

# Risk Management of Long Liabilities

**Industrial-Academic Forum on Financial Engineering and  
Insurance Mathematics**

**Robert R. Reitano**

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**Brandeis University** INTERNATIONAL BUSINESS SCHOOL

# Outline

- Introduction and General Theory
  - Some Long Liability Product Structures
  - Long Liability Risk Attributes
  - Risks Associated with Assets
  - Capital Risk
  - Hedged vs. Unhedged Liabilities
- Example: A Hedged Liability
  - Long-Term Care Insurance
- Example: An Unhedged Liability
  - Defined Benefit Pension Plan



# INTRODUCTION AND GENERAL THEORY



# Long Liability Product Structures

- Life Insurance

- Basic Idea:

- Guarantees a Death Benefit (maybe CSV)
    - Requires payment of Premiums

- Variations

- Term DB versus whole life
      - Limited vs. full period premiums payments
    - Fully guaranteed costs/benefits vs. participations
      - Narrow participation w/ guaranteed minimums: Par; UL
      - Wide participation w/ guaranteed minimums : VL



# Long Liability Product Structures (cont'd)

- Annuity Products

- Basic Idea:

- Guarantees an Income Stream(maybe DB, CSV)
    - Requires payment of “Considerations”

- Variations

- Single vs. periodic considerations
    - Immediate versus deferred annuity
      - Term Annuity vs. Life Contingent (w/guarantee)
    - Fully guaranteed costs/benefits vs. participations
      - Narrow participation w/ guaranteed minimums: IA/DA
      - Wide participation w/ guaranteed minimums : VA



# Long Liability Product Structures (cont'd)

- Health-Based Products

- Basic Idea:

- Guarantees Health-Contingent Payments
    - Requires payment of Premiums

- Variations

- Medical/Hospital expense, Disability Income, LTC
    - Fixed payments vs. reimbursements
    - Various levels of payments/reimbursements
      - Deductibles, co-pays, waiting periods, annual/lifetime caps
    - Fully guaranteed costs/benefits vs. participations
      - Non-cancelable, guaranteed renewable, optionally renewable



# Long Liability Product Structures (cont'd)

- Pension Plans (Defined Benefit)

- Basic Idea:

- Guarantees Post-Retirement Pension Payments
    - Requires Plan Contributions (from Employer/Employee)

- Variations

- Pension formula for Annual Payment Earned:
      - Minimum Service requirement for “vesting”
      - Service Years/Factors:  $N \times \% \times \text{“Final Salary”}$
      - Final Salary: Average 3-10 “Best” years or lifetime earnings
    - Single vs. multiple life contingent vs. term/lump sum
    - Indexed vs. discretionarily indexed vs. fixed rate



# Long Liability Risk Attributes

1. Return requirements on current and projected future deposits
  - Real vs. Nominal
  - Fixed vs. Variable
  - If Variable:
    - Indexed vs. participating vs. discretionary
    - With or without minimums
  - If Indexed:
    - What basis: fixed income, equity, commodity?



# Long Liability Risk Attributes (cont'd)

## 2. Life and Other Contingencies

- Mortality
  - Death risk, survival risk
  - Single vs. multiple life
- Health Related
  - Medical vs. Disability vs. LTC
- Lifestyle Choices
  - Retirement age
  - Annuity vs. lump sum



# Long Liability Risk Attributes (cont'd)

## 3. Embedded Options

- Call-like
  - Flexible/optional premiums
    - Call on Minimum rate guarantees, other benefits
- Put-like
  - Early contract termination – “lapsation”
  - Settlement options – lump sum vs. annuity
- Not All Financially Motivated
  - Even when financial – not efficient



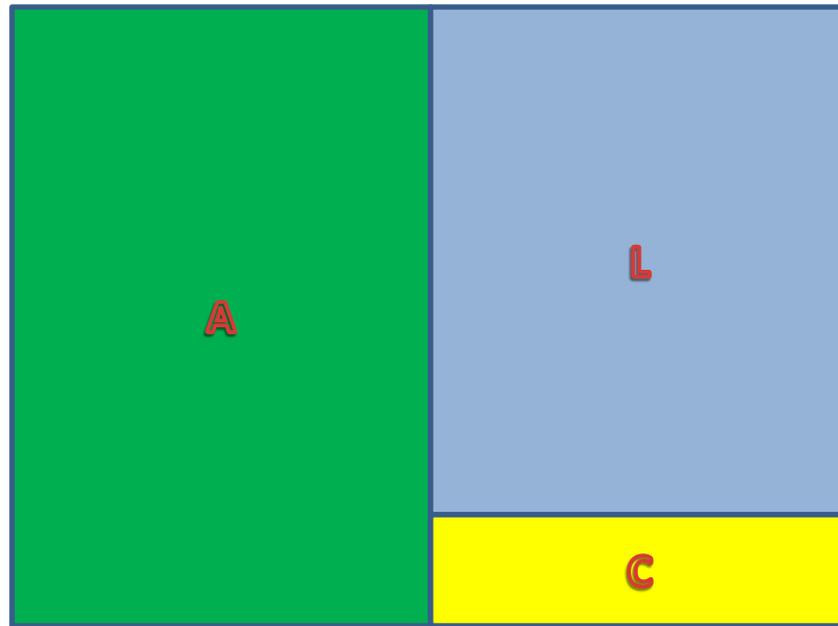
# Risks Associated with Assets

- Liability-Offsetting Risks
  - When related to rate guarantees and/or options
- Additive Risks
  - Return- or option-based
    - when not offsetting
  - Default losses
  - Exchange rates (currencies)
  - Liquidity losses



# From A/L Risks to Capital Risk

- The Balance Sheet



$$C = A - L$$

# Capital - Math

$$\begin{aligned}r^C &= \frac{A}{C} r^A - \frac{L}{C} r^L \\ &= r_A + \Lambda(r_A - r_L)\end{aligned}$$

$$\mu_C = \mu_A + \Lambda(\mu_A - \mu_L)$$

$$\sigma_C^2 = (1 + \Lambda)^2 \sigma_A^2 + \Lambda^2 \sigma_L^2 - 2\Lambda(1 + \Lambda)\sigma_A \sigma_L \rho$$

with “leverage”  $\Lambda = \frac{L}{C}$ .

# Capital Return - Observations

$$r_C = r_A + \Lambda(r_A - r_L)$$

1. Leverage is:

- Good when:  $r_A > r_L$

- Bad when:  $r_A < r_L$

2. More leverage is more good or more bad

3. Less leverage is less good or less bad



# Capital Risk - Observations

$$\sigma_C^2 = (1 + \Lambda)^2 \sigma_A^2 + \Lambda^2 \sigma_L^2 - 2\Lambda(1 + \Lambda)\sigma_A \sigma_L \rho$$

- $\sigma_C^2$  is a convex function of  $\Lambda$  and  $\sigma_A$ :

$$\left(\frac{\partial}{\partial \Lambda}\right)^2 \sigma_C^2 = 2(\sigma_A^2 + \sigma_L^2 - 2\sigma_A \sigma_L \rho) = 2\text{Var}[r_A - r_L]$$

$$\left(\frac{\partial}{\partial \sigma_A}\right)^2 \sigma_C^2 = 2(1 + \Lambda)^2$$

- $\sigma_C^2$  is linear in  $\rho$

# Capital Risk - Conclusions

## 1. As a Function of Asset Risk $\sigma_A$ :

$$\frac{\partial \sigma_C^2}{\partial \sigma_A} = 0 \quad \text{when} \quad \sigma_A = \frac{L}{A} \sigma_L \rho$$

- When  $\rho \leq 0$ , the minimum risk is for  $\sigma_A$  “non-positive”
  - All increases in  $\sigma_A$  increase  $\sigma_C$
- For  $\rho > 0$ , the minimum risk is at  $\sigma_A = L\sigma_L\rho/A > 0$ 
  - Increasing  $\sigma_A$  to this point decreases  $\sigma_C$

# Capital Risk – Conclusions (cont'd)

## 2. As a Function of Leverage $\Lambda$ :

$$\frac{\partial \sigma_C^2}{\partial \Lambda} = 0 \quad \text{when} \quad \Lambda = \frac{\sigma_A (\sigma_L \rho - \sigma_A)}{\text{Var}[r_A - r_L]}$$

- When  $\sigma_A < \sigma_L \rho$ , the minimum risk is at  $\Lambda_0 > 0$ 
  - Increasing  $\Lambda$  to  $\Lambda_0$  decreases  $\sigma_C$
- For  $\sigma_A > \sigma_L \rho$ , the minimum risk is at  $\Lambda_0 < 0$ 
  - All increases in  $\Lambda$  increases  $\sigma_C$

# Capital Risk – Conclusions (cont'd)

## 3. As a Function of Correlation $\rho$ :

$$\frac{\partial \sigma_C^2}{\partial \rho} = -2\Lambda(1 + \Lambda)\sigma_A\sigma_L < 0$$

- Any increase in  $\rho$  decreases capital risk
- *For  $\rho = 1$ , the minimum capital risk is:*

$$\sigma_C^2 = \left[ (1 + \Lambda)\sigma_A - \Lambda\sigma_L \right]^2$$

so  $\sigma_C^2 = 0$  when  $A\sigma_A = L\sigma_L$

# “Hedged” vs. Unhedged Liabilities

An Equivalent Capital Risk Expression:

$$\sigma_C^2 = \left( (1 + \Lambda)\sigma_A - \Lambda\sigma_L \right)^2 + 2\Lambda(1 + \Lambda)\sigma_A\sigma_L(1 - \rho)$$

- Given  $\Lambda$ , managing capital risk is all about  $\rho$  and  $\sigma_A$ 
  - $\rho \approx -1$  maximizes capital risk given  $\sigma_A$ 
    - Example: SPDAs hedged with MBS
  - $\rho > 0$  is usually achievable with diversified assets
  - Hedging involves deliberately managing
    - $\rho \rightarrow 1$  and  $\sigma_A \rightarrow L\sigma_L\rho/A$



# **EXAMPLE: A “HEDGED” LIABILITY - LONG-TERM CARE INSURANCE**



# Example: A “Hedged” Liability

## Long-Term Care Insurance

- Model: Closed Block Acquisition of LTC
- Initial Assets: \$200 million
- Initial Annual Premium: \$18.8 million

**35-45**

**50**

**55**

**60**

**65**

3.8%

4.6%

11.8%

22.8%

40.3%

- 3 year benefit, \$200/day, \$219,000 maximum
- 4% inflation adjustment to all benefits



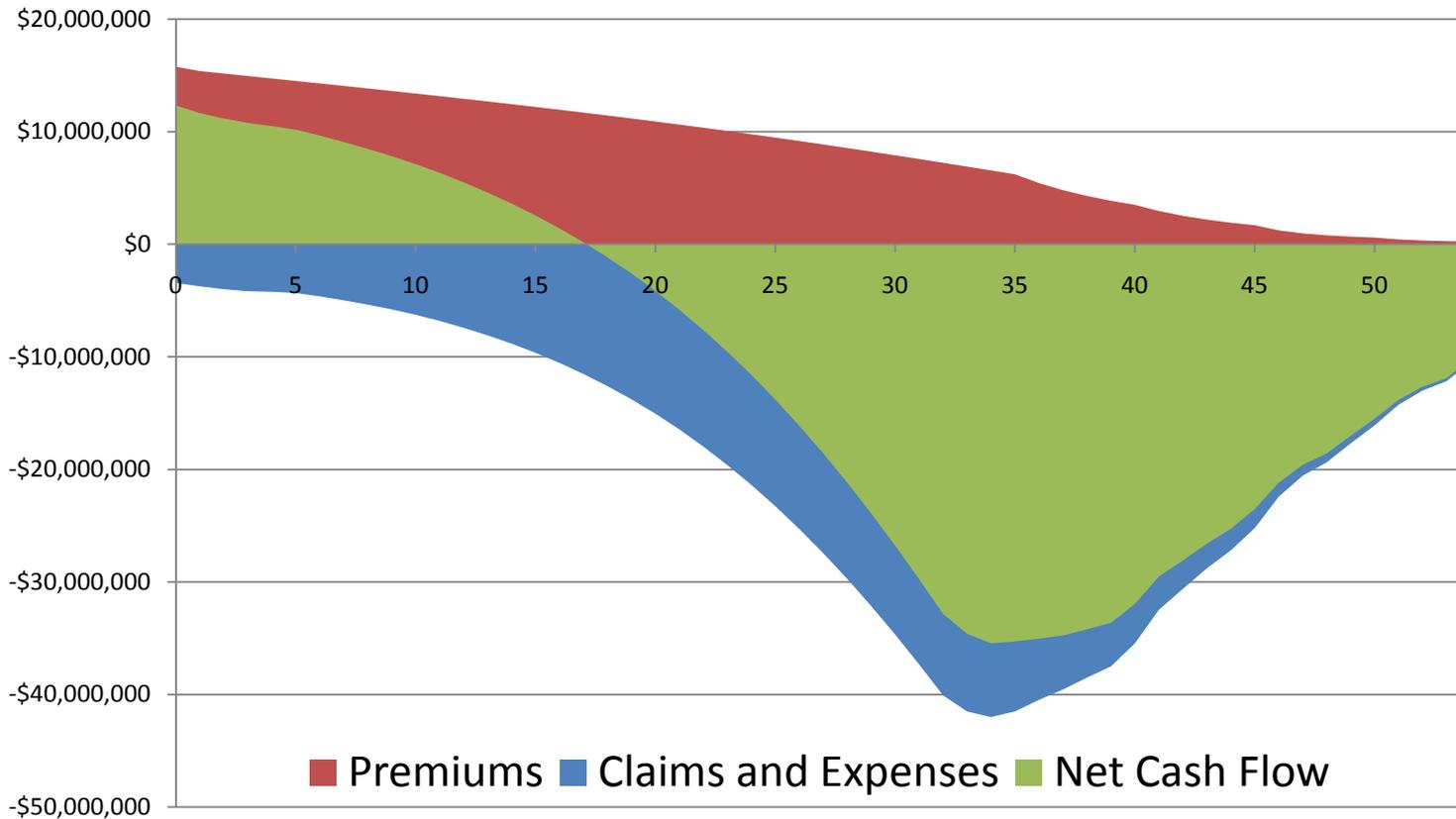
# Example: Long-Term Care (cont'd)

- LTC Benefits for “incurred expenses” for:
  - Home care
  - Assisted living care
  - Nursing home care
- Eligibility: Inability to perform 2 of 6 “Activities of Daily Living” (ADLs):
  1. Bathing, 2. Dressing, 3. Transferring (from bed to chair), 4. Continence, 5. Eating, 6. Toileting



# Example: Long-Term Care (cont'd)

## LTC Policy Cash Flows (55 Years)



# Example: Long-Term Care (cont'd)

Book Value Balance Sheet (@4%)			
Assets		Liabilities	
Bonds	200M	PV Claims	315M
PV Premiums	252M	PV Expenses	9M
		Equity	128M
<b>Total</b>	<b>452M</b>	<b>Total</b>	<b>452M</b>



# Example: Long-Term Care (cont'd)

- Risk Based Capital Assumptions

- LTC RBC =  $0.05V$   
 $+ [0.1 * P + 0.03 * (P - 50M)]$   
 $+ [0.25 * C + 0.08 * (C - 35M)]$

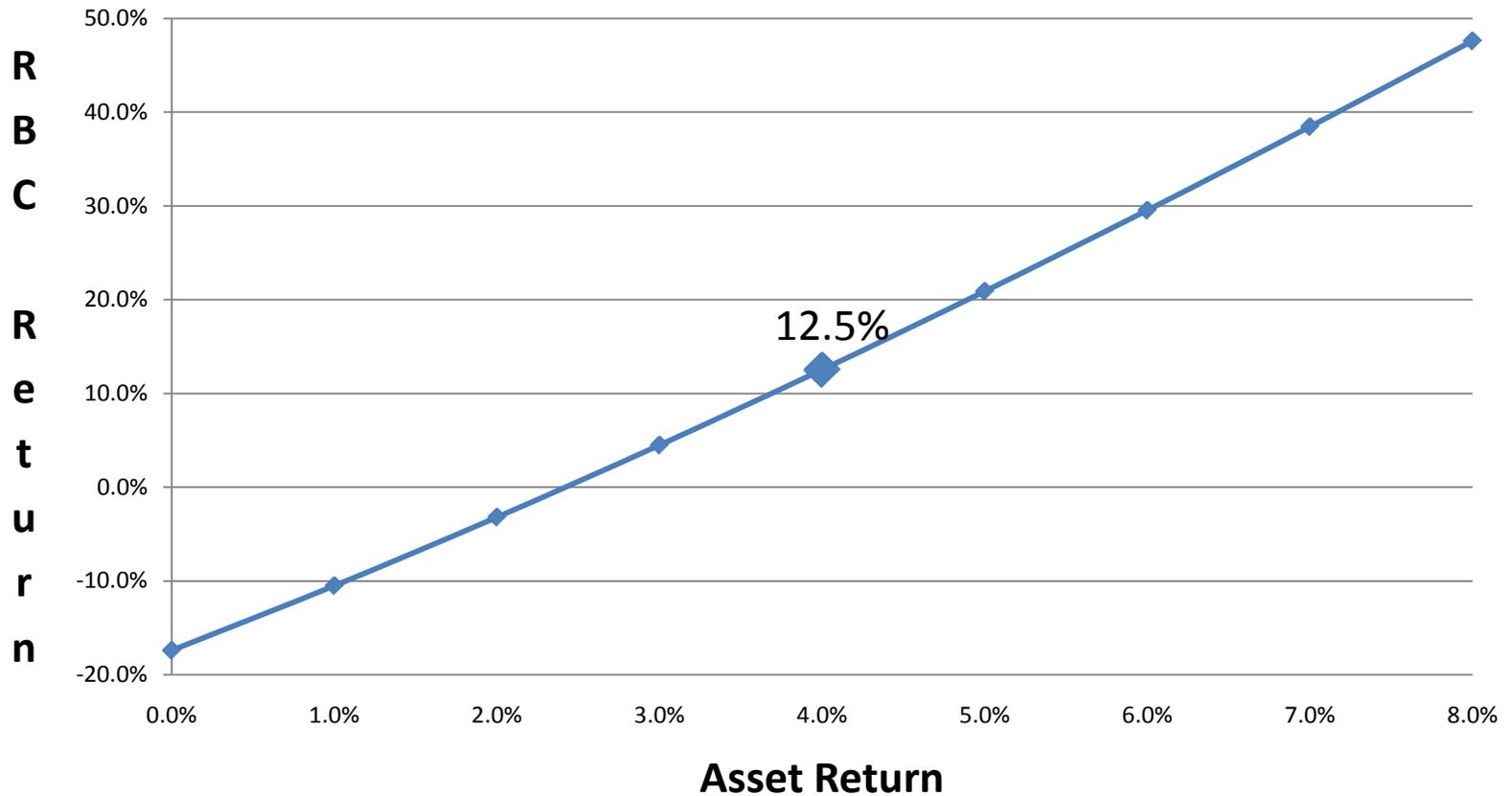
- Asset RBC =  $0.015A$

- Pricing RBC = 400% Total RBC



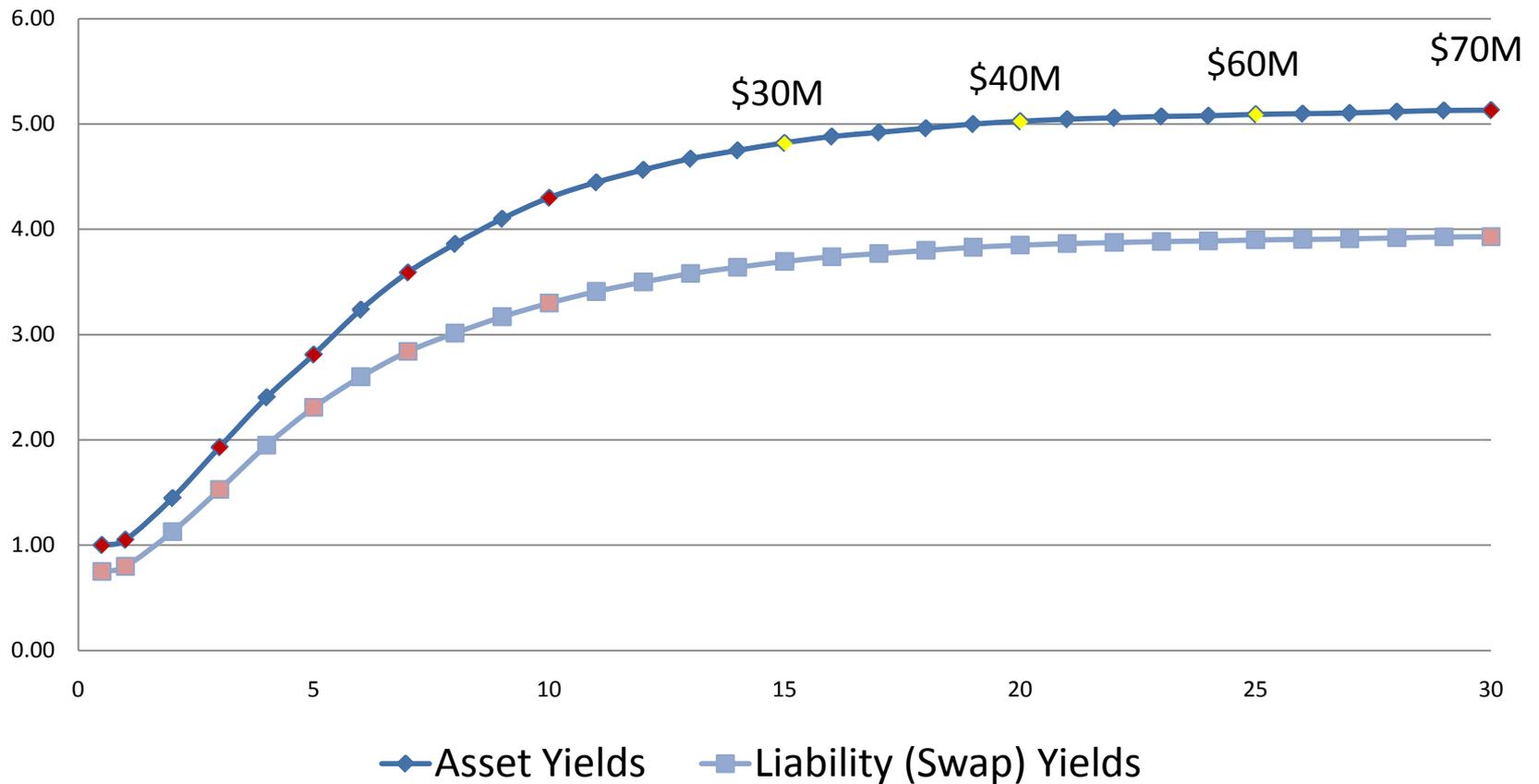
# Example: Long-Term Care (cont'd)

## RoRBC vs. ROA (55 year)



# Example: Long-Term Care (cont'd)

## Valuation Par Bond Yields



# Example: Long-Term Care (cont'd)

- Some Options for Extending the Yield Curve
  - Obtain swap rate quotes (up to 50 years?)
  - Define spots/forwards  $t > 30$ ,  $i(t) = f(i(30), t)$ 
    - Freeze spots/forwards for years  $t \geq 30$
    - Decrease forwards for years  $t \geq 30$  to some “Final” value
  - Extend with Vasicek, CIR, Hull/White, etc.
- Choice affects  $\Delta C$  pattern through time
  - Most choices not tradable



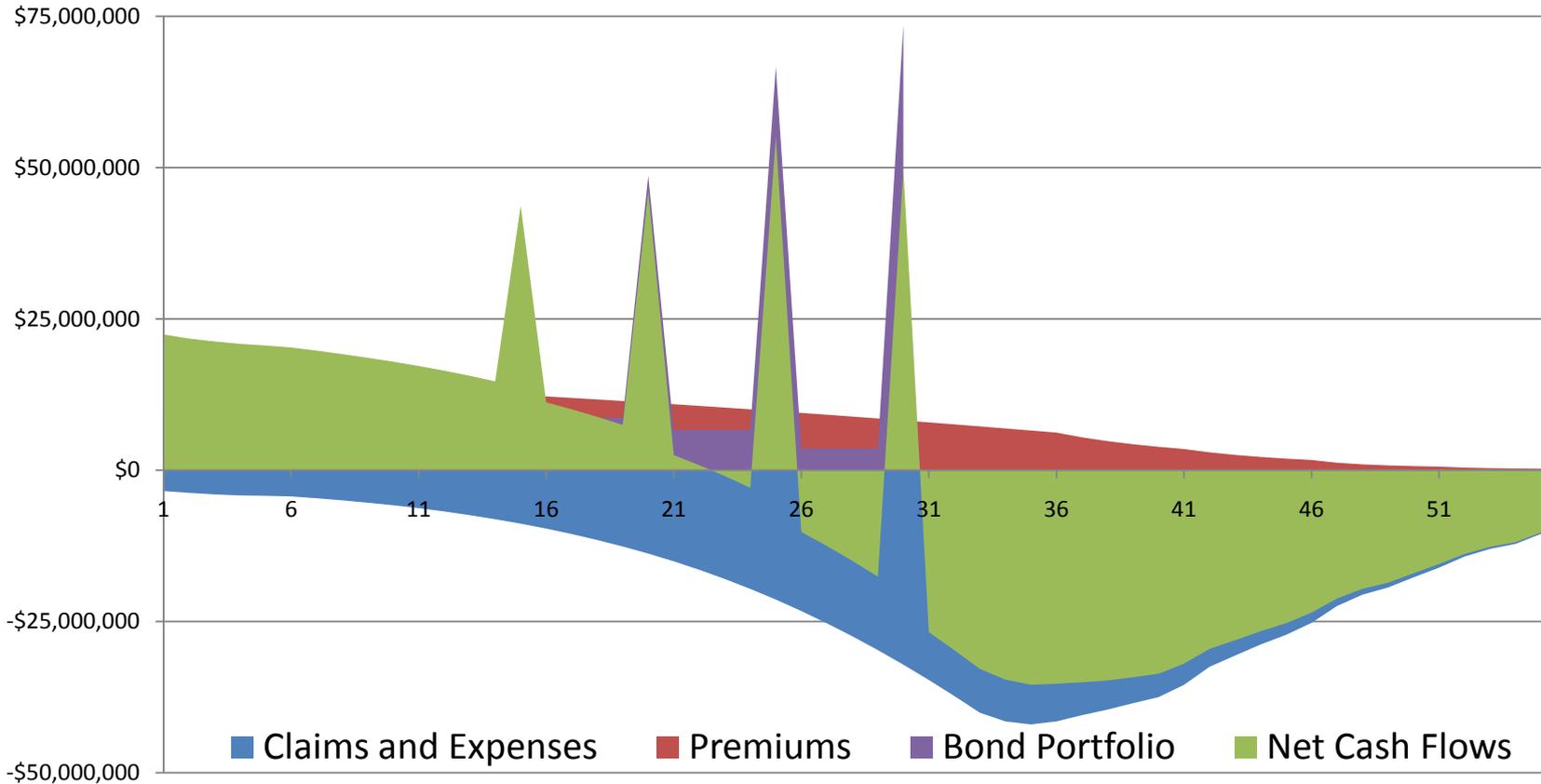
# Example: Long-Term Care (cont'd)

Fair Value Balance Sheet			
Assets		Liabilities	
Bonds @ Bond Rates	200.0M	PV Claims @ Swap	305.9M
PV Premium @ Swap	249.9M	PV Expense @ Swap	8.9M
		Equity	135.1M
Total	449.9M	Total	449.9M



# Example: Long-Term Care (cont'd)

## LTC Portfolio Cash Flows



# Example: Long-Term Care (cont'd)

## Partial Duration Review (see Reitano, JPM)

### Price Model for Assets/Liabilities

$A(i_1, i_3, i_5, i_7, i_{10}, i_{30})$ :

$i_k$  denotes reference par bond yields

$L(j_1, j_3, j_5, j_7, j_{10}, j_{30})$ :

$j_k$  denotes swap yields

# Example: Long-Term Care (cont'd)

## Partial Duration Review (see JPM)

Price Model for Assets/Liabilities

$$A(i_1, i_3, i_5, i_7, i_{10}, i_{30})$$

$$L(j_1, j_3, j_5, j_7, j_{10}, j_{30}):$$

Partial Durations:

$$D_k^A = -\frac{\partial A}{\partial i_k} / A$$

$$D_k^L = -\frac{\partial L}{\partial j_k} / L$$

Dollar Partial (per .01)

$$D_k^{A\$} = -AD_k^A (.01)$$

$$D_k^{L\$} = -LD_k^L (.01)$$



# Example: Long-Term Care (cont'd)

Dollar Partial Duration Analysis (\$PD = Per .01)				
Maturity	Prem. \$PD	Bond \$PD	Claims/Exp \$PD	Net CF \$PD
1	-142,388	0	269,720	-412,108
3	-366,778	0	718,125	-1,084,902
5	-594,602	0	1,219,033	-1,813,635
7	-1,005,706	0	2,148,103	-3,153,809
10	-4,081,939	-2,296,065	8,133,611	-14,511,615
30	-26,175,427	-25,580,276	-99,922,756	48,167,053
Sum	-\$32,366,840	-\$27,876,341	-\$87,434,164	\$27,190,983



# Example: Long-Term Care (cont'd)

## Partial Duration Review (cont'd)

From Calculus:

$$\Delta A \approx -A \sum D_k^A \Delta i_k = -A \left( \overline{D}^A \cdot \overline{\Delta i} \right)$$

- Similar formula for  $\Delta L$  but with  $\Delta j$

From Probability:

$$E[\Delta A] \approx -A \left( \overline{D}^A \cdot E[\overline{\Delta i}] \right)$$

$$Var[\Delta A] \approx A^2 \left( \overline{D}^A \right)^T V(\overline{\Delta i}) \overline{D}^A$$

- Similar formulas for  $\Delta L$

# Example: Long-Term Care (cont'd)

## Partial Duration Review (cont'd)

As Random Variables:

$$\Delta i_k = \Delta j_k + \Delta s_k$$

Here, we assume spreads are fixed (NOT necessary)

$$V[\overline{\Delta i}] = V[\overline{\Delta j}]$$

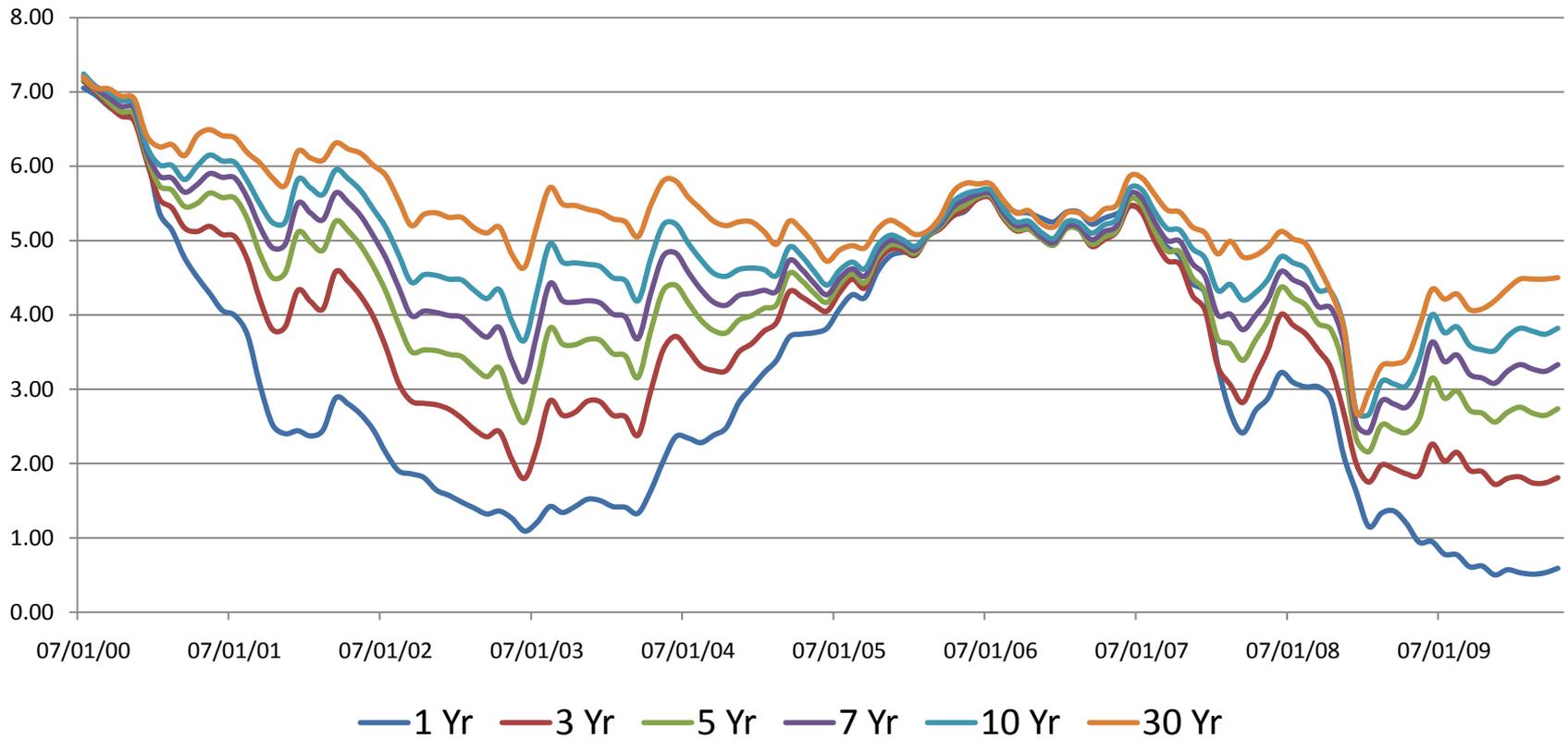
$$Var[\Delta A] \approx A^2 \left( \overline{D}^L \right)^T V(\overline{\Delta j}) \overline{D}^L$$

$$Cov[\Delta A, \Delta L] \approx AL \left( \overline{D}^L \right)^T V(\overline{\Delta j}) \overline{D}^A$$



# Example: Long-Term Care (cont'd)

## Monthly Swap Rates 7/2000 - 4/2010



# Example: Long-Term Care (cont'd)

Swap Rates 7/2000 - 4/2010

Average Monthly Deltas (Basis Points)					
1	3	5	7	10	30
-5.52	-4.56	-3.79	-3.31	-2.92	-2.31

Monthly Delta Correlation Matrix						
	1	3	5	7	10	30
1	1.000	0.851	0.743	0.675	0.611	0.490
3	0.851	1.000	0.971	0.935	0.893	0.792
5	0.743	0.971	1.000	0.991	0.970	0.886
7	0.675	0.935	0.991	1.000	0.993	0.927
10	0.611	0.893	0.970	0.993	1.000	0.958
30	0.490	0.792	0.886	0.927	0.958	1.000



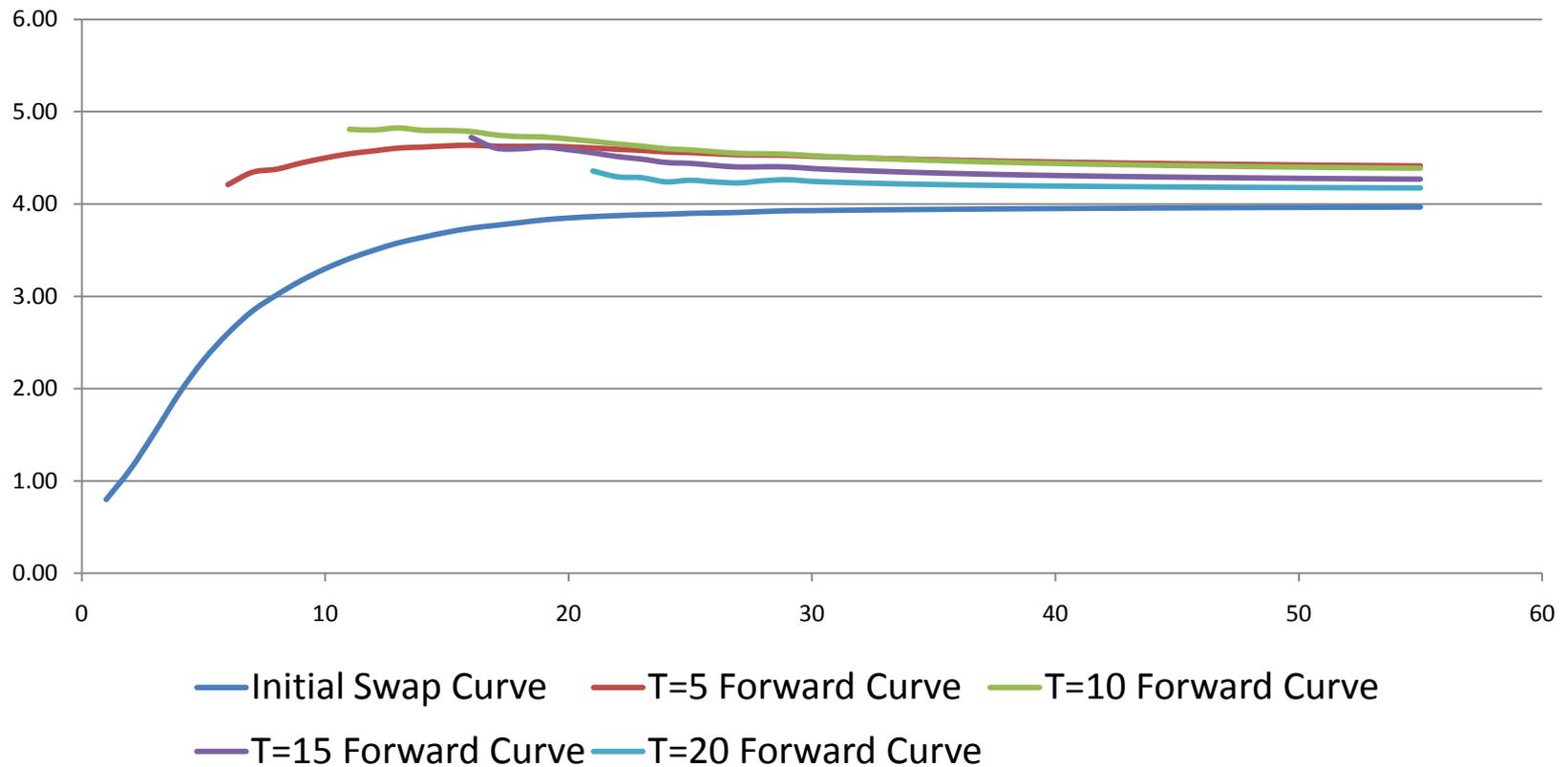
# Example: Long-Term Care (cont'd)

<b>LTC Model w/ Bonds</b>			
	<b>Monthly Swap Rates 7/2000 - 4/2010</b>		
	<b>Assets</b>	<b>Liabilities</b>	<b>Capital</b>
<b>Average Monthly Delta</b>	\$1,461,146	\$1,903,342	-\$442,196
<b>Std. Dev. Of Delta</b>	\$13,777,824	\$19,707,377	\$6,176,554
<b>Covariance (A,L)</b>	270,029,658,812,433		
<b>Correlation (A,L)</b>	0.9944936		



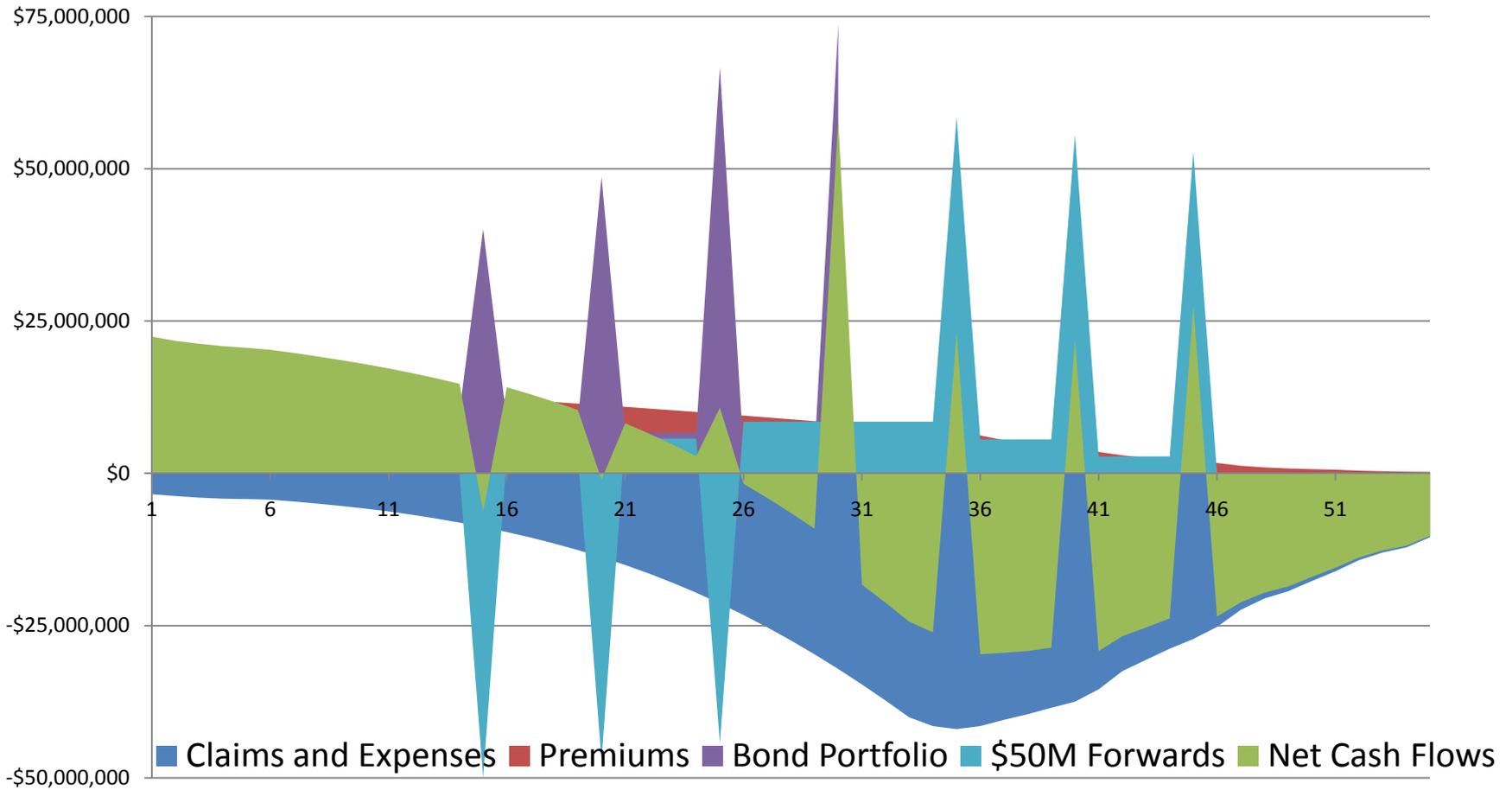
# Example: Long-Term Care (cont'd)

## Forward Swap Curves



# Example: Long-Term Care (cont'd)

## LTC Portfolio Cash Flows w/ 3x50M Forwards



# Example: Long-Term Care (cont'd)

Dollar Partial Duration Analysis (\$PD = Per .01)					
Mty	Prem. \$PD	Bond \$PD	Forward \$PD	Claims/Exp \$PD	Net CF \$PD
1	-142,388	0	36508	269,720	-412,108
3	-366,778	0	100486	718,125	-1,084,902
5	-594,602	0	174089	1,219,033	-1,813,635
7	-1,005,706	0	320297	2,148,103	-3,153,809
10	-4,081,939	-2,296,065	7898296	8,133,611	-14,511,615
30	-26,175,427	-25,580,276	-22057701	-99,922,756	48,167,053
Sum	-\$32,366,840	-\$27,876,341	-\$13,528,024	-\$87,434,164	\$13,662,959



# Example: Long-Term Care (cont'd)

<b>LTC Model w/ Bonds and Forwards</b>			
	<b>Monthly Swap Rates 7/2000 - 4/2010</b>		
	<b>Assets</b>	<b>Liabilities</b>	<b>Capital</b>
<b>Average Monthly Delta</b>	\$1,715,517	\$1,903,342	-\$187,825
<b>Std. Dev. Of Delta</b>	\$16,691,749	\$19,707,377	\$3,195,893
<b>Covariance (A,L)</b>	328,390,720,220,899		
<b>Correlation (A,L)</b>	0.9982980		



# Example: Long-Term Care (cont'd)

## Marginal Effect of New Business Issue

- For a new business model office:  
Net Cash Flow Duration = -24.8
- Per \$100 of Year One Premium:  
Dollar Duration (per .01)  $\approx$  110
- Per \$100 of Par, 30 year non-call bonds:  
Dollar Duration  $\approx$  15



**EXAMPLE: AN UNHEDGED LIABILITY**  
**- DEFINED BENEFIT PENSION PLAN**  
**- FIXED INCOME MODEL**



# Example: An “Un-Hedged” Liability

## Defined Benefit Pension Plan

- Model: Open Group DB Plan
- Initial Assets: \$750 million

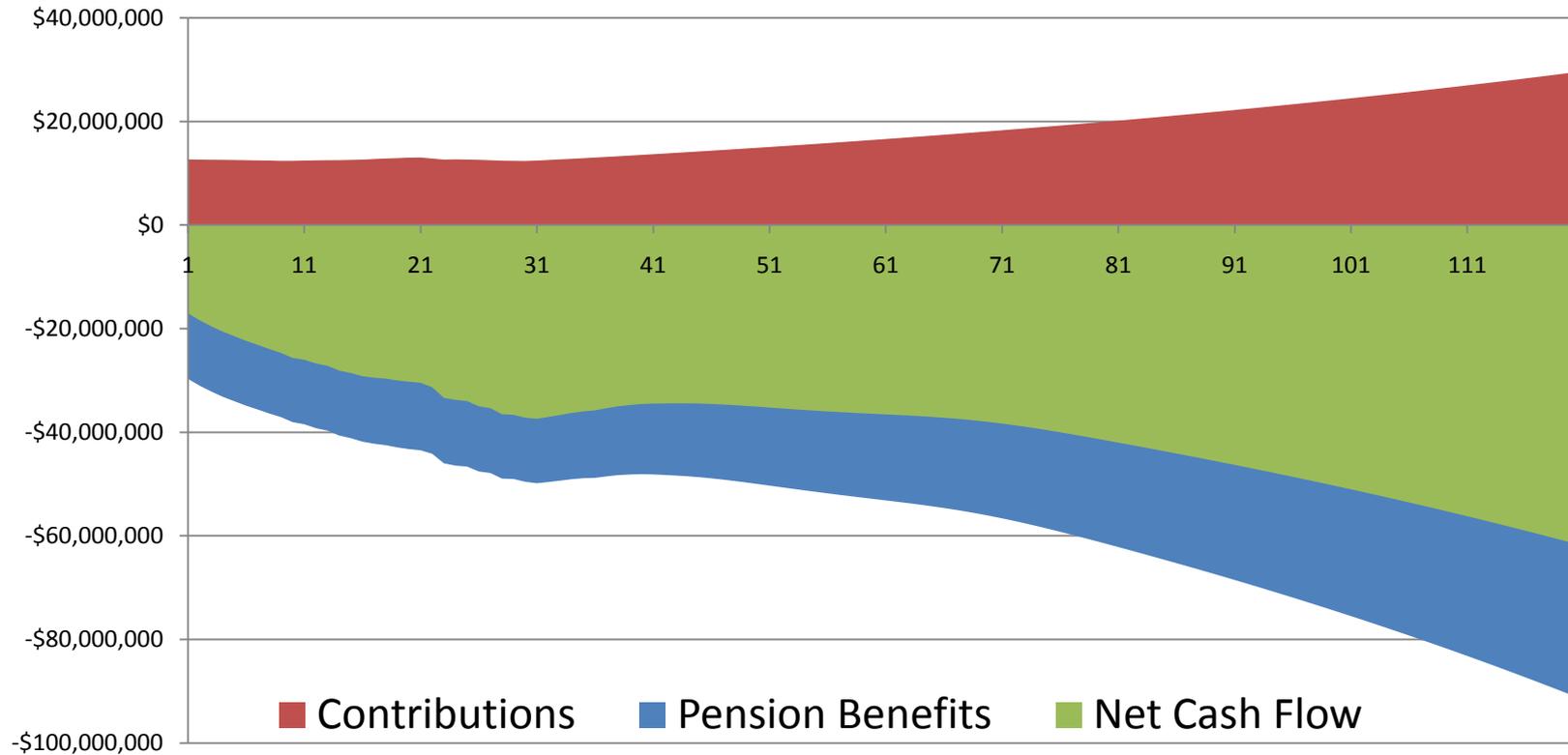
### Book Value Balance Sheet (@4%)

Assets		Liabilities	
Bonds	750M	PV Pension Benefits	1092M
PV 8.5% Contributions	342M	Equity	0
Total	1092M	Total	1092M



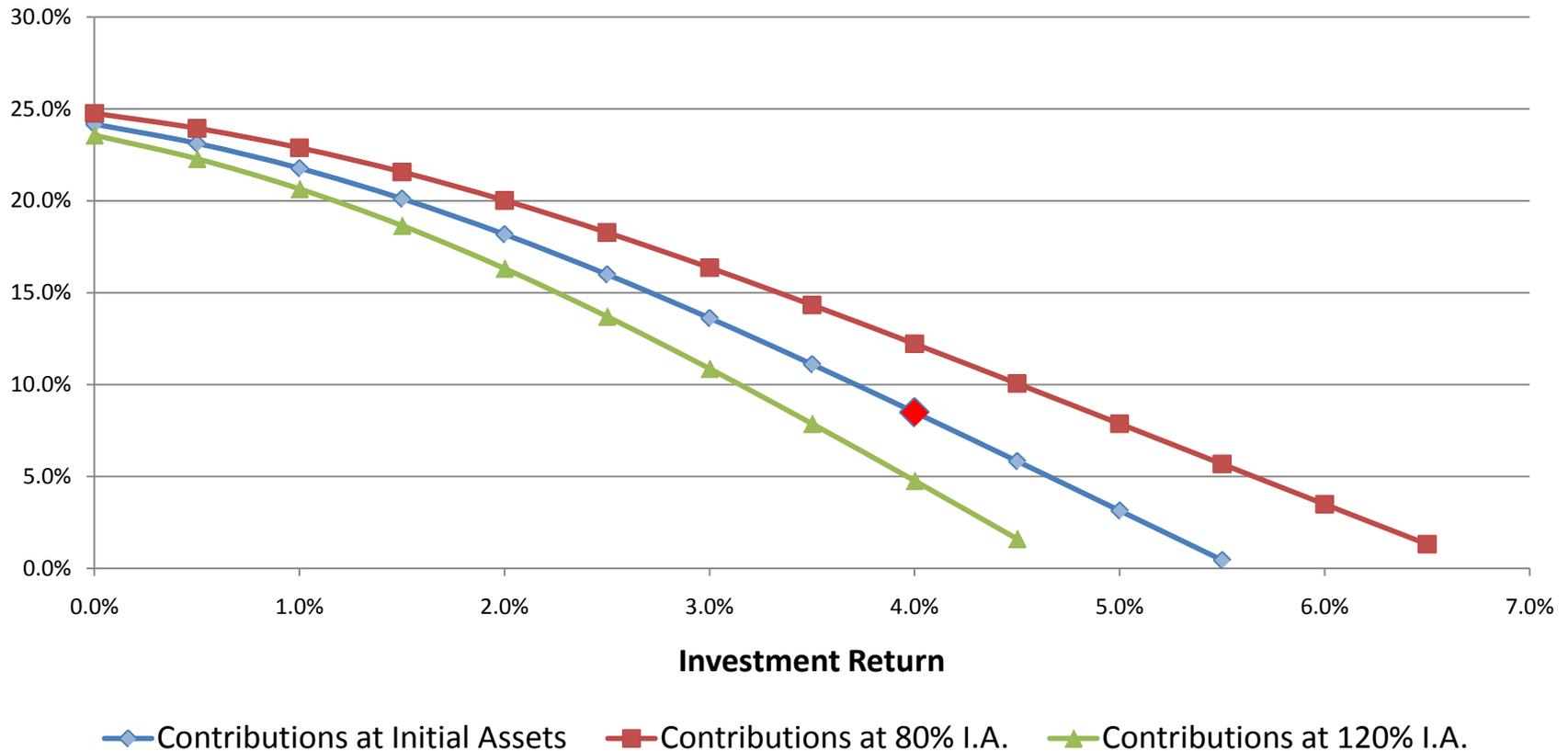
# Example: Defined Benefit Pension(cont'd)

## Open Group Pension Cash Flows (120 Years)



# Example: Defined Benefit Pension(cont'd)

## Break-Even Contribution Rate



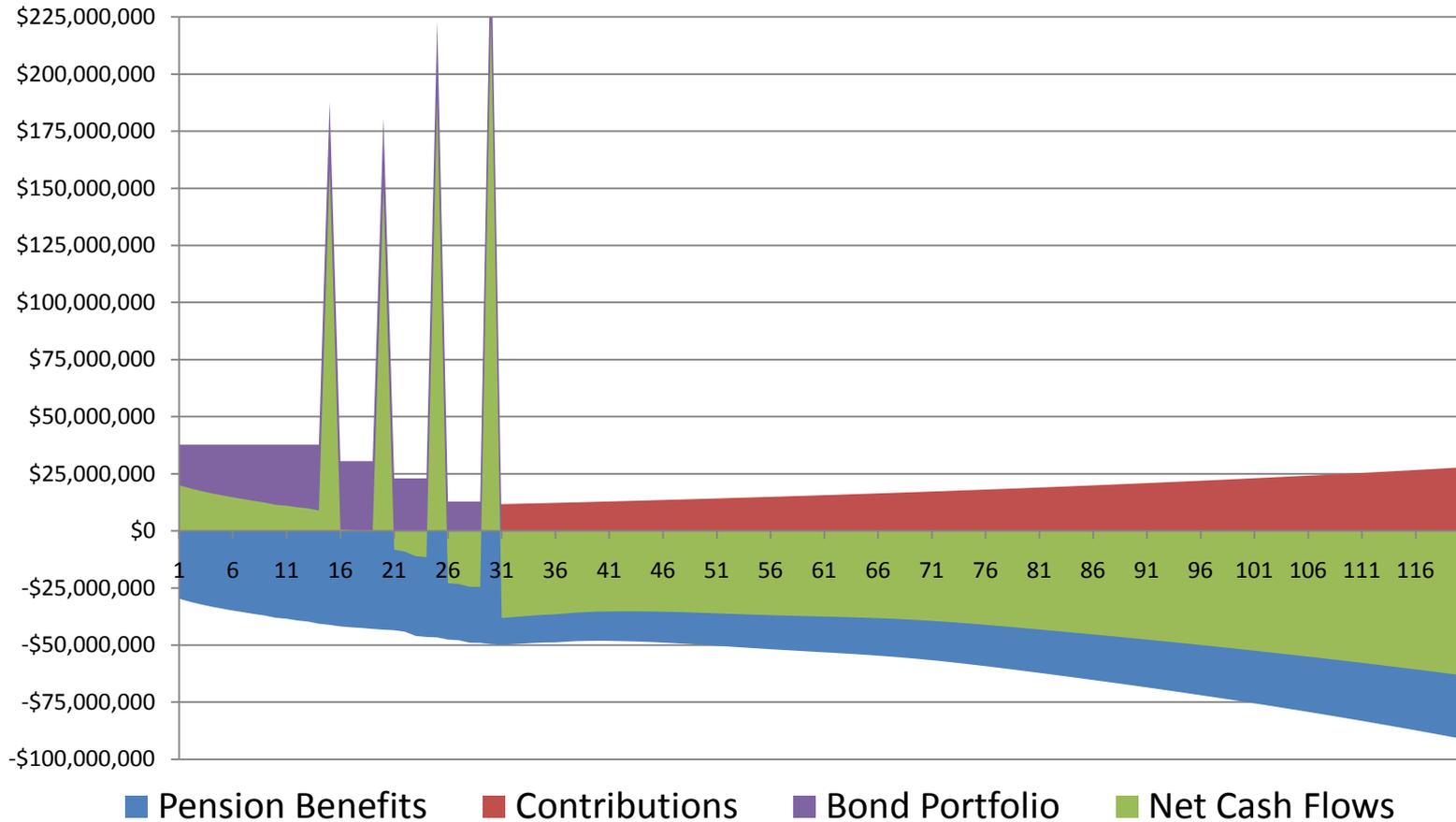
# Example: Defined Benefit Pension(cont'd)

Book Value Balance Sheet (@4%)			
Assets		Liabilities	
<b>Closed Group</b>			
Bonds	750M	PV Benefits	854M
PV 8.5% Contributions	145M	Equity	41M
<b>Future Group</b>			
Bonds	0	PV Benefits	238M
PV 8.5% Contributions	197M	Equity	(41M)



# Example: Defined Benefit Pension(cont'd)

## Open Group Pension Portfolio Cash Flows



# Example: Defined Benefit Pension(cont'd)

Fair Value Balance Sheet			
Assets		Liabilities	
Bonds @ Bond Rates	750M	PV Benefits @ Swap	1101M
PV Contrib. @ Swap	326M		
		Equity	(25M)
Total	1076M	Total	1076M



# Example: Defined Benefit Pension(cont'd)

## Dollar Partial Duration Analysis (\$PD = Per .01)

Mty	Contrib. \$PD	Bond \$PD	Benefits \$PD	Net CF \$PD
1	147,272	0	732,713	-585,441
3	407,685	0	1,907,306	-1,499,621
5	709,082	0	3,128,608	-2,419,526
7	1,313,808	0	5,435,409	-4,121,601
10	6,926,332	-9,972,761	25,390,611	-28,437,039
30	-103,679,274	-93,162,618	-366,071,779	169,229,887
Sum	-94,175,094	-103,135,378	-329,477,132	132,166,660



# Example: Defined Benefit Pension(cont'd)

<b>Open Group Pension Model</b>			
	<b>Monthly Swap Rates 7/2000 - 4/2010</b>		
	<b>Assets</b>	<b>Liabilities</b>	<b>Capital</b>
<b>Average Monthly Delta</b>	\$4,534,546	\$7,280,033	-\$2,745,487
<b>Std. Dev. Of Delta</b>	\$44,769,269	\$74,313,306	\$29,633,827
<b>Covariance (A,L)</b>	3,324,295,614,272,540		
<b>Correlation (A,L)</b>	0.9992014		



# Example: Defined Benefit Pension(cont'd)

## Open Group Pension Model w/ \$1.5B Forwards

	Monthly Swap Rates 7/2000 - 4/2010		
	Assets	Liabilities	Capital
Average Monthly Delta	\$7,078,257	\$7,280,033	-\$201,776
Std. Dev. Of Delta	\$74,363,064	\$74,313,306	\$3,424,084
Covariance (A,L)	5,520,304,169,237,350		
Correlation (A,L)	0.9989394		



**EXAMPLE: AN UNHEDGED LIABILITY**  
**- DEFINED BENEFIT PENSION PLAN**  
**- GENERAL ASSET MODEL PART I**  
**-FAILURE PROBABILITY MODEL**



# Example: Defined Benefit Pension(cont'd)

- Asset Model:

$$dA_t = \mu A_t dt + \sigma A_t dB_t$$

$$A_0 = 750M$$

- Return Model:

$$A_{t+s} = A_t \exp\left(s\left[\mu - 0.5\sigma^2\right] + \sqrt{s}\sigma z\right), \quad z \sim N(0,1)$$

- Pension Asset Model:

$$A_{t+1} = A_t \exp\left(\mu - 0.5\sigma^2 + \sigma z_t\right) + CF_t^{Net}$$



# Example: Defined Benefit Pension(cont'd)

## Model Used for Estimation of:

### 1. Asset Adequacy Failure

$$p_t^A \equiv \Pr[A_t < 0]$$

### 2. Capital Adequacy Failure

$$p_t^{V,r} \equiv \Pr[A_t < V_t],$$

where

$$V_t \equiv -\sum_{s>0} CF_{t+s}^{Net} (1+r)^{-s}$$

# Example: Defined Benefit Pension(cont'd)

- Model Parameters

Geometric Brownian Motion Parameters						
Equity Allocation	0%	20%	40%	60%	80%	100%
$\mu$	3.0%	4.3%	5.7%	7.2%	8.8%	10.6%
$\sigma$	0.0%	3.6%	7.2%	10.8%	14.4%	18.0%

- 20% Equity model: approximates Bond Portfolio volatility using swap market data: 7/2000 - 4/2010



# Example: Defined Benefit Pension(cont'd)

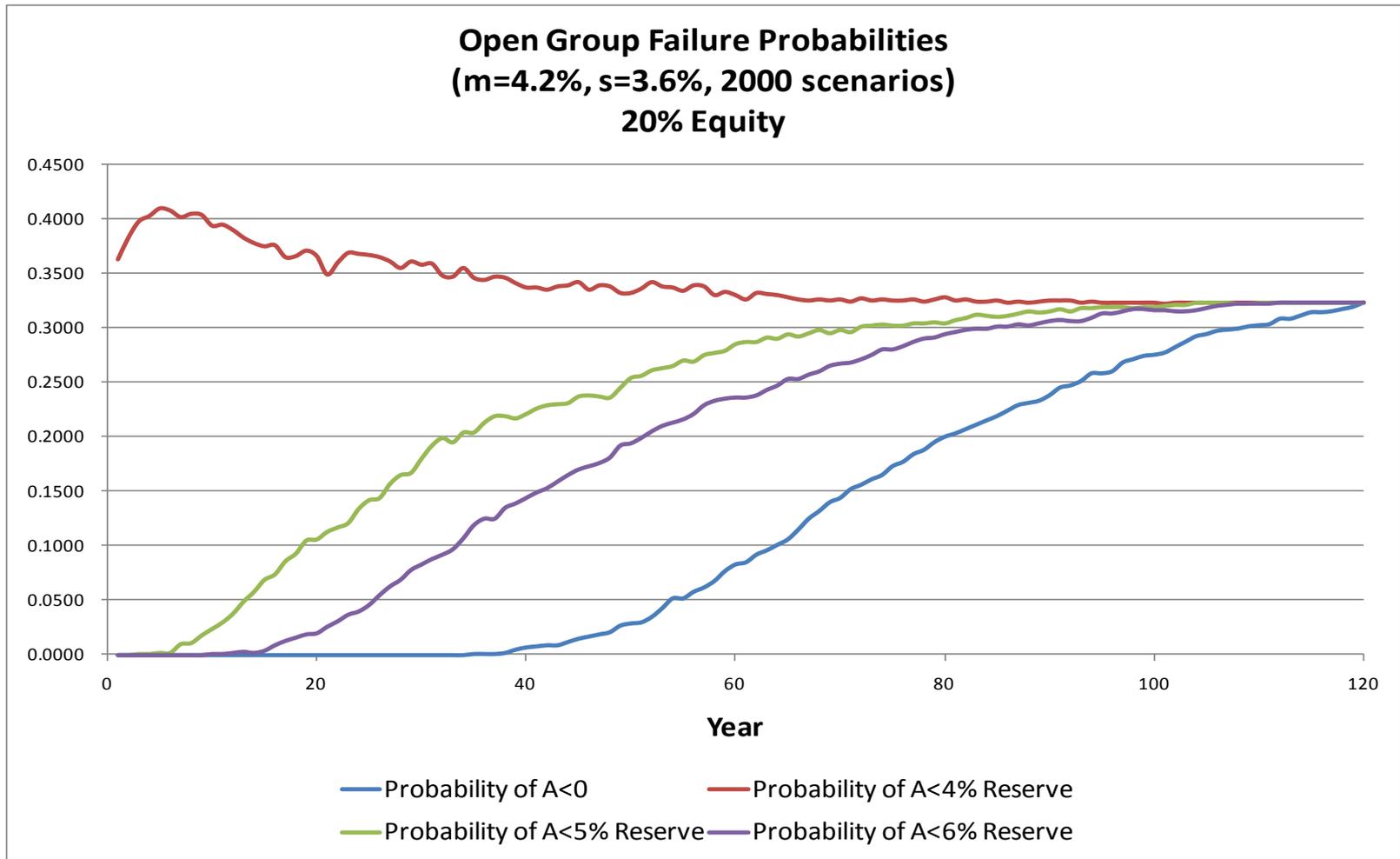
## Probability of Failure for Some t

2000 Scenarios

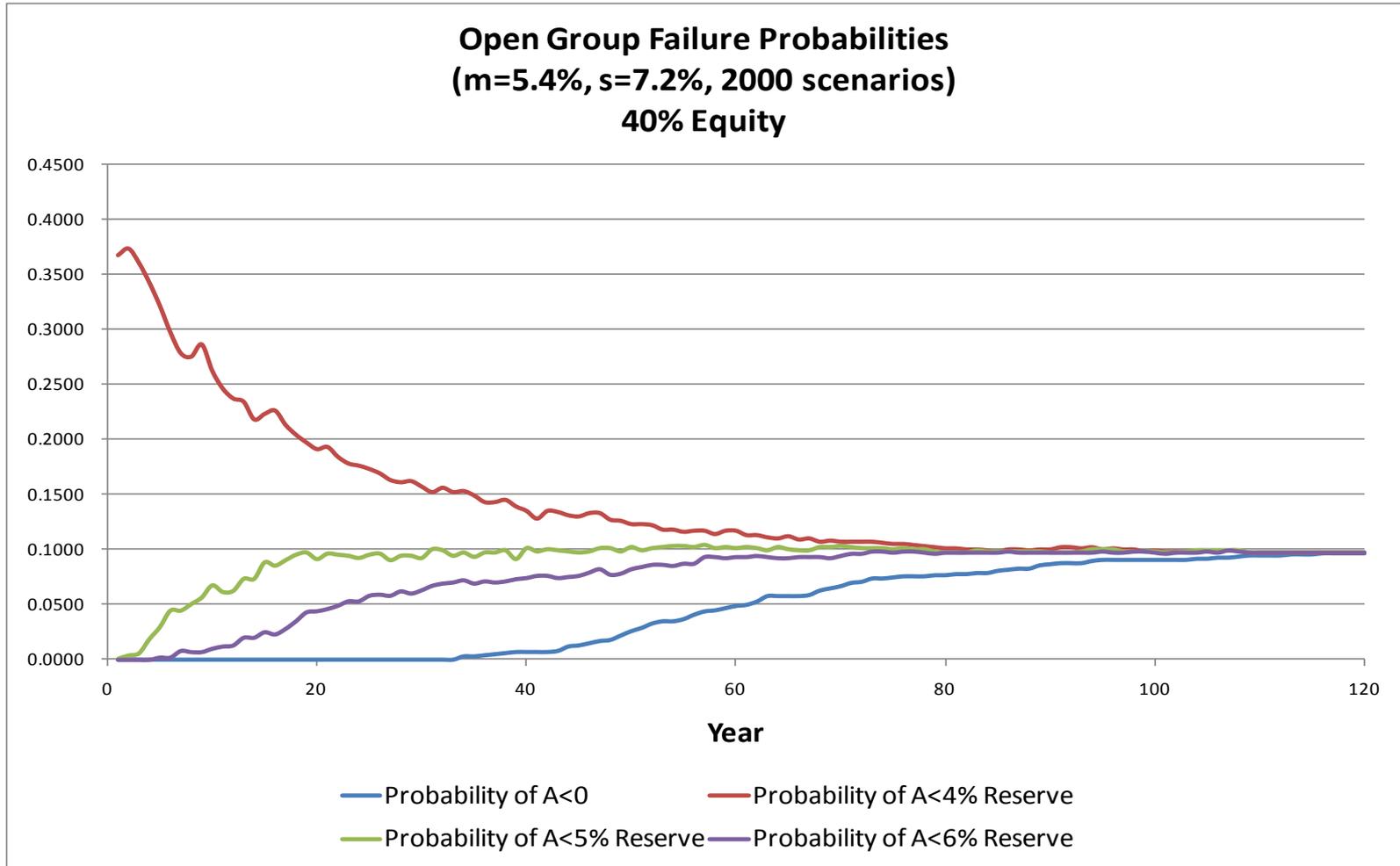
Equity Percent	Asset Test	4% Reserve Test	5% Reserve Test	6% Reserve Test
0%	100.0%	100.0%	100.0%	100.0%
20%	32.3%	77.9%	35.0%	32.5%
40%	9.7%	67.5%	20.9%	13.0%
60%	6.2%	66.2%	24.6%	12.9%
80%	4.9%	62.4%	27.2%	17.3%
100%	4.3%	64.8%	31.5%	18.7%



# Example: Defined Benefit Pension(cont'd)



# Example: Defined Benefit Pension(cont'd)



**EXAMPLE: AN UNHEDGED LIABILITY**  
**- DEFINED BENEFIT PENSION PLAN**  
**- GENERAL ASSET MODEL PART II**  
**- INITIAL ASSETS VS. CTE**  
**- MANISTRE (2009)**



# Example: Defined Benefit Pension(cont'd)

- Asset Model: Same as before
- For each Brownian path,  $\{z^\alpha(j)\}$ , define:

$$A_0^\alpha \equiv -\sum_{j=1}^N CF_j^{Net} \exp\left[-j(\mu - 0.5\sigma^2) - \sigma z^\alpha(j)\right]$$

- Each  $A_0^\alpha$  value identifies the adequate level of initial assets for path  $\alpha$

# Example: Defined Benefit Pension(cont'd)

Model Used for Estimation of:

## 1. Asset Adequacy Failure

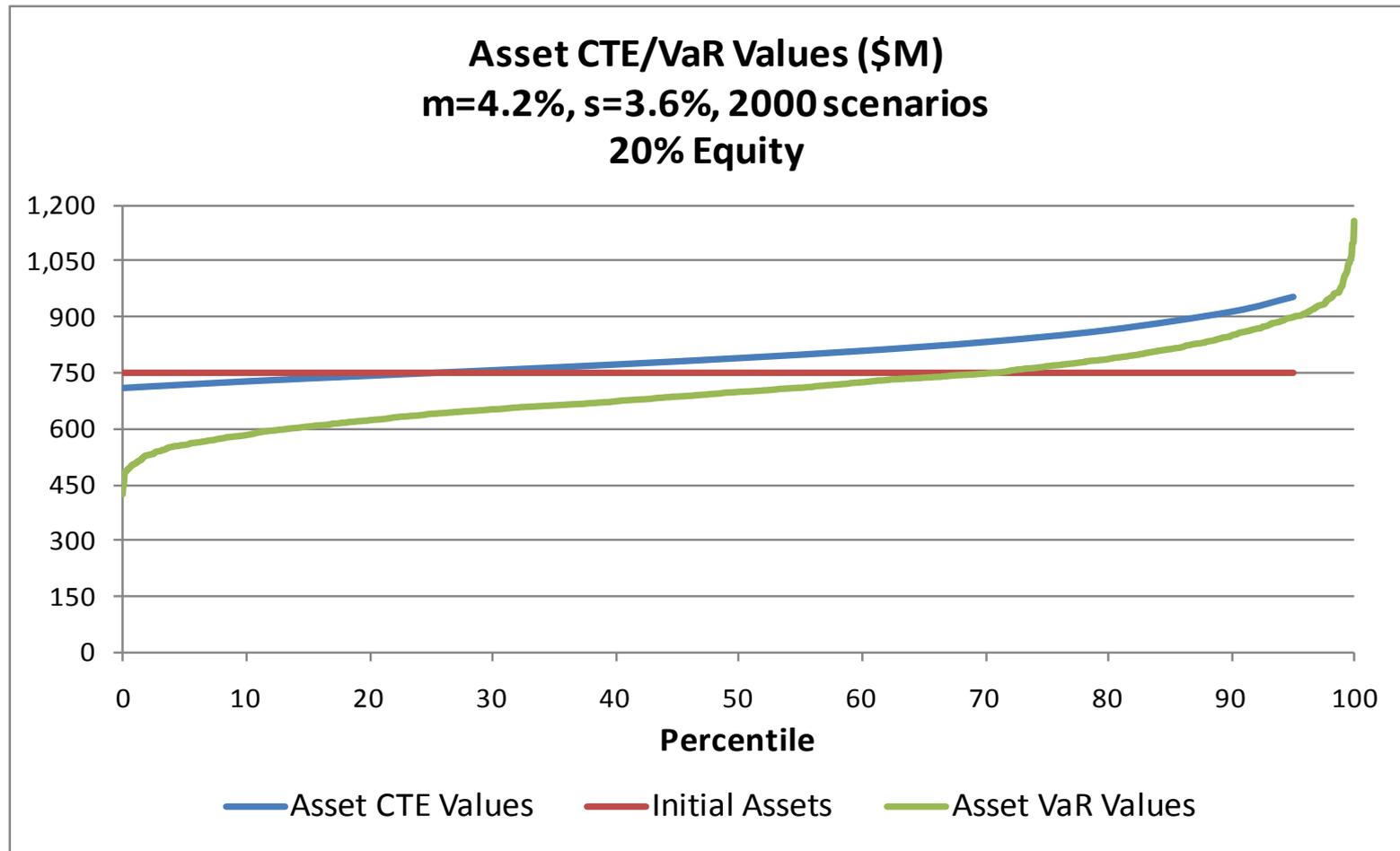
$$\Pr[A_0^\alpha > A_0]$$

## 2. Needed Initial Assets, based on *VaR* or *CTE*

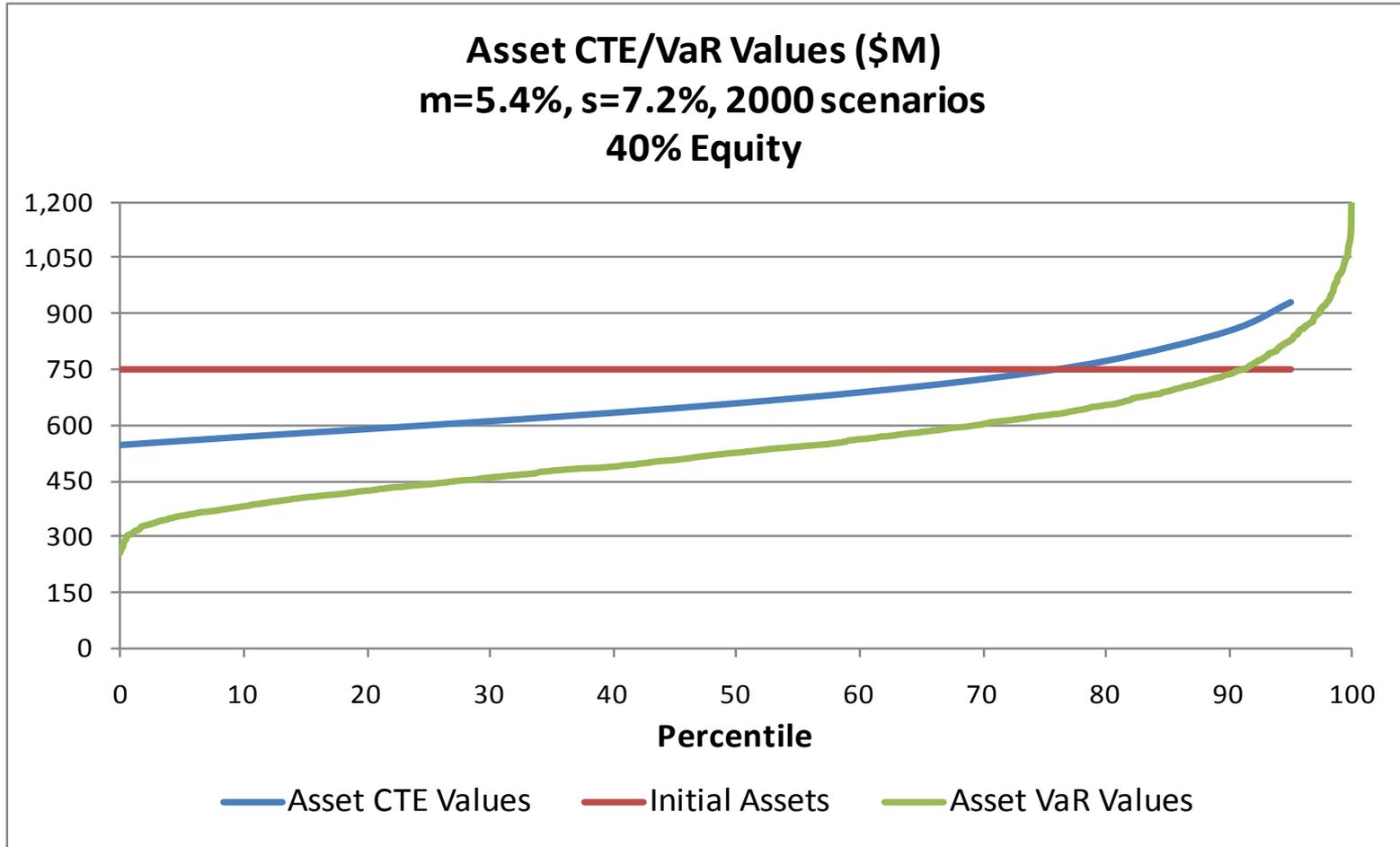
$$A_0^{VaR_x} = \min\{A \mid \Pr[A_0^\alpha > A] \leq 1 - x\}$$

$$A_0^{CTE_x} = E\left\{A_0^\alpha \mid A_0^\alpha > A_0^{VaR_x}\right\}$$

# Example: Defined Benefit Pension(cont'd)



# Example: Defined Benefit Pension(cont'd)



# Example: Defined Benefit Pension(cont'd)

Model Used for Estimation of:

3. Evaluation of the “marginal cost yield curve”

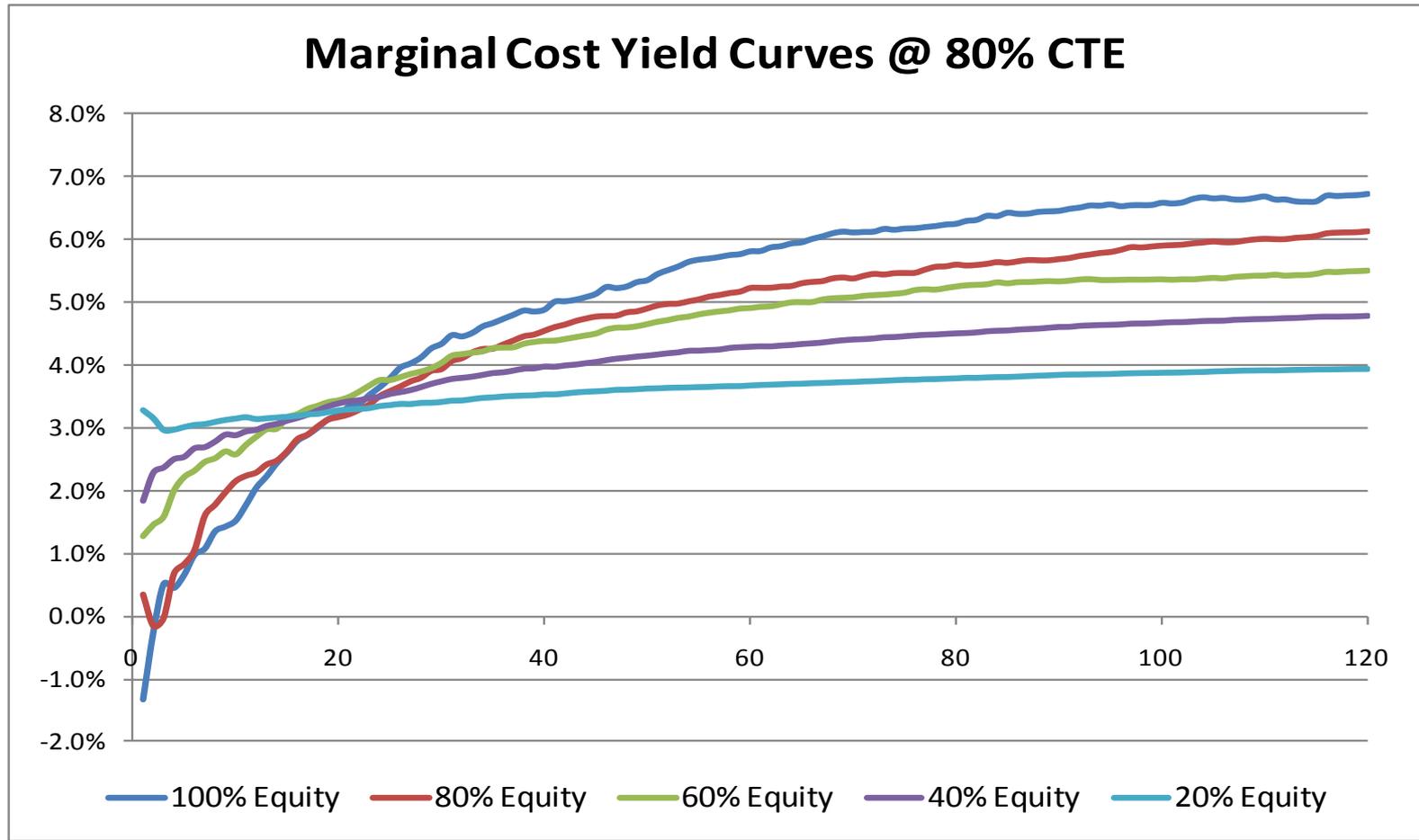
$$\{y_j(x)\}_{j=1}^N$$

so that:

$$A_0^{CTE_x} \equiv -\sum_{j=1}^N CF_j^{Net} \left(1 + y_j(x)\right)^{-j}$$

- The MCYC identifies a reserving basis consistent with the targeted confidence level

# Example: Defined Benefit Pension(cont'd)



# Example: Defined Benefit Pension(cont'd)

## MCYC Math

1. As a function of  $(CF_1, \dots, CF_N)$ ,  $A_0^{CTE_x}$  is homogeneous of degree 1:

$$A_0^{CTE_x}(\lambda CF_1, \dots, \lambda CF_N) = \lambda A_0^{CTE_x}(CF_1, \dots, CF_N)$$

– Need to think a minute because

$$A_0^{CTE_x} = E \left\{ A_0^\alpha \mid A_0^\alpha > A_0^{VaR_x} \right\}$$

and  $A_0^{VaR_x}$  also depends on  $(CF_1, \dots, CF_N)$

# Example: Defined Benefit Pension(cont'd)

## MCYC Math

2. By Euler's theorem for homogeneous functions:

$$A_0^{CTE_x}(CF_1, \dots, CF_N) = - \sum_{j=1}^N CF_j \frac{\partial A_0^{CTE_x}}{\partial CF_j}$$

– So  $\left\{ \frac{\partial A_0^{CTE_x}}{\partial CF_j} \right\}_{j=1}^N$  are pricing factors which discount  $\{CF_j\}_{j=1}^N$  to equal  $A_0^{CTE_x}$

# Example: Defined Benefit Pension(cont'd)

## MCYC Math

### 3. Recall

$$A_0^{CTE_x} = - \sum_{j=1}^N CF_j E \left[ \exp(-j(\mu - 0.5\sigma^2) - \sigma z^\alpha(j)) \mid A_0^\alpha > A_0^{VaR_x} \right]$$

- Partial derivatives not obvious because  $A_0^{VaR_x}$  depends on  $(CF_1, \dots, CF_N)$

# Example: Defined Benefit Pension(cont'd)

## MCYC Math

4. But (Tasche(2000)) derives:

$$\frac{\partial A_0^{CTE_x}}{\partial CF_j} = -E \left[ \exp(-j(\mu - 0.5\sigma^2) - \sigma z^\alpha(j)) \mathbb{1}_{A_0^\alpha > A_0^{VaR_x}} \right]$$

– The MCYC is then given by:

$$(1 + y_j(x))^{-j} = \frac{\partial A_0^{CTE_x}}{\partial CF_j}$$