Analytical approximations for Operational Risk Capital

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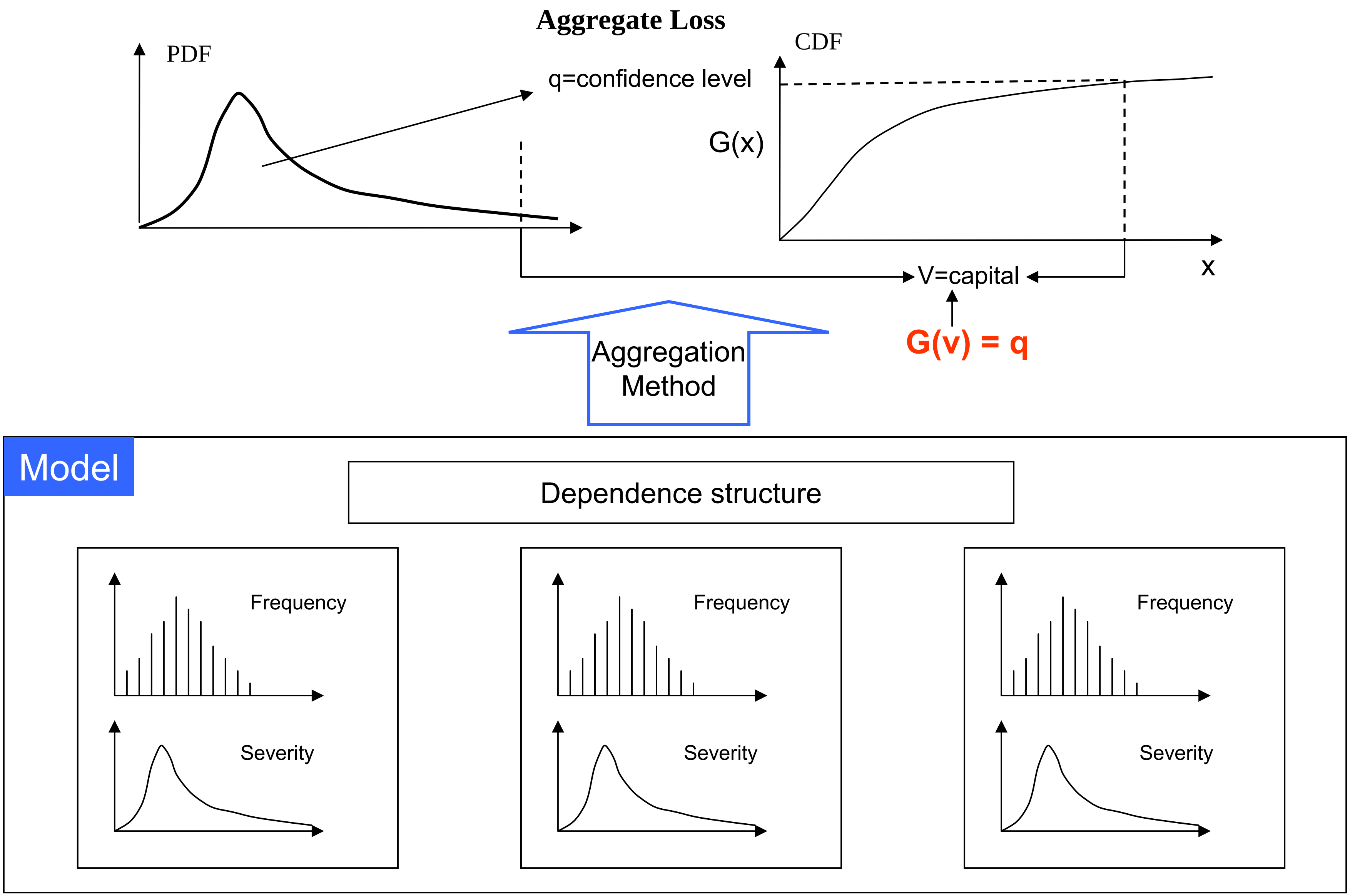
Standard LDA



Aggregation Methods

- Closed-form of G(x) not available

- Numerical approximations (e.g. Monte Carlo simulation, Fast Fourier, Panjer recursion)

  Advantage
  Easy to implement
  Robust

  Disadvantage
  Time consuming
  Black box

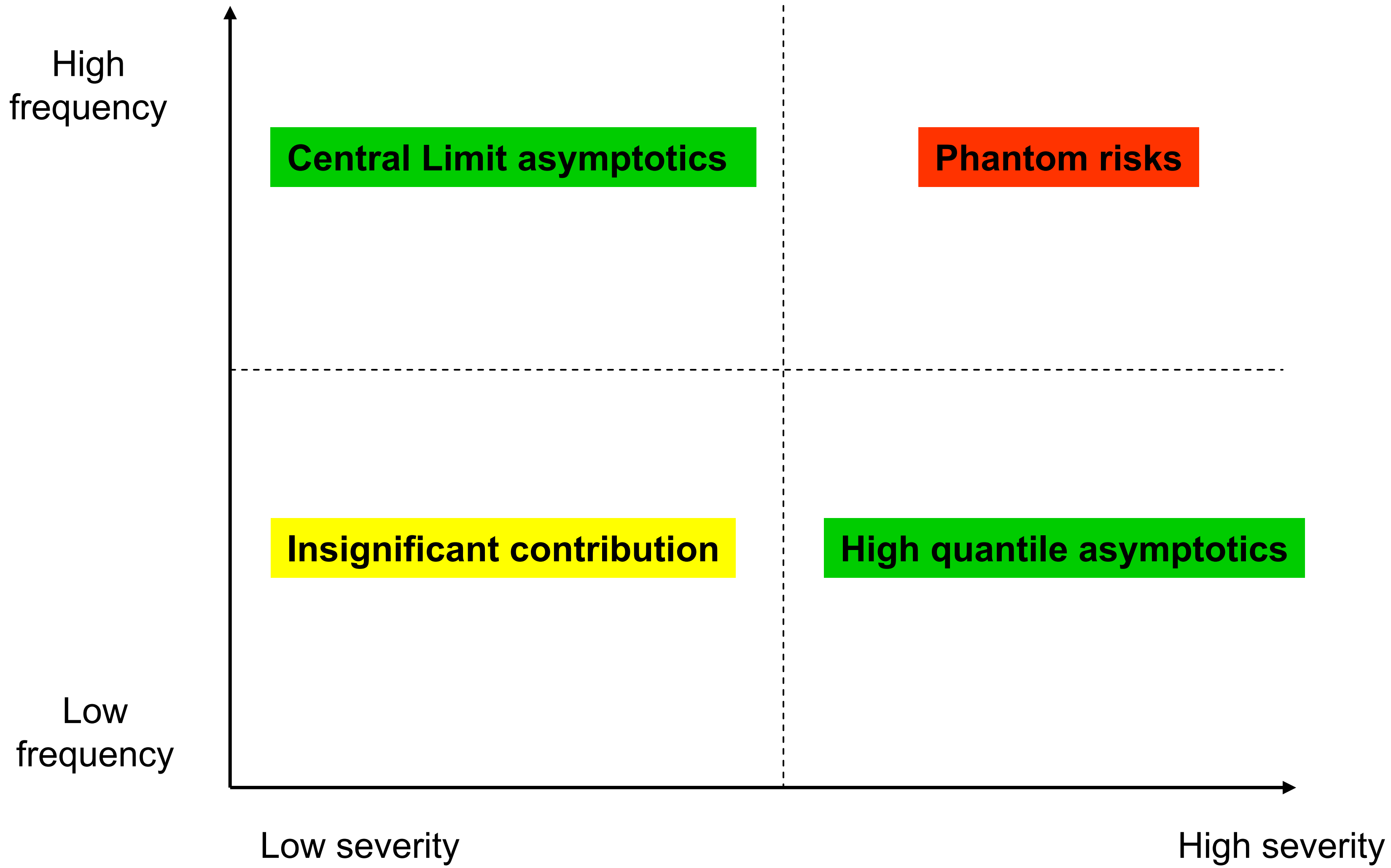
V = f (frequency, severity, dependence)

- Analytical approximations

  Advantage
  Not time consuming
  Makes relation with underlying characteristics of model explicit

  Disadvantage
  Valid for limited set of underlying parameters

Approximation regimes



Single Cell High Quantile Approximation

Reference: Anupam sahay, Zailong Wan and Brian Keller. Operational risk capital: asymptotics in the case of heavy-tailed severity. J Operational Risk, Vol 2 No 2, 2007.

Background

$$G(x) = P(L \le x) = \sum_{n=0}^{\infty} p(n) F^{*n}(x),$$

L aggregate loss over certain time horizon (random variable)
p(n) frequency mass function
F(x) severity distribution function; $F^{*n}(x)$ n-fold convolution
G(x) aggregate loss distribution function

For severity distributions with power-law tail (more generally heavy-tailed distributions)

$$\overline{F}(x) = 1 - F(x) \sim a x^{-1/\xi} \quad a, \xi > 0,\ x \to \infty$$

ξ tail index

Well-known first order approximation (a.k.a single-loss approximation)

$$\overline{G}(x) = 1 - G(x) \sim E[N]\overline{F}(x), \quad x \to \infty,$$

E[N] frequency mean

Can this approximation be improved by developing higher-order asymptotic terms?
YES.

Second-order approximation

Seek asymptotic approximation of remainder *R(x)*

$$\bar{G}(x) = E[N]\bar{F}(x) + R(x)$$

Key underlying mathematical result (Omey86, 87),

$$\lim_{x\to\infty}\left(\overline{F^{*(n)}}(x) - n\bar{F}(x)\right) = \begin{cases} \frac{n(n-1)}{2}F_I(\infty)f(x) & \xi < 1 \ (\textit{finite-mean}) \\ \frac{n(n-1)}{2}\frac{c(\xi)}{2}F_I(x)f(x) & \xi \geq 1 \ (\textit{infinite-mean}) \end{cases}$$

f(x) severity density function
$F_I(x)$ integrated tail function

leads to second-order correction

Finite mean

$$R(x) \sim E[N(N-1)]F_I(\infty)f(x)$$

Infinite mean

$$R(x) \sim \begin{cases} E[N(N-1)]\frac{c(\xi)}{2}F_I(x)f(x) & \xi > 1 \\ E[N(N-1)]F_I(x)f(x) & \xi = 1, \end{cases}$$

Model for comparison with simulation

Frequency

Poisson
Negative Binomial

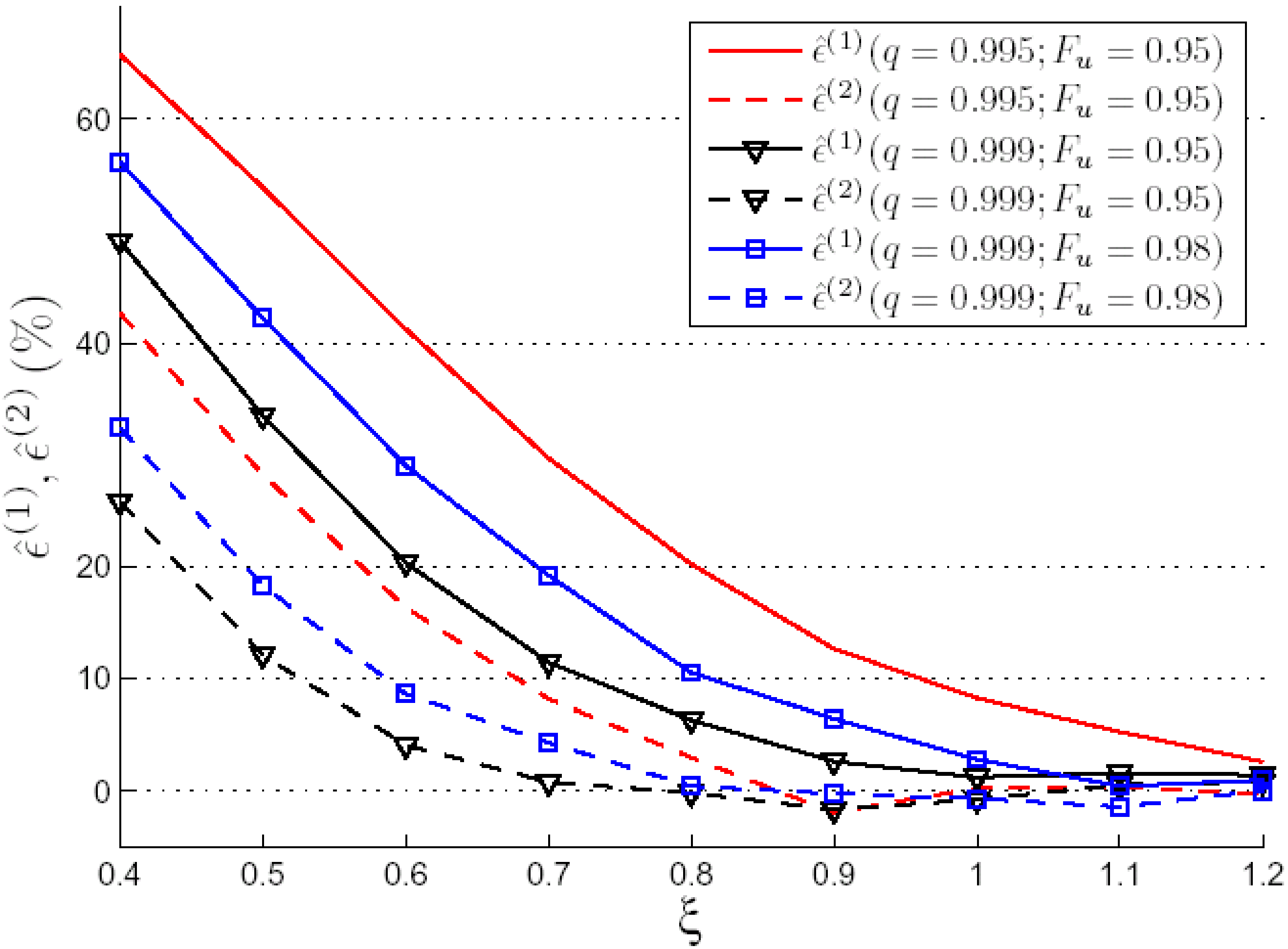
Severity

Lognormal (body) – GPD (tail)

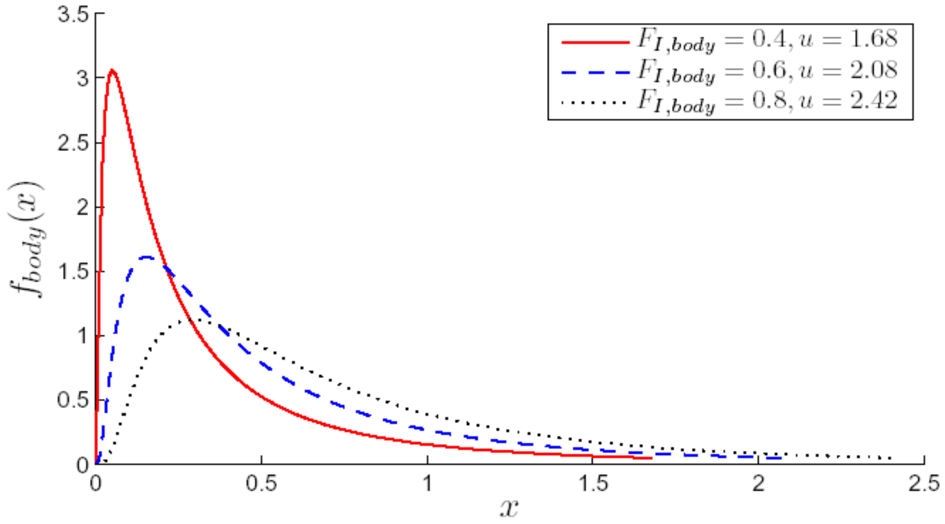
$$F(x) = \begin{cases} F_{\text{body}}(x) & 0 < x \le u \\ 1 - (1 - F_u)\left(1 + \xi \dfrac{x-u}{\theta}\right)^{-1/\xi} & x > u, \end{cases}$$

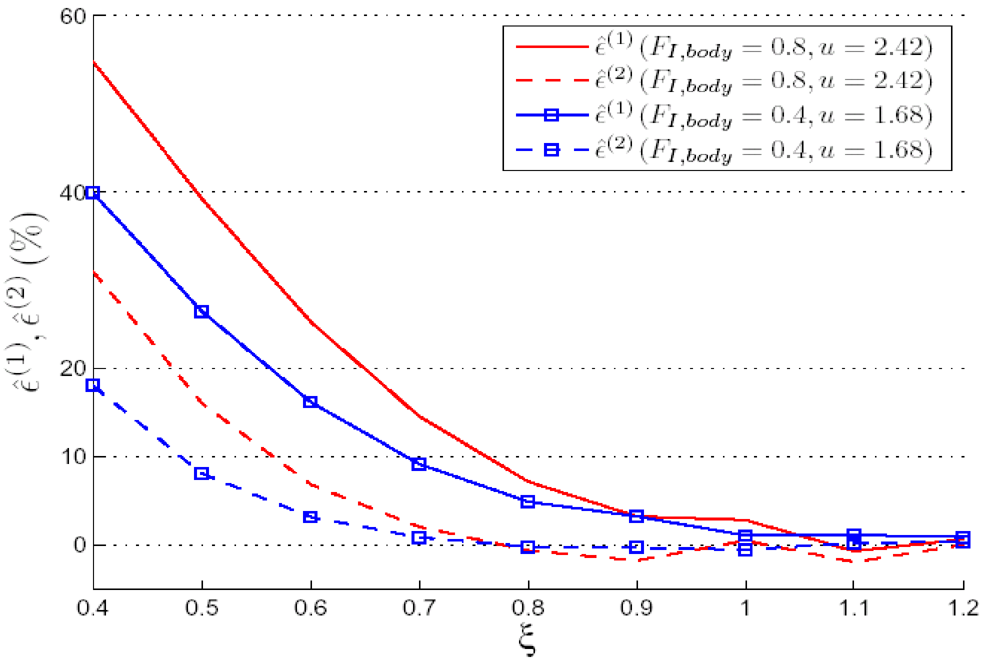
$$f_{\text{body}}(u) = \frac{1 - F_u}{\theta}$$

Quality of approximation

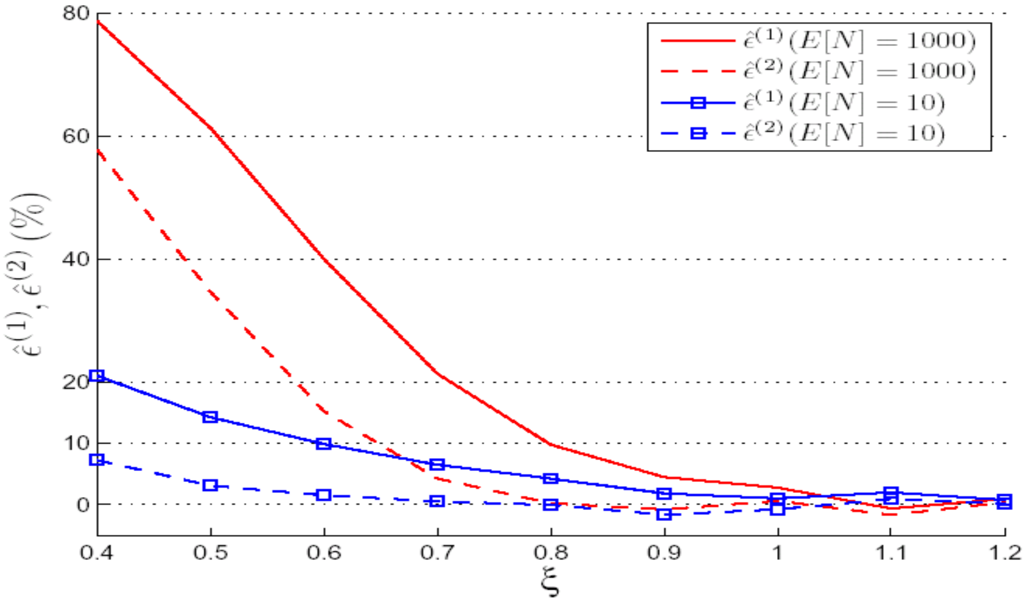


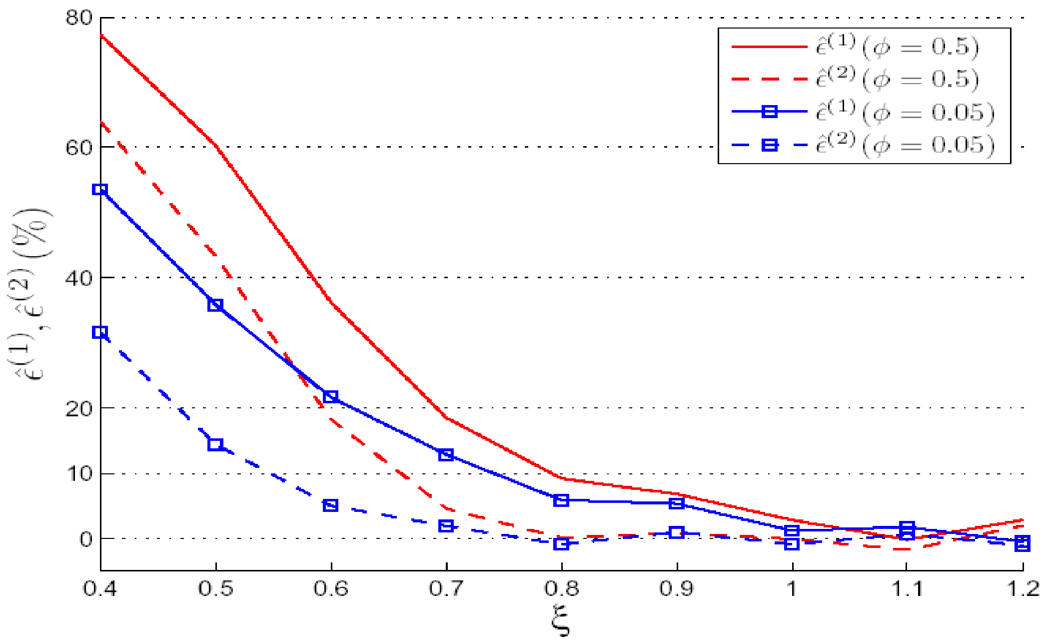
Quality of approximation





Quality of approximation





Single Cell Approximation Recap

- First order approximation is good for very heavy severity (infinite mean)
- Second order approximation significantly improves upon the first order in the range $0.5 < \xi < 1$
- Simulation studies map variation of approximation error with characteristics of underlying frequency and severity distribution, identifying ranges of practical use

Multi Cell High Quantile Approximation

Frequency dependence assumption

Assume dependence only among frequency of cells

Rationale

1. Analytical tractibility
2. Intuition: To a first approximation, underlying drivers affect frequency and not severity; co-movement of drivers creates dependence among cells
3. Indirect empirical evidence: aggregate loss correlation implied from empirical frequency correlation is a significant part of the empirical loss correlation

Two-cell approximation

Assuming only frequency dependence

$$\bar{G}(x) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} P(N_1 = n_1, N_2 = n_2) \, \overline{F_1^{*(n_1)} * F_2^{*(n_2)}}(x).$$

P joint probability mass function of frequencies in cell 1 and cell 2
G(x) aggregate loss distribution function of losses in cell 1 and cell 2

Key underlying mathematical result (Omey86, 87, 94) is the asymptotic expansion of the severity convolutions.

Three cases arise:

(1) Finite mean case ($\xi_1 < 1$, $\xi_2 < 1$)

(2) Infinite mean case ($\xi_1 \geq 1$, $\xi_2 \geq 1$)

(3) Mixed mean case ($\xi_1 < 1$, $\xi_2 \geq 1$)

Finite Mean Case

$$\overline{F_1^{*(n_1)} * F_2^{*(n_2)}}(x) = n_1\overline{F_1}(x) + \binom{n_1}{2} F_{1,I}(\infty) f_1(x) + n_2\overline{F_2}(x) + \binom{n_2}{2} F_{2,I}(\infty) f_2(x) + n_1 F_{1,I}(\infty) n_2 f_2(x) + n_2 F_{2,I}(\infty) n_1 f_1(x).$$

leading to

$$\bar{G}(x) = \underbrace{E[N_1]\bar{F_1}(x)}_{I} + \underbrace{E[N_1(N_1-1)]F_{1,I}(\infty)f_1(x)}_{II} + \underbrace{E[N_2]\bar{F_2}(x)}_{III} + \underbrace{E[N_2(N_2-1)]F_{2,I}(\infty)f_2(x)}_{IV} + \underbrace{E[N_1N_2]F_{1,I}(\infty)f_2(x) + E[N_1N_2]F_{2,I}(\infty)f_1(x)}_{V}$$

I, II: First and Second order approximation for cell 1
III, IV: First and Second order approximation for cell 2
V: Mixing terms

Implied aggregate loss dependence (linear correlation)

$$\rho_L = \left(\sqrt{ \left(1 + \frac{V[X_1]}{E^2[X_1]} \frac{1}{(1+\phi_1 E[N_1])} \right) \left(1 + \frac{V[X_2]}{E^2[X_2]} \frac{1}{(1+\phi_2 E[N_2])} \right) } \right)^{-1} \rho$$

1. $0 \leq abs(\rho_L) \leq abs(\rho) \leq 1$; $sgn(\rho_L) = sgn(\rho)$.
2. $\rho_L = \rho$ if and only if either $V[X_1] = V[X_2] = 0$ or $\phi_1 E[N_1] = \phi_2 E[N_2] = \infty$.
3. $\phi = 0$ corresponds to the Poisson case as reported in Frachot et al. (2004).
4. An increase in $V[X_j] \;/\, E^2[X_j]$, for either j=1 or j=2, implies a decrease in ρ_L / ρ .
5. An increase in $\phi_1 E[N_1]$ or $\phi_2 E[N_2]$ implies an increase in ρ_L / ρ .

Implied aggregate loss dependence (tail dependence)

Analytical approximations

Tail Dependence Function $\lambda(q) = P(Y > F_Y(q) \,|\, X > F_X(q)) = \frac{P(Y > F_Y(q), X > F_X(q))}{P(X > F_X(q))}$

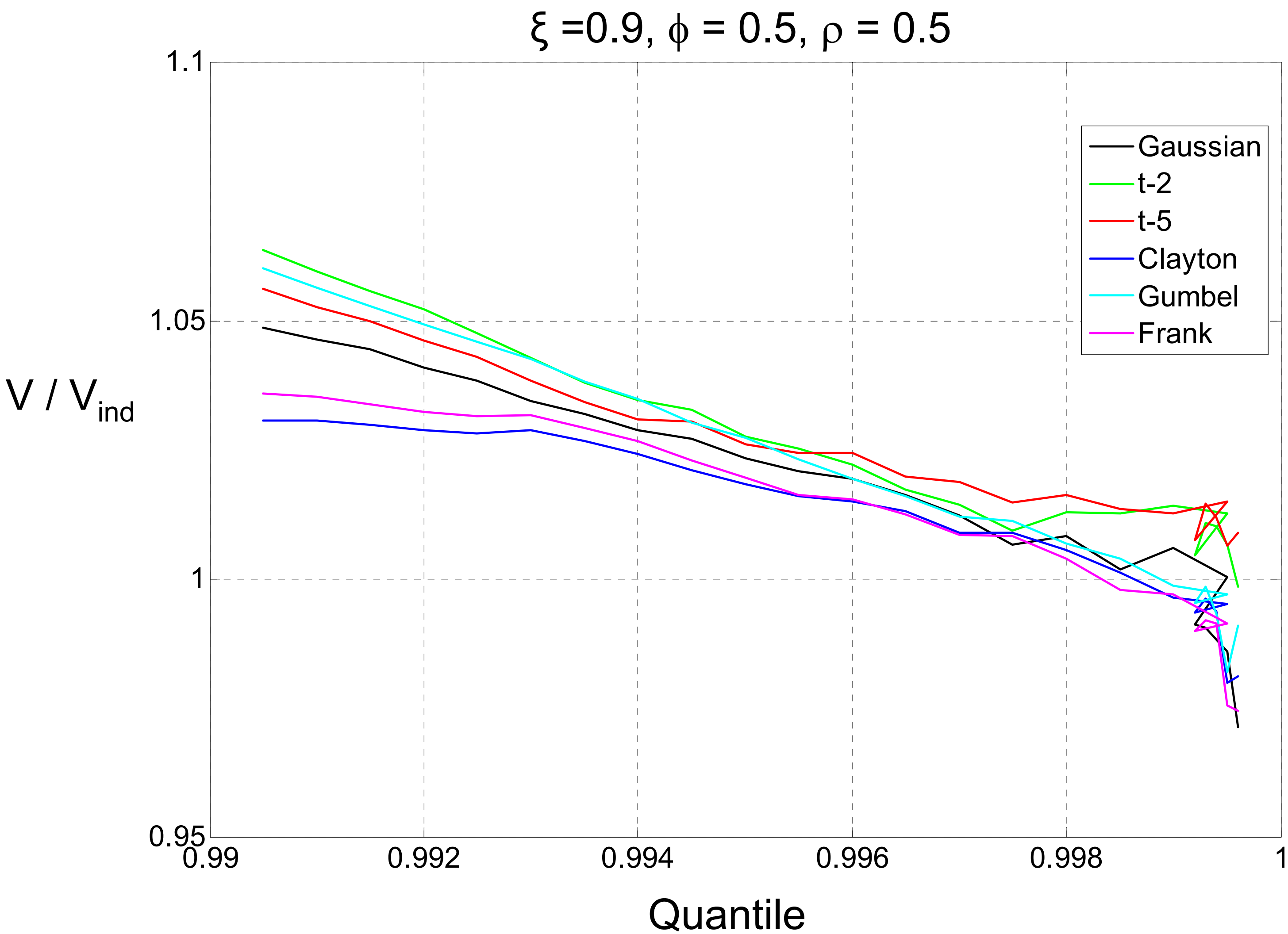
Asymptotic Tail Dependence $\lambda \doteq \lim_{q \to 1} \lambda(q)$

$$\lambda(q) = \left[\sqrt{\left(\frac{1}{E[N_1]} + \phi_1\right)\left(\frac{1}{E[N_2]} + \phi_2\right)\rho + 1}\right](1-q)$$

1. $\lambda = 0$
2. $\lambda(q)$ increases as ϕ_j increases or ρ increases
3. $\lambda(q)$ decreases as $E[N_j]$ increases

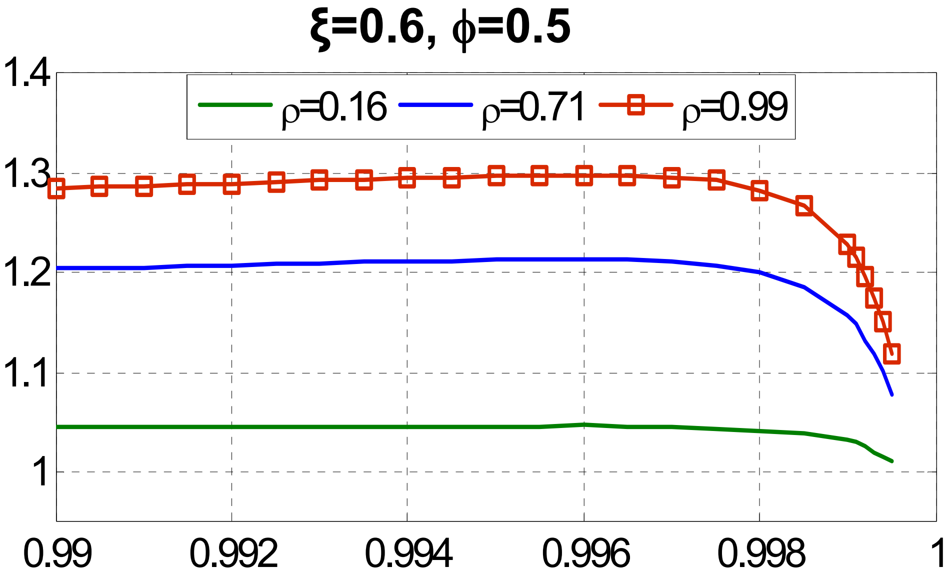
Two cell V / V_{ind} versus quantile (simulation)

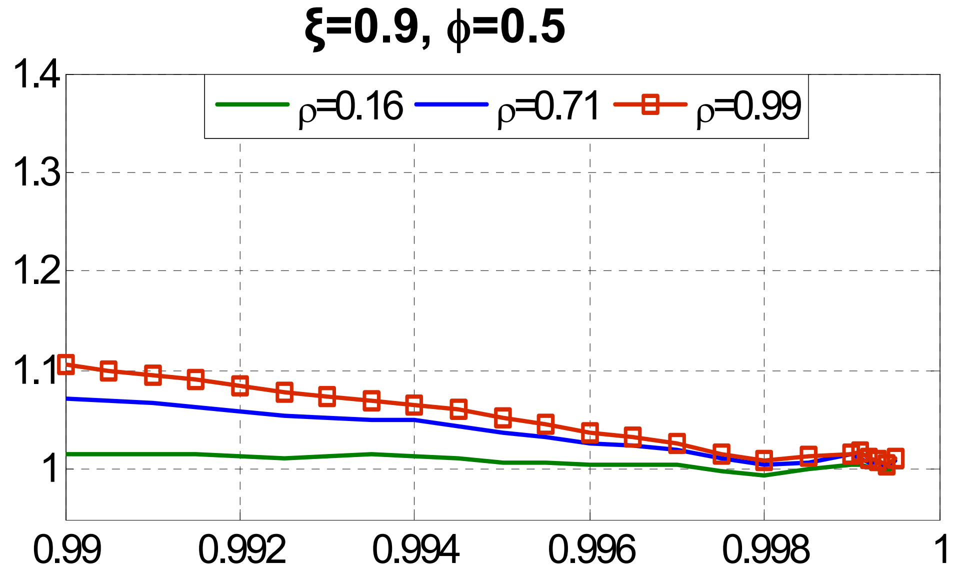
Result depends only on ρ; largely independent of choice of copula

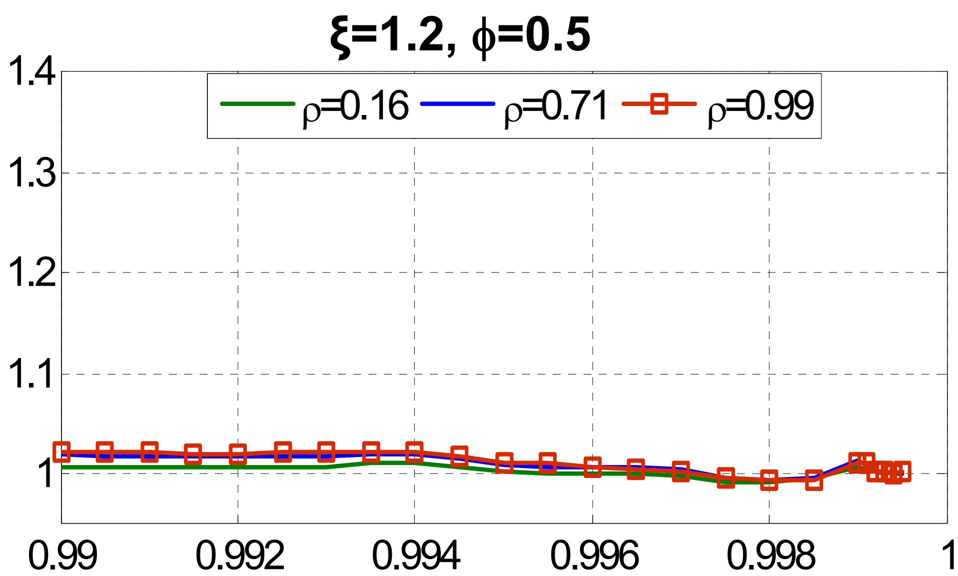


Two cell V / V_{ind} versus quantile (simulation)

Effect of dependence, both absolute value and sensitivity, decreases as tail becomes heavier





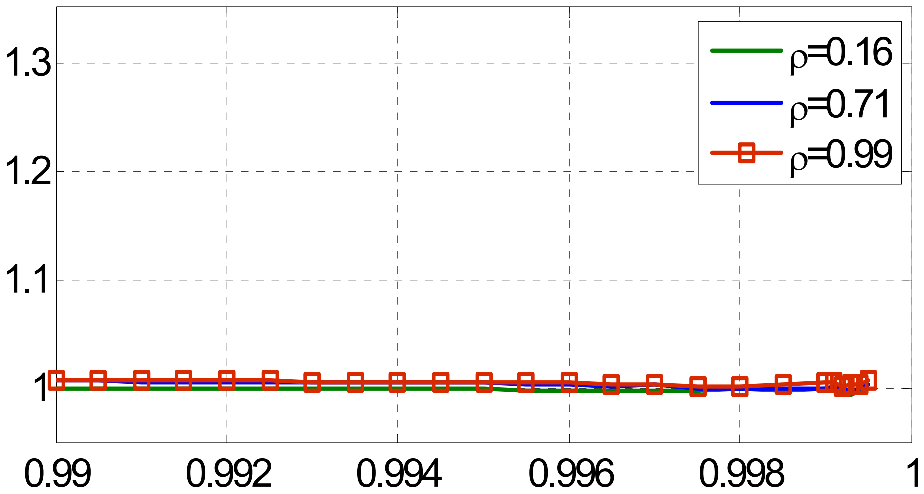


Two cell V / V_{ind} versus quantile (simulation)

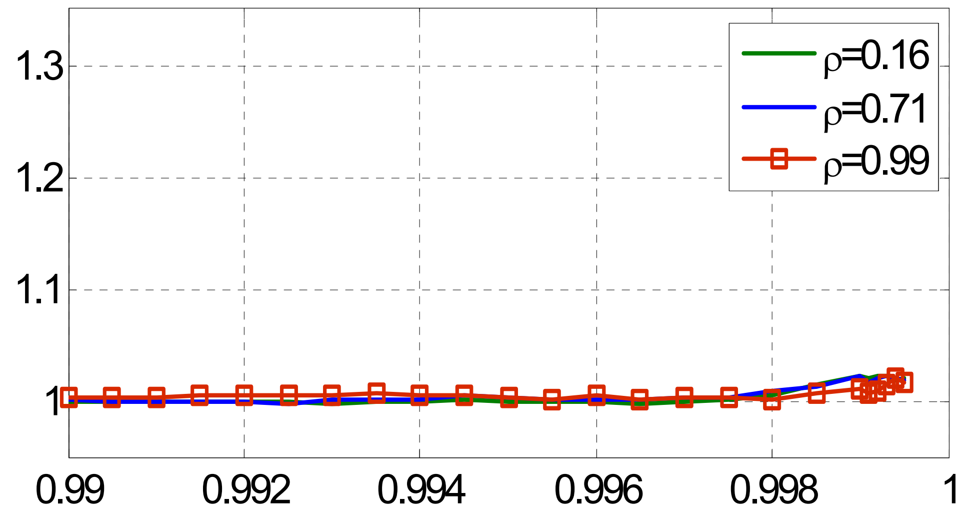
Analytical approximations

Effect of dependence is negligible for Poisson frequency distribution

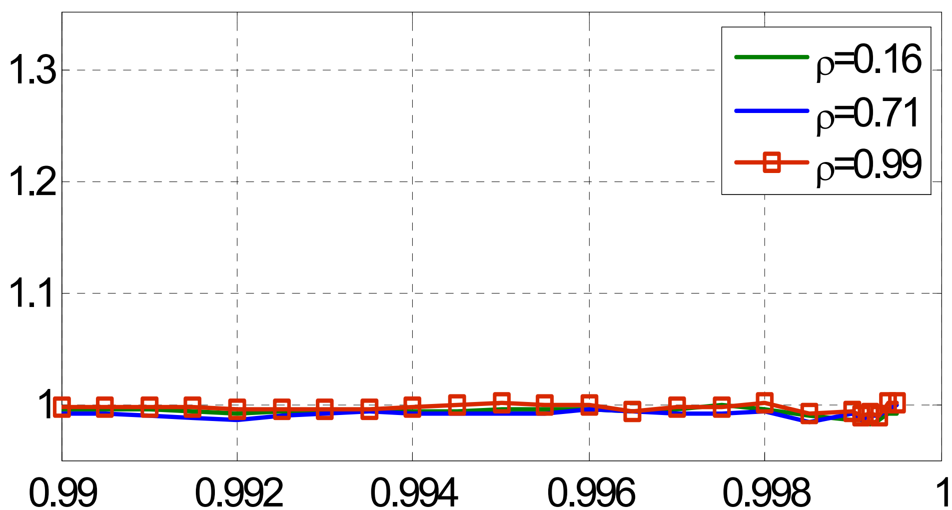
ξ=0.6, ϕ=0



ξ=0.9, ϕ=0

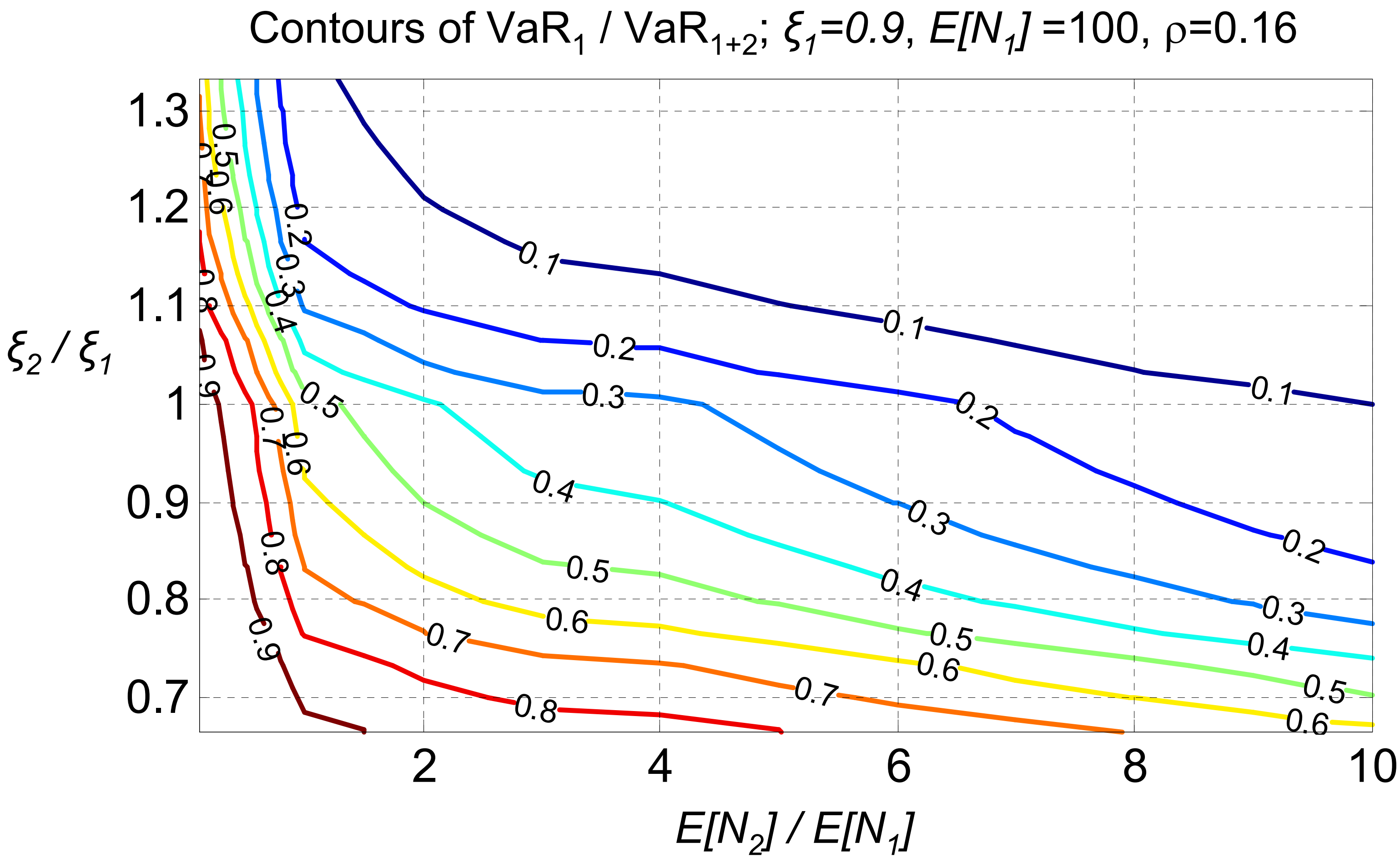


ξ=1.2, ϕ=0

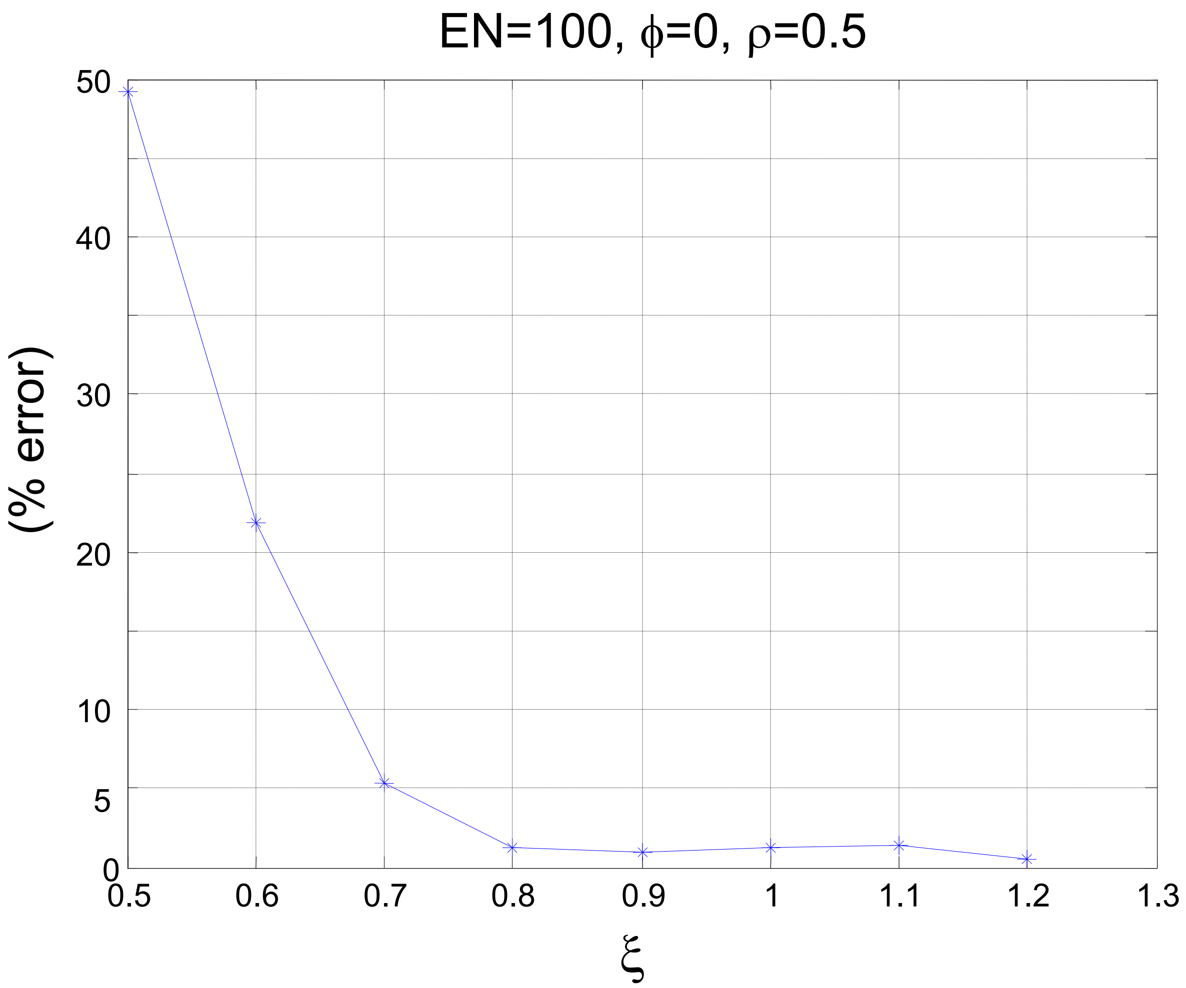


Single cell dominance (simulation)

Analytical approximations



Quality of two-cell approximation; identical cells



Multi Cell Approximation Recap

- Approximation depends only on the correlation, and is independent of any other characteristic of the copula (dependence structure)
- Effect of dependence decreases as severity tail(s) become heavier
- Insignificant effect of dependence in the case of Poisson
- Quality of approximation is similar to single cell result