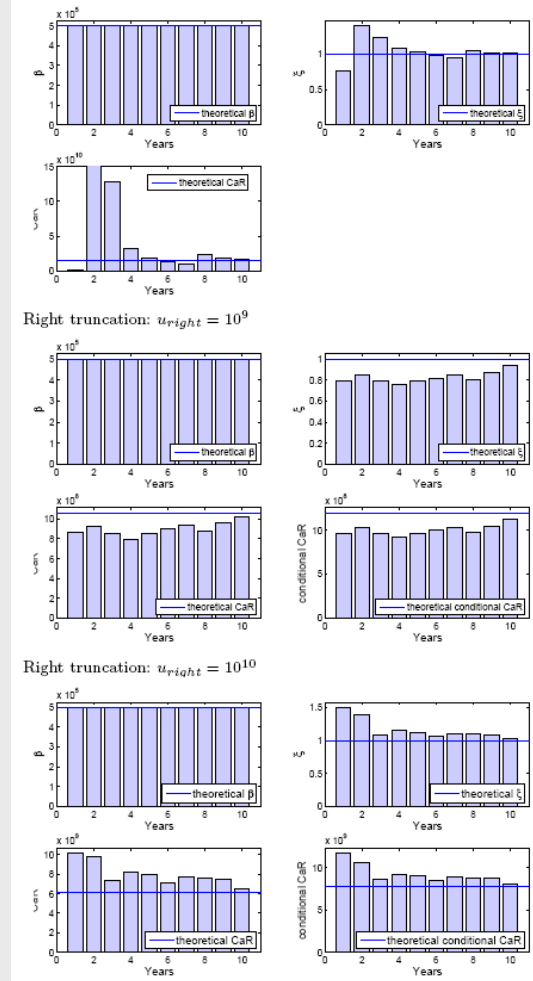
Robustness vs. sensitivity

Risk measures need to be

- **Sensitive**, so that it captures changes in the risk profile of the institution.

- **Robust**, so it is not affected by spurious fluctuations in the data sample.

These are **conflicting objectives**.

Need to strike a **balance** between robustness and sensitivity.

Simulations: robustness vs. sensitivity:

Original sample: N = 1,000 events (5 years of data

- **Case 0: original sample.**
- **Case 1: bootstrap sample** (resampling with replacement).
- **Case 2: eliminate maximum loss from** the original sample.
- **Case 3: double maximum loss** in the original sample.
- **Case 4: repeat maximum loss** in the original sample.

Fit model:

- **Frequency:** Poisson (λ = N/5 = 200)
- **Severity:** Fit to data assuming form of true model known.

Report statistics (median, interquartile range) for M=100 simulations.

Robustness vs. sensitivity: Lognormal data

Sample distribution: Lognormal

Case	μ		σ		CaR $\times 10^{-9}$		cCaR $\times 10^{-9}$	
theoretical	10.00		2.50		1.48		2.87	
0	9.99	[9.94,10.04]	2.50	[2.47,2.53]	1.47	[1.26,1.72]	2.76	[2.36,3.28]
1	9.98	[9.93,10.06]	2.50	[2.44,2.56]	1.44	[1.13,1.85]	2.72	[2.09,3.60]
2	9.98	[9.94,10.03]	2.48	[2.46,2.52]	1.37	[1.20,1.60]	2.57	[2.22,3.06]
3	9.99	[9.95,10.04]	2.50	[2.47,2.54]	1.48	[1.27,1.74]	2.79	[2.39,3.32]
4	10.00	[9.95,10.05]	2.51	[2.48,2.55]	1.55	[1.35,1.83]	2.92	[2.54,3.51]

Right truncation: $u_{right} = 10^9$

Case	μ		σ		CaR $\times 10^{-9}$		cCaR $\times 10^{-9}$	
theoretical	10.00		2.50		0.88		0.99	
0	10.00	[9.96,10.06]	2.50	[2.46,2.53]	0.88	[0.82,0.93]	0.99	[0.95,1.03]
1	10.01	[9.94,10.09]	2.50	[2.43,2.54]	0.88	[0.79,0.96]	0.99	[0.92,1.06]
2	9.99	[9.95,10.05]	2.48	[2.45,2.52]	0.87	[0.80,0.91]	0.98	[0.93,1.01]
3	10.00	[9.96,10.06]	2.50	[2.46,2.54]	0.89	[0.83,0.93]	1.00	[0.95,1.03]
4	10.01	[9.96,10.07]	2.51	[2.48,2.54]	0.90	[0.84,0.95]	1.01	[0.96,1.05]

Right truncation: $u_{right} = 10^{10}$

Case	μ		σ		CaR $\times 10^{-9}$		cCaR $\times 10^{-9}$	
theoretical	10.00		2.50		1.47		2.56	
0	10.01	[9.95,10.07]	2.50	[2.45,2.54]	1.46	[1.22,1.75]	2.54	[2.14,3.01]
1	10.01	[9.93,10.08]	2.49	[2.44,2.55]	1.42	[1.11,1.84]	2.48	[1.96,3.13]
2	10.00	[9.94,10.06]	2.49	[2.44,2.53]	1.37	[1.15,1.66]	2.40	[2.03,2.87]
3	10.01	[9.95,10.06]	2.50	[2.45,2.54]	1.47	[1.23,1.77]	2.57	[2.16,3.04]
4	10.01	[9.96,10.07]	2.51	[2.46,2.55]	1.54	[1.31,1.87]	2.67	[2.30,3.19]

Robustness vs. sensitivity: g-and-h tail

Sample distribution: Lognormal body & g-and-h tail

Case	g		h		CaR $\times 10^{-9}$		cCaR $\times 10^{-9}$	
theoretical	2.10		0.25		3.49		9.75	
0	2.23	[1.86,2.50]	0.18	[0.00,0.46]	2.57	[1.51,7.88]	6.17	[2.86,31.29]
1	2.32	[1.89,2.54]	0.00	[0.00,0.33]	2.53	[1.47,7.20]	5.41	[2.78,25.12]
2	2.37	[2.24,2.48]	0.00	[0.00,0.13]	1.41	[0.94,2.25]	2.82	[1.69,4.76]
3	2.08	[1.68,2.42]	0.30	[0.00,0.56]	4.19	[1.70,12.36]	12.60	[3.56,60.41]
4	1.98	[1.42,2.43]	0.40	[0.01,0.72]	8.39	[2.17,24.83]	36.70	[4.28,220.60]

Right truncation: $u_{right} = 10^9$

Case	g		h		CaR $\times 10^{-9}$		cCaR $\times 10^{-9}$	
theoretical	2.10		0.25		1.00		1.10	
0	2.13	[1.37,2.40]	0.22	[0.00,0.69]	0.99	[0.87,1.13]	1.09	[0.98,1.30]
1	2.19	[1.36,2.46]	0.05	[0.00,0.56]	0.98	[0.76,1.12]	1.07	[0.90,1.29]
2	2.26	[1.87,2.44]	0.00	[0.00,0.35]	0.88	[0.68,1.01]	0.99	[0.83,1.10]
3	2.02	[1.24,2.31]	0.29	[0.03,0.72]	1.03	[0.89,1.16]	1.13	[1.00,1.34]
4	1.83	[0.79,2.27]	0.47	[0.08,1.01]	1.09	[0.95,1.31]	1.24	[1.05,1.51]

Right truncation: $u_{right} = 10^{10}$

Case	g		h		CaR $\times 10^{-9}$		cCaR $\times 10^{-9}$	
theoretical	2.10		0.25		3.09		5.00	
0	2.25	[1.74,2.47]	0.15	[0.00,0.43]	2.47	[1.48,4.52]	4.12	[2.63,6.50]
1	2.20	[1.27,2.48]	0.03	[0.00,0.71]	2.11	[1.06,6.83]	3.57	[1.90,8.29]
2	2.36	[2.19,2.49]	0.00	[0.00,0.05]	1.34	[0.86,1.97]	2.34	[1.50,3.37]
3	2.11	[1.56,2.43]	0.25	[0.00,0.54]	3.37	[1.73,5.54]	5.20	[3.03,7.36]
4	2.01	[1.30,2.45]	0.33	[0.00,0.75]	4.29	[2.12,7.34]	6.28	[3.57,8.64]

Robustness vs. sensitivity: GP tail

Sample distribution: Lognormal body & GP tail

Case	ξ		CaR $\times 10^{-9}$		cCaR $\times 10^{-9}$	
theoretical	1.00		15.14		∞	
0	0.99	[0.93,1.09]	14.25	[8.18,34.75]	162.80	[62.20,∞]
1	0.97	[0.82,1.08]	12.00	[3.05,31.42]	130.50	[12.88, ∞]
2	0.93	[0.84,1.02]	7.60	[3.50,17.50]	54.54	[16.19,∞]
3	1.01	[0.95,1.10]	16.24	[9.41,39.16]	∞	[81.74, ∞]
4	1.06	[0.99,1.16]	26.27	[14.13,67.63]	∞	[160.70,∞]

Right truncation: $u_{right} = 10^9$

Case	ξ		CaR $\times 10^{-9}$		cCaR $\times 10^{-9}$	
theoretical	1.00		1.06		1.20	
0	1.00	[0.91,1.09]	1.06	[1.00,1.16]	1.19	[1.09,1.34]
1	0.98	[0.84,1.11]	1.05	[0.93,1.19]	1.17	[1.02,1.38]
2	0.92	[0.82,1.00]	1.00	[0.90,1.06]	1.09	[1.00,1.20]
3	1.00	[0.91,1.08]	1.07	[1.00,1.15]	1.20	[1.09,1.33]
4	1.07	[0.98,1.17]	1.13	[1.05,1.28]	1.31	[1.17,1.48]

Right truncation: $u_{right} = 10^{10}$

Case	ξ		CaR $\times 10^{-9}$		cCaR $\times 10^{-9}$	
theoretical	1.00		6.15		7.83	
0	0.98	[0.90,1.06]	5.69	[4.04,7.19]	7.48	[6.11,8.54]
1	0.95	[0.85,1.08]	5.16	[3.01,7.39]	7.08	[5.05,8.68]
2	0.91	[0.83,0.98]	4.19	[2.68,5.81]	6.25	[4.67,7.58]
3	0.99	[0.91,1.07]	5.95	[4.36,7.29]	7.68	[6.40,8.61]
4	1.05	[0.98,1.13]	6.97	[5.69,8.09]	8.39	[7.48,9.14]

Lognormal vs. g-and-h tail vs. GP tail

$$CaR_{woMax} < CaR_0 \approx CaR_{bootstrap} < CaR_{doubleMax} < CaR_{repeatMax}$$

Loss cap reduces uncertainty in model choice, parameter estimates and therefore in **risk measures**

CaR $\times10^{-9}$		cCaR $\times10^{-9}$	
1.48		2.87	
1.47	[1.26,1.72]	2.76	[2.36,3.28]
1.44	[1.13,1.85]	2.72	[2.09,3.60]
1.37	[1.20,1.60]	2.57	[2.22,3.06]
1.48	[1.27,1.74]	2.79	[2.39,3.32]
1.55	[1.35,1.83]	2.92	[2.54,3.51]

CaR $\times10^{-9}$		cCaR $\times10^{-9}$	
3.49		9.75	
2.57	[1.51,7.88]	6.17	[2.86,31.29]
2.53	[1.47,7.20]	5.41	[2.78,25.12]
1.41	[0.94,2.25]	2.82	[1.69,4.76]
4.19	[1.70,12.36]	12.60	[3.56,60.41]
8.39	[2.17,24.83]	36.70	[4.28,220.60]

CaR $\times10^{-9}$		cCaR $\times10^{-9}$	
15.14		∞	
14.25	[8.18,34.75]	162.80	[62.20,∞]
12.00	[3.05,31.42]	130.50	[12.88, ∞]
7.60	[3.50,17.50]	54.54	[16.19,∞]
16.24	[9.41,39.16]	∞	[81.74, ∞]
26.27	[14.13,67.63]	∞	[160.70,∞]

CaR $\times10^{-9}$		cCaR $\times10^{-9}$	
0.88		0.99	
0.88	[0.82,0.93]	0.99	[0.95,1.03]
0.88	[0.79,0.96]	0.99	[0.92,1.06]
0.87	[0.80,0.91]	0.98	[0.93,1.01]
0.89	[0.83,0.93]	1.00	[0.95,1.03]
0.90	[0.84,0.95]	1.01	[0.96,1.05]

CaR $\times10^{-9}$		cCaR $\times10^{-9}$	
1.00		1.10	
0.99	[0.87,1.13]	1.09	[0.98,1.30]
0.98	[0.76,1.12]	1.07	[0.90,1.29]
0.88	[0.68,1.01]	0.99	[0.83,1.10]
1.03	[0.89,1.16]	1.13	[1.00,1.34]
1.09	[0.95,1.31]	1.24	[1.05,1.51]

CaR $\times10^{-9}$		cCaR $\times10^{-9}$	
1.06		1.20	
1.06	[1.00,1.16]	1.19	[1.09,1.34]
1.05	[0.93,1.19]	1.17	[1.02,1.38]
1.00	[0.90,1.06]	1.09	[1.00,1.20]
1.07	[1.00,1.15]	1.20	[1.09,1.33]
1.13	[1.05,1.28]	1.31	[1.17,1.48]

CaR $\times10^{-9}$		cCaR $\times10^{-9}$	
1.47		2.56	
1.46	[1.22,1.75]	2.54	[2.14,3.01]
1.42	[1.11,1.84]	2.48	[1.96,3.13]
1.37	[1.15,1.66]	2.40	[2.03,2.87]
1.47	[1.23,1.77]	2.57	[2.16,3.04]
1.54	[1.31,1.87]	2.67	[2.30,3.19]

CaR $\times10^{-9}$		cCaR $\times10^{-9}$	
3.09		5.00	
2.47	[1.48,4.52]	4.12	[2.63,6.50]
2.11	[1.06,6.83]	3.57	[1.90,8.29]
1.34	[0.86,1.97]	2.34	[1.50,3.37]
3.37	[1.73,5.54]	5.20	[3.03,7.36]
4.29	[2.12,7.34]	6.28	[3.57,8.64]

CaR $\times10^{-9}$		cCaR $\times10^{-9}$	
6.15		7.83	
5.69	[4.04,7.19]	7.48	[6.11,8.54]
5.16	[3.01,7.39]	7.08	[5.05,8.68]
4.19	[2.68,5.81]	6.25	[4.67,7.58]
5.95	[4.36,7.29]	7.68	[6.40,8.61]
6.97	[5.69,8.09]	8.39	[7.48,9.14]

Interpretation of risk analysis

Risk measures are just **numbers**, they need to be interpreted

- **Data sources**
  - Reliability: Correctness / completeness
  - Relevance

- **Limitations** of the analysis.

  - **Model uncertainty**

  - **Uncertainty in** the estimates of the **model parameters**
    - Fits using different criteria (likelihood, probability weighted moments, robust fitting techniques, etc.)
    - Multiple local optima.

- **Robustness** and **stability** of the results

Economic sense in economic capital

Desirable properties of risk measures

- **Sensitive** to changes in the risk profile.
- **Robust** to spurious fluctuations in data used to fit models.
- Reasonably **stable with time**.

Alternatives

- Use a **lower percentile** (e.g. operational VaR at 99%)
- Assume a **loss cap**: Sharp / exponential
  - Set the loss cap on the basis of economic analysis
  - Use the loss cap as a control / sensitivity parameter
- **Generative models** for operational risk events (???)

Single loss approximation

Single loss approximation [Böcker + Klüppelberg (2005)]

$$VaR_\alpha = F^\leftarrow\left(1 - \frac{1-\alpha}{E[N]}\right)$$

Single loss approximation + mean correction

[Böcker + Sprittulla (2005)]

$$VaR_\alpha = F^\leftarrow\left(1 - \frac{1-\alpha}{E[N]}\right) + \left(E[N]-1\right)\mu$$

Second order asymptotic approximation

[Omey & Willekens (1986-7)], [Sahay, Wan & Keller (2007)]

$$VaR_\alpha = F^\leftarrow \left(1 - \frac{1-\alpha}{E[N]} + \left(\frac{E[N^2]}{E[N]} - 1 \right) \mu\, f\left(VaR_\alpha\right) \right)$$

Iterative algorithm $\qquad VaR_\alpha^{[0]} = F^\leftarrow \left(1 - \frac{1-\alpha}{E[N]} \right)$

$$VaR_\alpha^{[k+1]} = F^\leftarrow \left(1 - \frac{1-\alpha}{E[N]} + \left(\frac{E[N^2]}{E[N]} - 1 \right) \mu\, f\left(VaR_\alpha^{[k]}\right) \right); \quad k = 0,1,2,...$$

Second order estimate for Pareto severity

Poisson $\quad E[N] = \lambda; \quad E[N^2] = \lambda(\lambda+1); \qquad \dfrac{E[N^2]}{E[N]} - 1 = \lambda$

Pareto $\quad f(x) = \dfrac{1}{\xi}\dfrac{u^{1/\xi}}{x^{1+1/\xi}}; \quad F(x) = 1 - \dfrac{u^{1/\xi}}{x^{1/\xi}}; \quad F^{\leftarrow}(p) = (1-p)^{-\xi}u$

$$VaR_\alpha^{[0]} = F^{\leftarrow}\left(1 - \dfrac{1-\alpha}{\lambda}\right) = \left(\dfrac{1-\alpha}{\lambda}\right)^{-\xi} u$$

$$VaR_\alpha^{[1]} = F^{\leftarrow}\left(1 - \dfrac{1-\alpha}{\lambda} + \lambda\mu\, f\left(VaR_\alpha^{[0]}\right)\right) = VaR_\alpha^{[0]}\left(1 - \dfrac{1}{\xi}\dfrac{\lambda\mu}{VaR_\alpha^{[0]}}\right)^{-\xi}$$

Mean-corrected single-loss approximation

$$VaR_{\alpha}^{[0]} = \left(\frac{1-\alpha}{\lambda} \right)^{-\xi} u$$

$$VaR_{\alpha}^{[1]} = VaR_{\alpha}^{[0]} \left(1 - \frac{1}{\xi} \frac{\lambda\mu}{VaR_{\alpha}^{[0]}} \right)^{-\xi} =$$

$$= VaR_{\alpha}^{[0]} \left(1 + \frac{\lambda\mu}{VaR_{\alpha}^{[0]}} + O\left(\frac{\lambda\mu}{VaR_{\alpha}^{[0]}} \right)^{2} \right)$$

$$VaR_{\alpha}^{[1]} \approx VaR_{\alpha}^{[0]} + \lambda\mu \quad [\text{Böcker} + \text{Sprittulla}, 2006]$$

Second order correction [Degen, 2010]

$$f(x) = \frac{1}{\xi} \frac{u^{1/\xi}}{x^{1+1/\xi}} \qquad \mu = \int_u^\infty dx \, x \, f(x) = \frac{1}{1-\xi} u; \quad VaR_\alpha^{[0]} = \left(\frac{1-\alpha}{\lambda}\right)^{-\xi} u$$

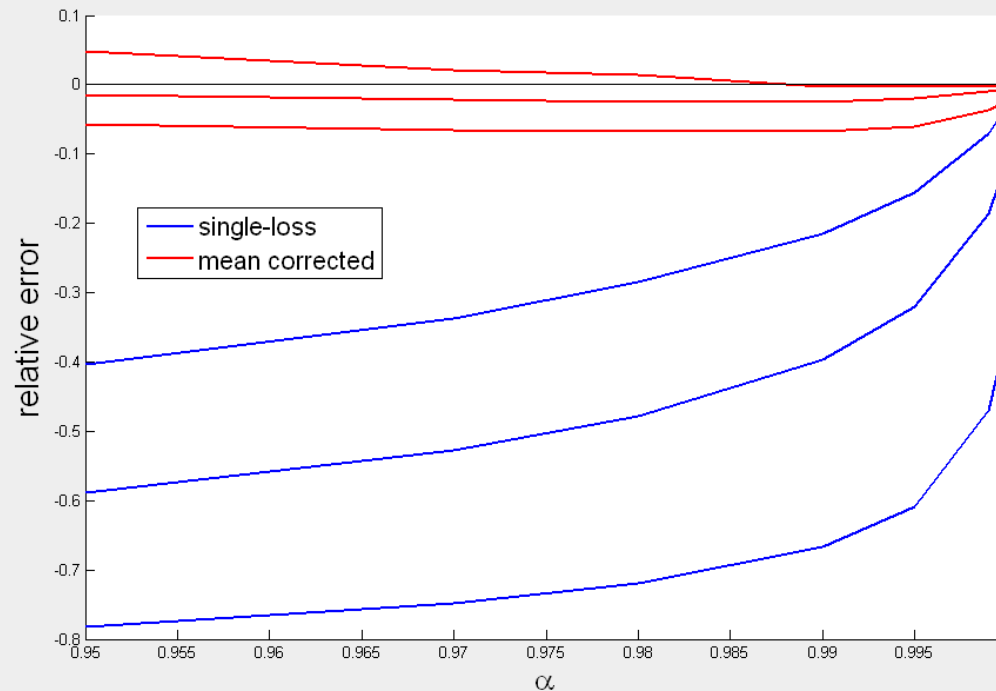$$\frac{VaR_\alpha^{[0]}}{VaR_\alpha^{[1]}} - 1 \approx K \, A(\alpha) \Rightarrow$$

$$VaR_\alpha^{[1]} \approx VaR_\alpha^{[0]} \left(1 + K \, A(\alpha)\right)^{-1} \approx VaR_\alpha^{[0]} \left(1 - K \, A(\alpha)\right)$$

$$- K \, A(\alpha) = \lambda^{1-\xi} \frac{(1-\alpha)^\xi}{1-\xi} u = \lambda \left(\frac{1-\alpha}{\lambda}\right)^\xi \frac{1}{1-\xi} u = \frac{\lambda \mu}{VaR_\alpha^{[0]}}$$
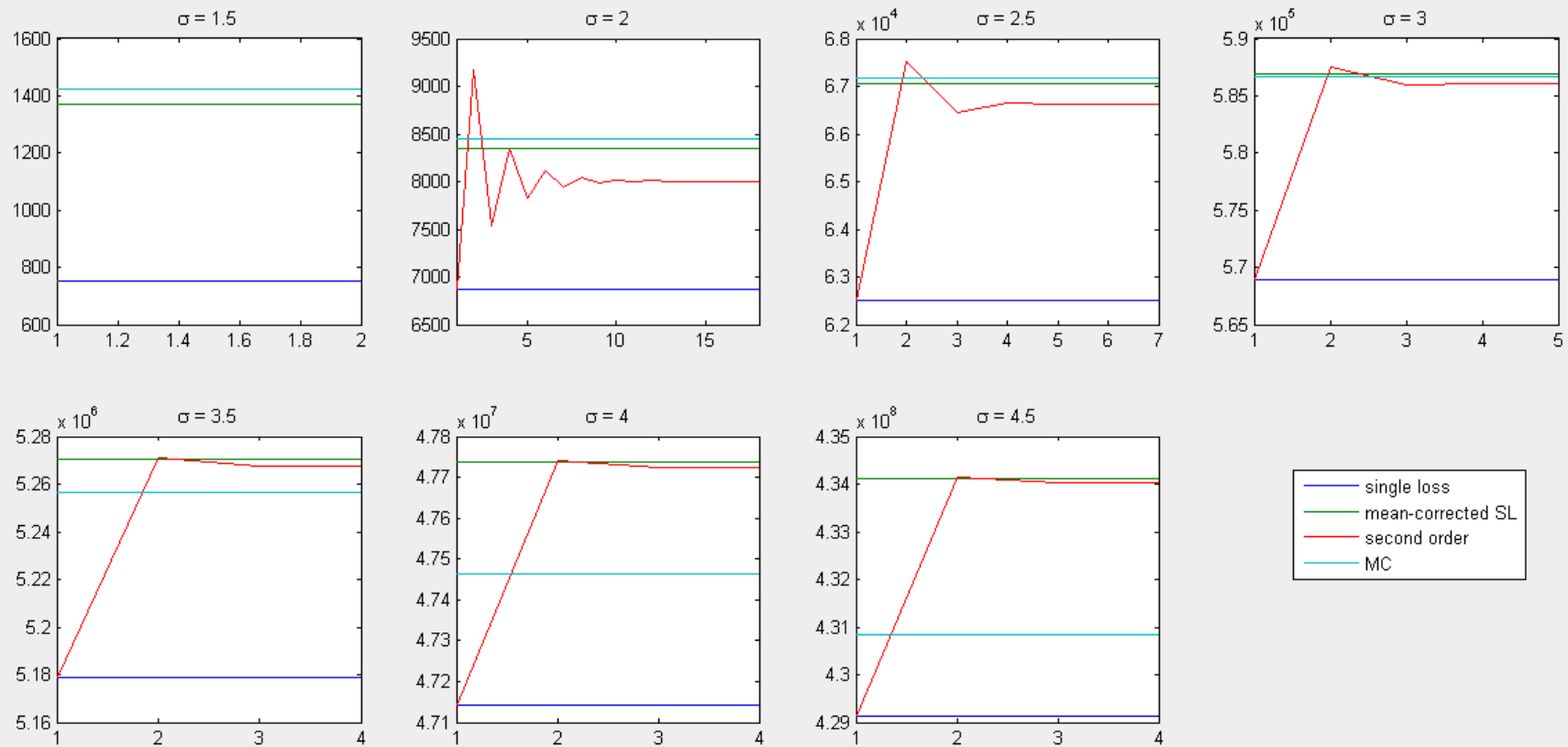
$$\boxed{VaR_\alpha^{[1]} \approx VaR_\alpha^{[0]} + \lambda \mu}$$

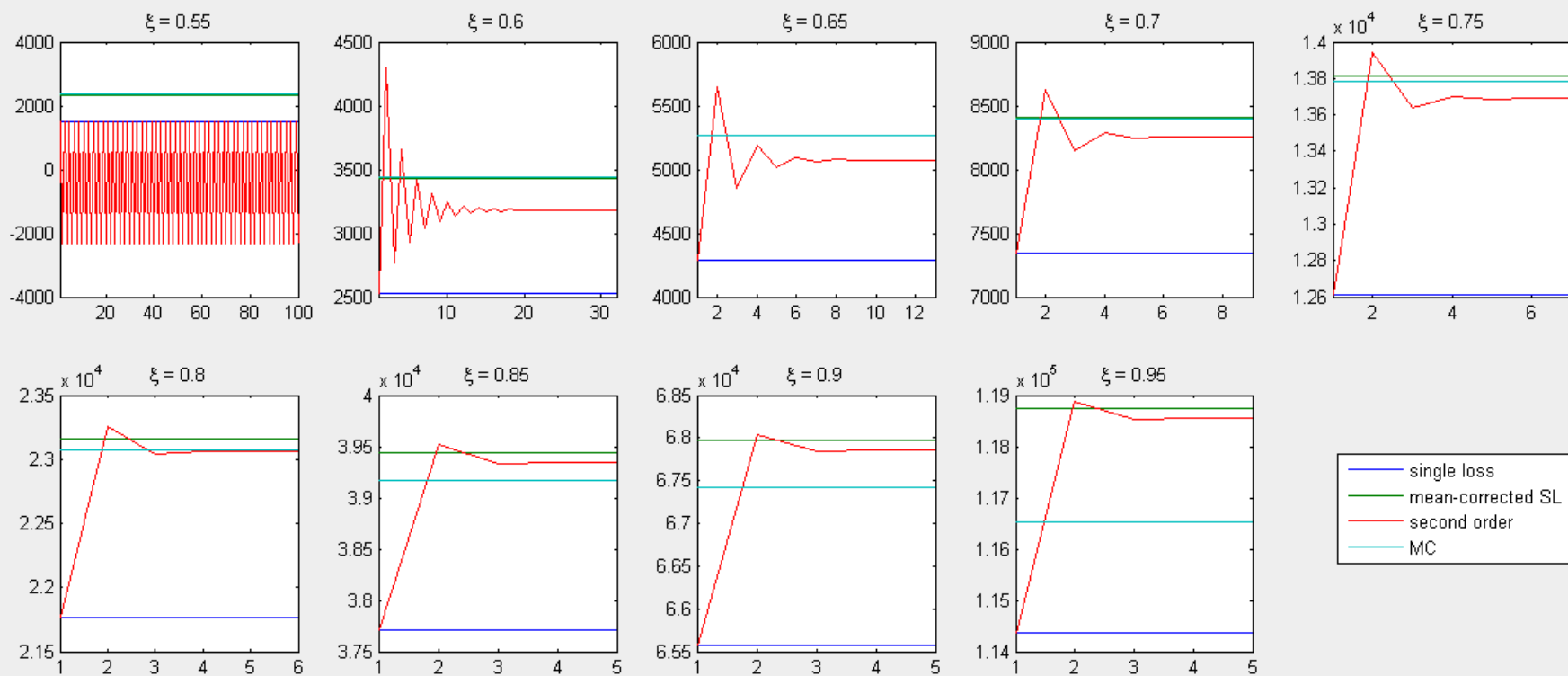# Performance of the correction by the mean

- **Poisson:** $\lambda = 200$
- **Longnormal:** $\sigma = 1.5, 2, 2,5$

Poisson (λ = 200) + LN (μ = 200)

Poisson (λ = 200) + GP (u = 2, θ = 1)

Asymptotic formulas to operational VaR

- The **single-loss** approximation is insufficient, specially with
  - Lower percentiles.
  - Less heavy tails.
  - Higher frequencies.
- The **second order asymptotic approximation**
  - Improves estimate and is easy to compute.
  - Can diverge.
- The **single loss formula corrected by the mean**
  - Can be derived from the second order asymptotic.
  - Accurate in a wide range of cases

Estimation of the mean can be difficult
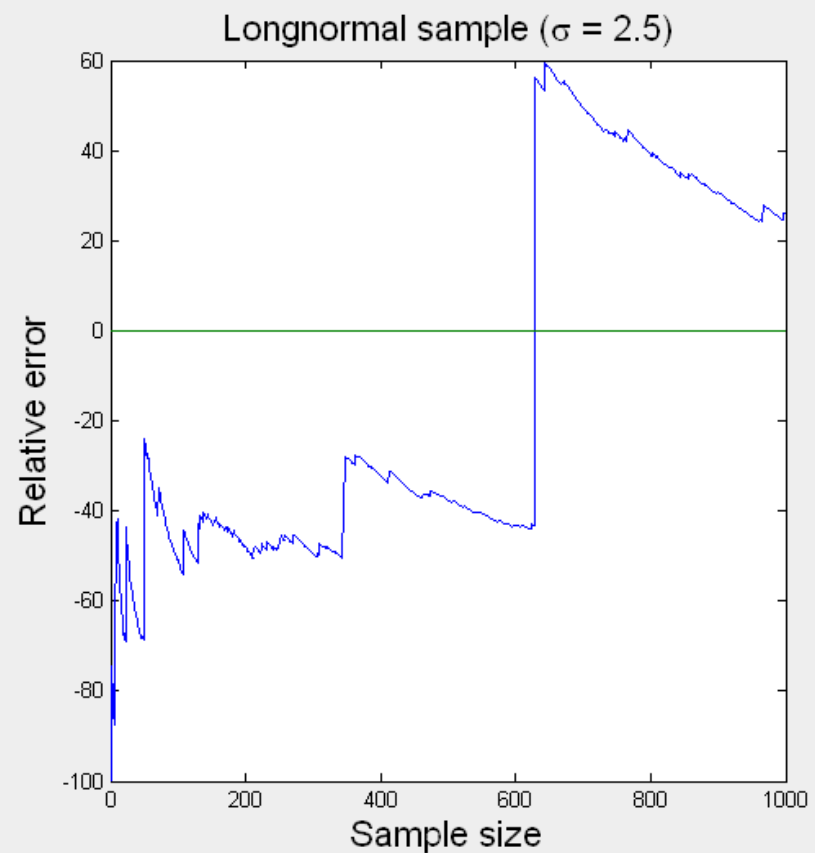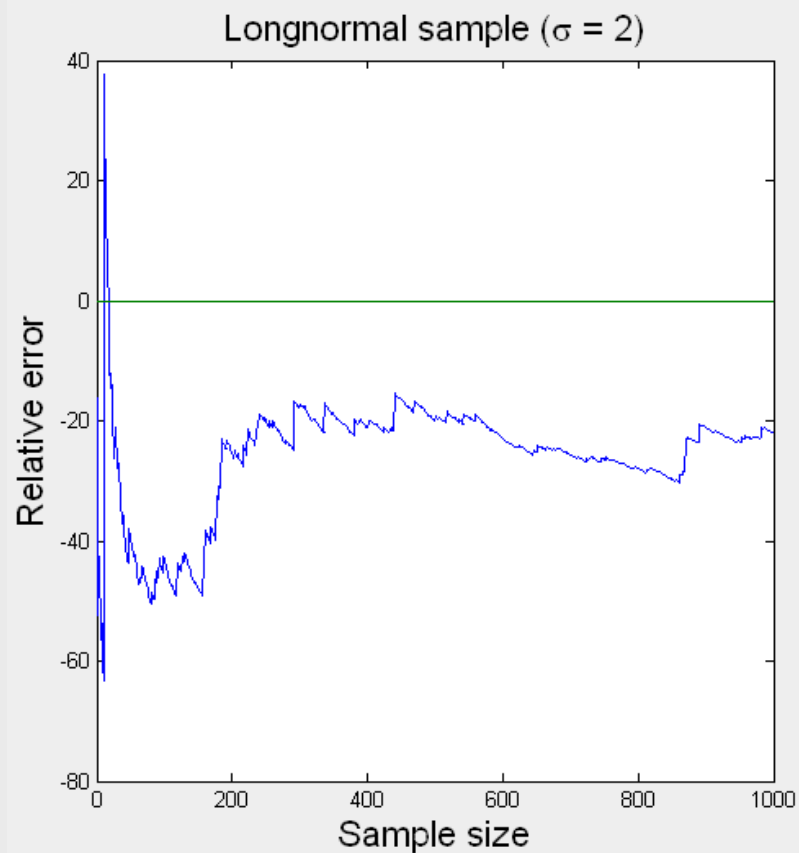
State of things

- **Economic sense** is needed in OR measurements
  - Imposing a cap (soft / hard) on OR losses introduces a scale in the data.
  - Generative models.

- Challenges.
  - Back-testing
  - Benchmarking