Optimal Life Insurance, Consumption and Investment

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A Financial Planning Problem

- A young wage earner is just starting a career and a family
- He wants to allocate anticipated income among
  - Consumption
  - Investments
  - Life Insurance Purchase
- His “concerns” are
  - Level of consumption during working years
  - Size of estate at retirement (if he lives that long)
  - Size of estate (bequest) if he dies before retirement

What is the optimal allocation of income?

Literature Review

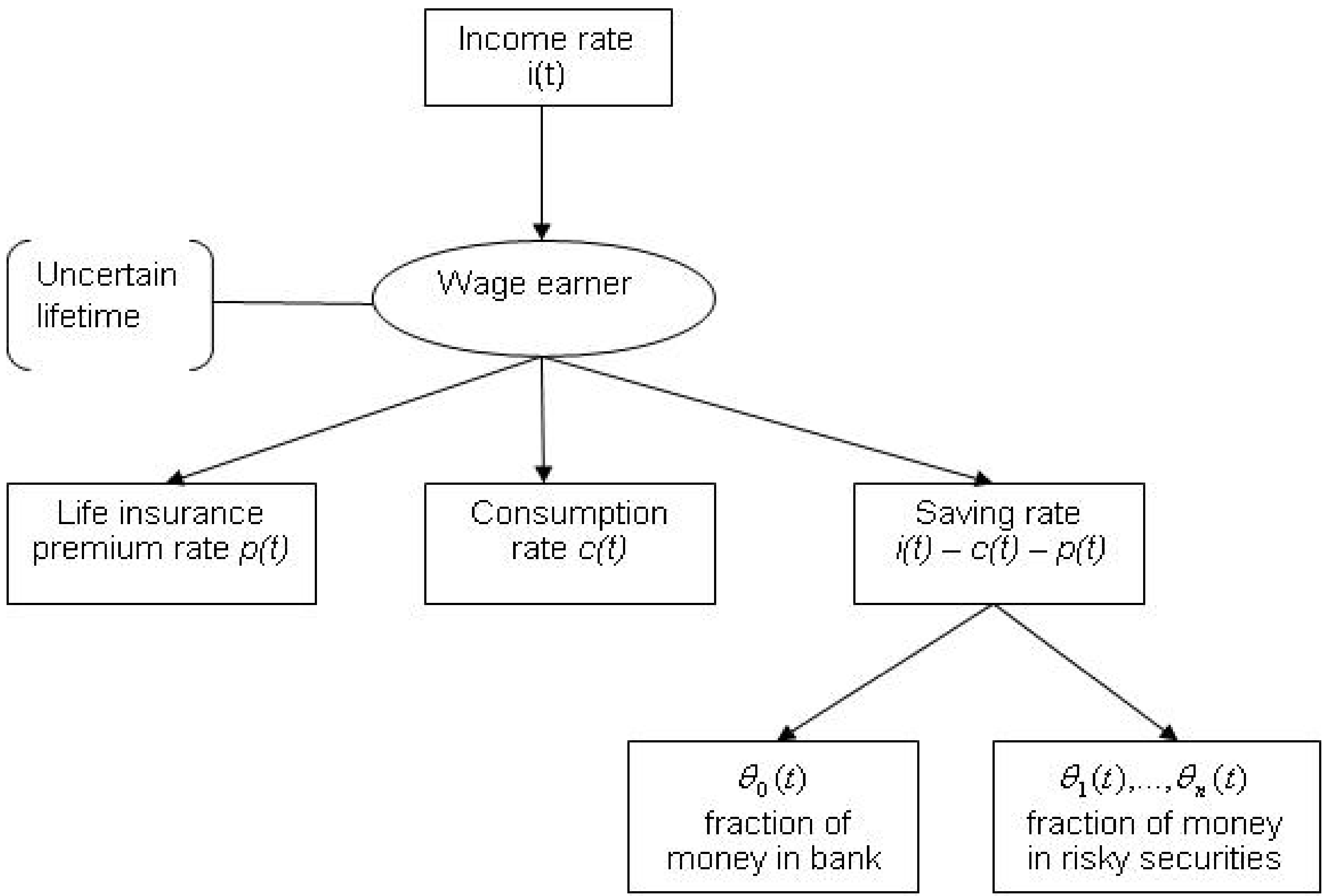
- Yaari (1965) considered a problem of optimal financial planning (including insurance purchase) decisions for an individual with an uncertain lifetime. A key idea was showing how to convert the optimization problem having a random planning horizon to one with a fixed time horizon.

- Merton (1969, 1971) studied the problem of optimal consumption and portfolio investment for a fixed planning horizon, but with life insurance playing no role. He formulated a stochastic model in a continuous-time setting and used dynamic programming to derive the optimal consumption and portfolio strategies for particular cases.

- Richard (1975) extended Merton's work to include a random lifetime and optimal decisions about life insurance purchase in addition to decisions about consumption and investment
  - He obtained explicit solutions for certain cases
  - However, he relied on questionable assumptions about a boundary condition associated with the dynamic program
  - Economic interpretation of the planning horizon is unclear
  - Besides, his solutions do not satisfy one of his assumptions

- Ye (2005) studied the same model, except that, in contrast to Richard's confusing set-up, the planning horizon T is fixed and should be interpreted as the time when the wage earner retires
  - He obtained explicit solutions for certain cases
  - But he assumed only a single risky security, in addition to the riskless bank account, was available for investment

Our Basic Model



Lifetime of the Wage Earner

- The wage earner is alive and starts working at time $t = 0$
- His lifetime is a non-negative random variable τ which is independent of the risky securities
- This random variable has the distribution function F and the density function f, which are known to the wage earner
- Corresponding to τ is the hazard rate function λ, that is

$$\lambda(t) = f(t)/[1 - F(t)]$$

The (Continuous-Time) Insurance Market

- At any point in time t the wage earner will be covered by a life insurance policy if he is paying the premium payment rate $p(t)$ to the insurance company

- Then if he dies at time t, his beneficiaries will collect

$$p(t)/\eta(t),$$

where η is a strictly positive, continuous function called the insurance premium payout ratio

- Both $p(t)$ and $\eta(t)$ are known in advance to the wage earner

- No insurance will be purchased after the start of retirement

The Decision Variables

- $c(t)$ = consumption rate at time t
- $p(t)$ = premium payment rate at time t
- $\theta_n(t)$ = proportion of portfolio value invested in security n, $n = 0, 1, \ldots, N$ (so $\theta_0(t) + \theta_1(t) + \ldots + \theta_N(t) = 1$)
    - Security $n = 0$ is a riskless bank account having interest rate $r(t)$, a specified, deterministic function
    - Securities $n = 1, \ldots, N$ are modeled as multivariate, geometric Brownian motions, the dynamics of which are specified and known to the wage earner

Other Variables of Interest

- $X(t)$ = current wealth = initial wealth + cumulative income - cumulative consumption - cumulative insurance payments + time-t value of investments
- $Z(t) = X(t) + p(t)/\eta(t)$ = wage earner's legacy or bequest upon death if $t = \tau < T$
- $b(t)$ = time-t value of human capital, that is, a discounted value of the wage earner's future income, with the discount rate being the interest rate r combined with the insurance premium payout ratio η

The Wage Earner's Objective

- Maximize the expected utility of:

  1. Cumulative consumption while working
  2. The time-T value of the investments, if $\tau > T$
  3. The time- τ value of the bequest, if $\tau < T$

- $V(t,x)$ denotes the maximum expected utility if the time-t value of the investments is x

Stochastic Dynamic Programming

- With mild assumptions the maximum expected utility *V(t,x)* will be the unique solution of a Hamilton-Jacobi-Bellman equation (details later, if time permits)
- From such a solution, the optimal strategy (c^*, p^*, θ^*) can readily be determined
- Unfortunately, it is almost impossible to obtain explicit solutions, except for some rare special cases
- Hence one usually needs to use numerical methods to compute *V(t,x)*

Discounted CRRA Utilities

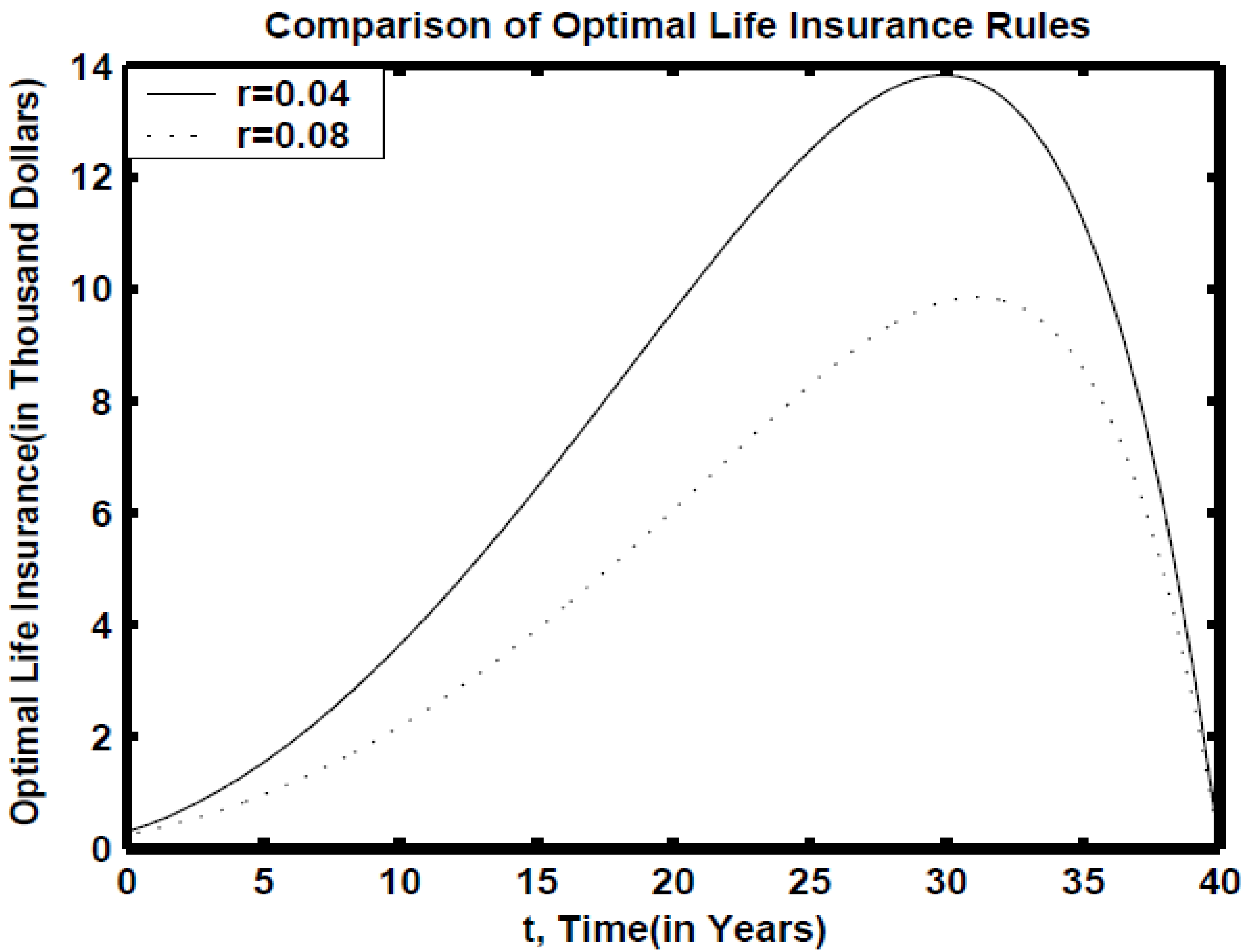
- From now on we restrict ourselves to the special case where the wage earner has the same discounted Constant Relative Risk Aversion (CRRA) utility function for:
  - the consumption of his family
  - the size of his legacy
  - his wealth at retirement

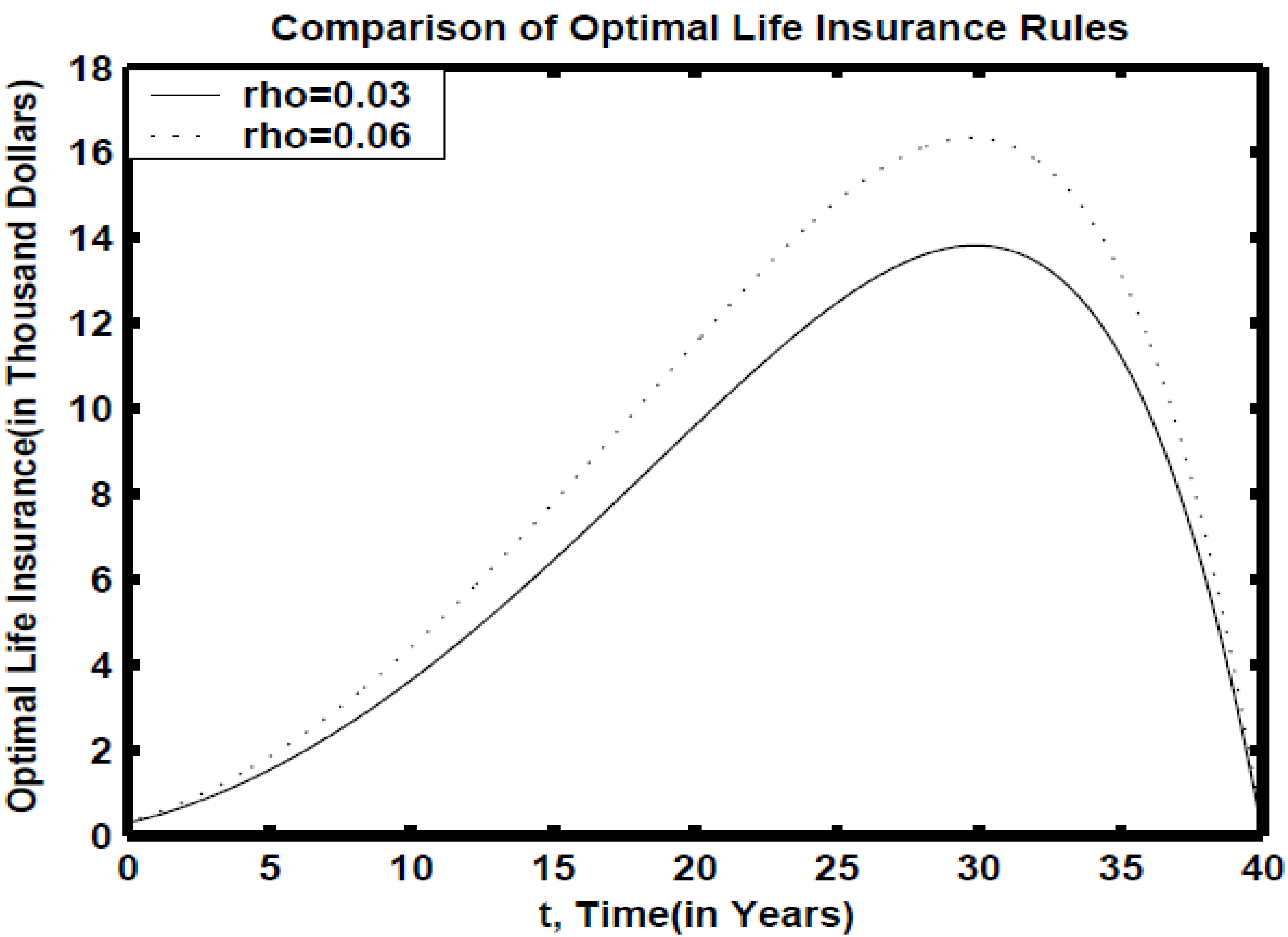
- We assume that $\gamma < 1$, $\gamma \neq 0$ and $\rho > 0$ and let
  - $U(c; t) = e^{-\rho t} c^{\gamma}/\gamma$ (this is for the integrand)
  - $B(Z; t) = e^{-\rho t} Z^{\gamma}/\gamma$
  - $W(X) = e^{-\rho T} X^{\gamma} /\gamma$

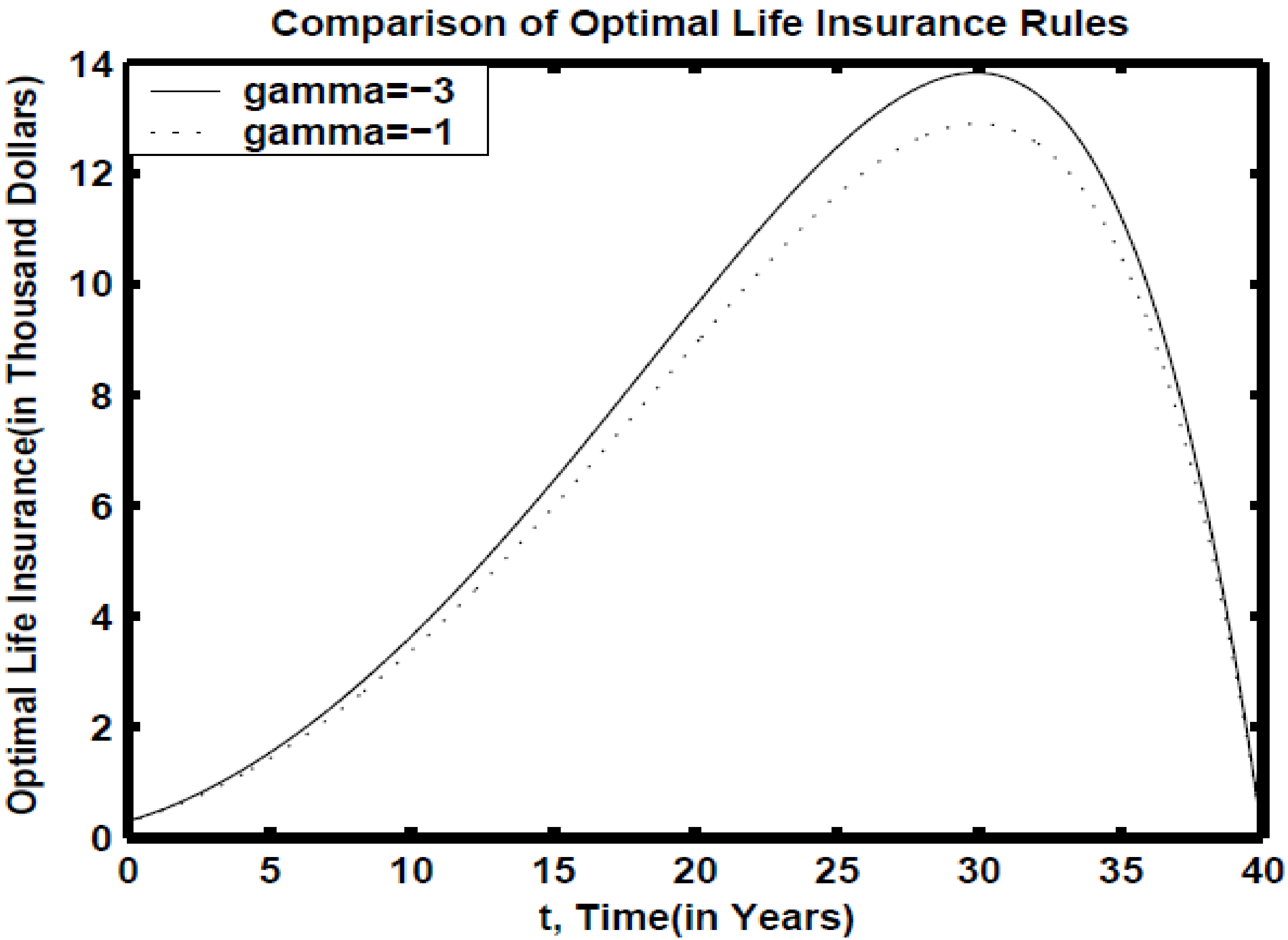
The Optimal Strategies

$$
\begin{aligned}
c^*(t,x) &= \frac{1}{e(t)}(x+b(t)) \\
p^*(t,x) &= \eta(t)\,((D(t)-1)\,x + D(t)b(t)) \\
\theta^*(t,x) &= \frac{1}{x(1-\gamma)}(x+b(t))\xi\alpha(t)\ ,
\end{aligned}
$$

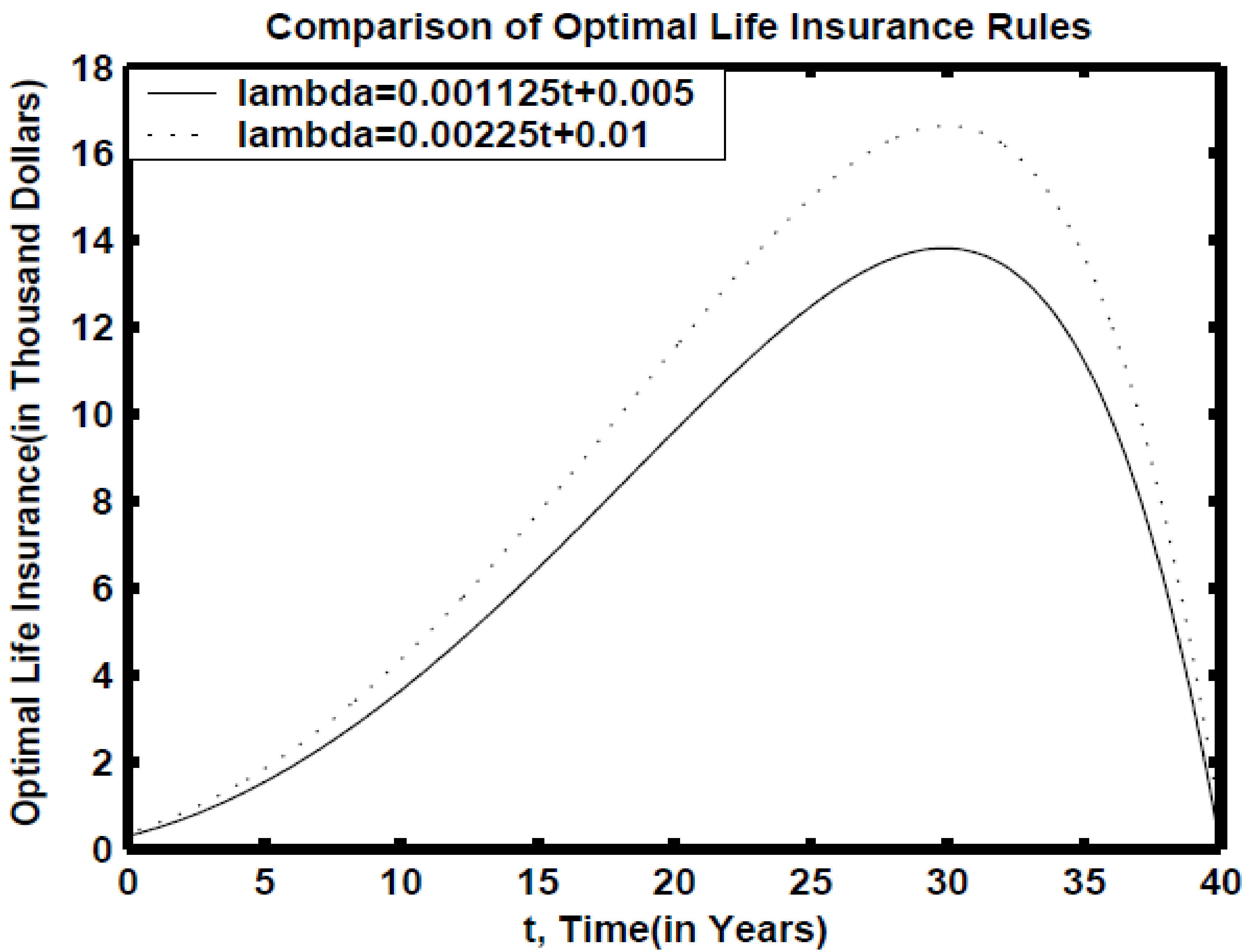
where *b(t)* (the human capital), *D(t)*, and *e(t)* are various deterministic, scalar-valued functions, α(t) is a vector-valued, deterministic function, and ξ is a matrix


Comparison of Optimal Life Insurance Rules
r=0.04
r=0.08
Optimal Life Insurance(in Thousand Dollars)
t, Time(in Years)
0
2
4
6
8
10
12
14
0
5
10
15
20
25
30
35
40


Comparison of Optimal Life Insurance Rules
rho=0.03
rho=0.06
Optimal Life Insurance(in Thousand Dollars)
t, Time(in Years)
0
2
4
6
8
10
12
14
16
18
0
5
10
15
20
25
30
35
40



Comparison of Optimal Life Insurance Rules
gamma=−3
gamma=−1
Optimal Life Insurance(in Thousand Dollars)
t, Time(in Years)
0
2
4
6
8
10
12
14
0
5
10
15
20
25
30
35
40



Comparison of Optimal Life Insurance Rules
lambda=0.001125t+0.005
lambda=0.00225t+0.01
Optimal Life Insurance(in Thousand Dollars)
t, Time(in Years)
0
2
4
6
8
10
12
14
16
18
0
5
10
15
20
25
30
35
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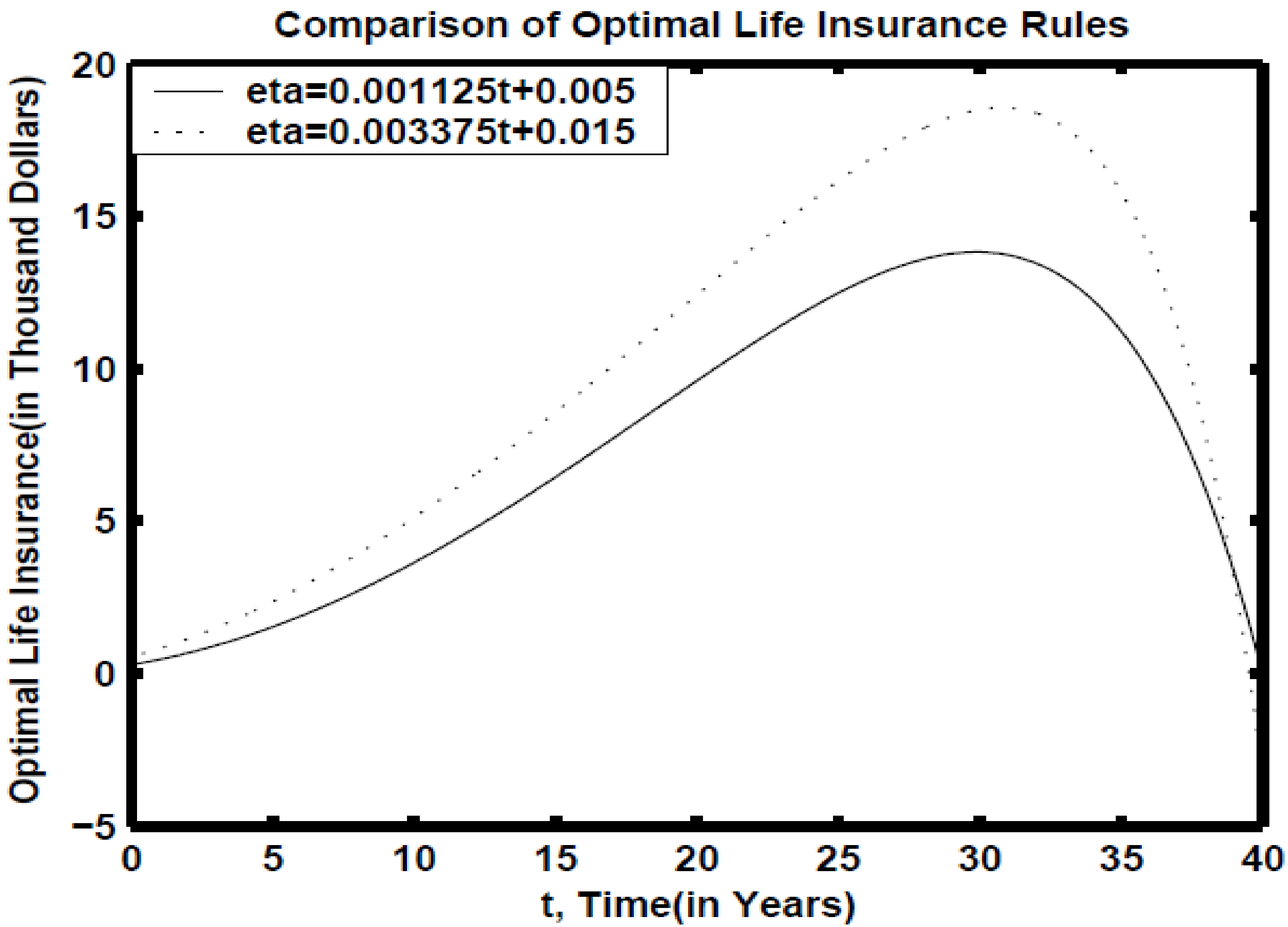
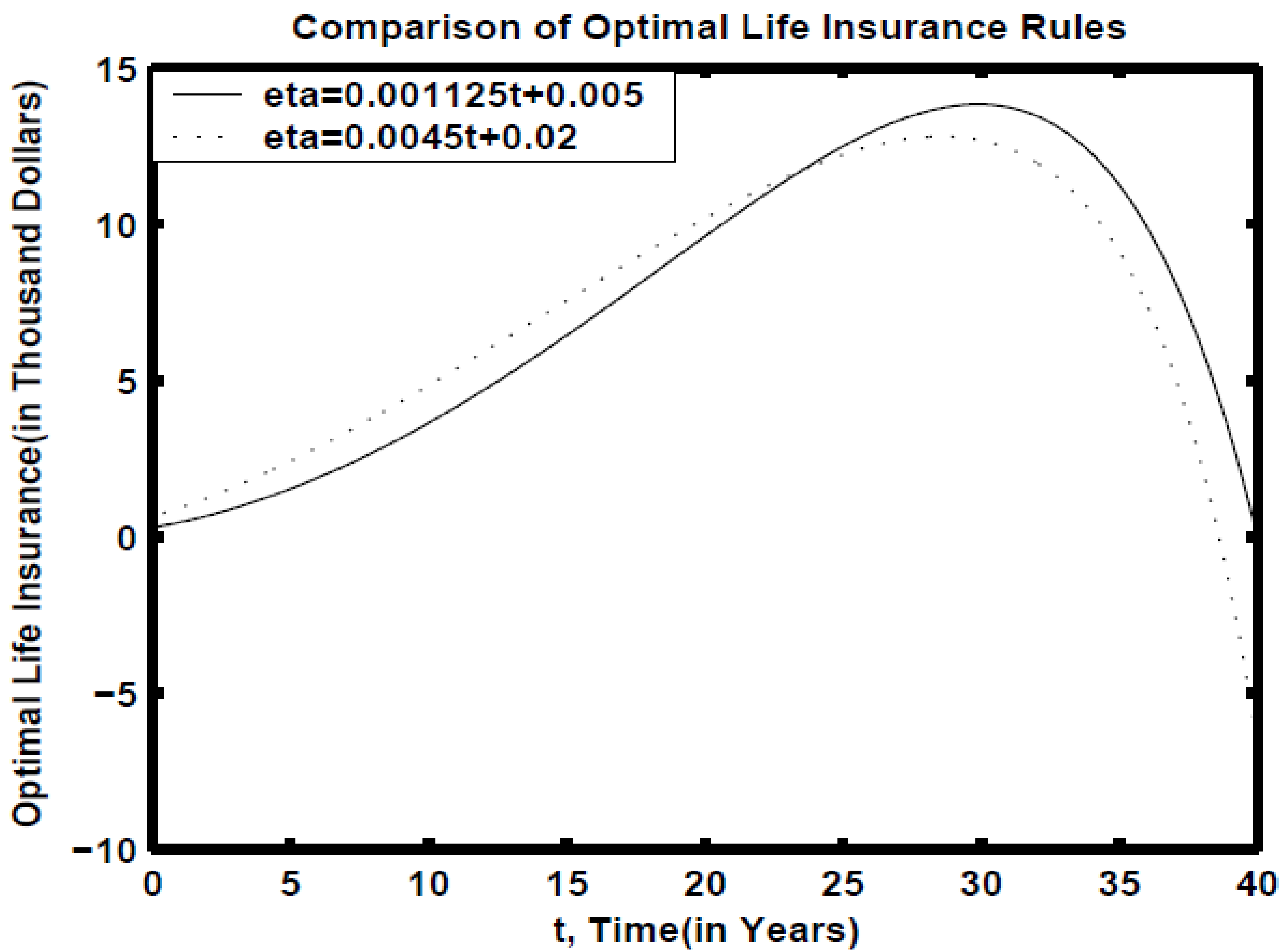
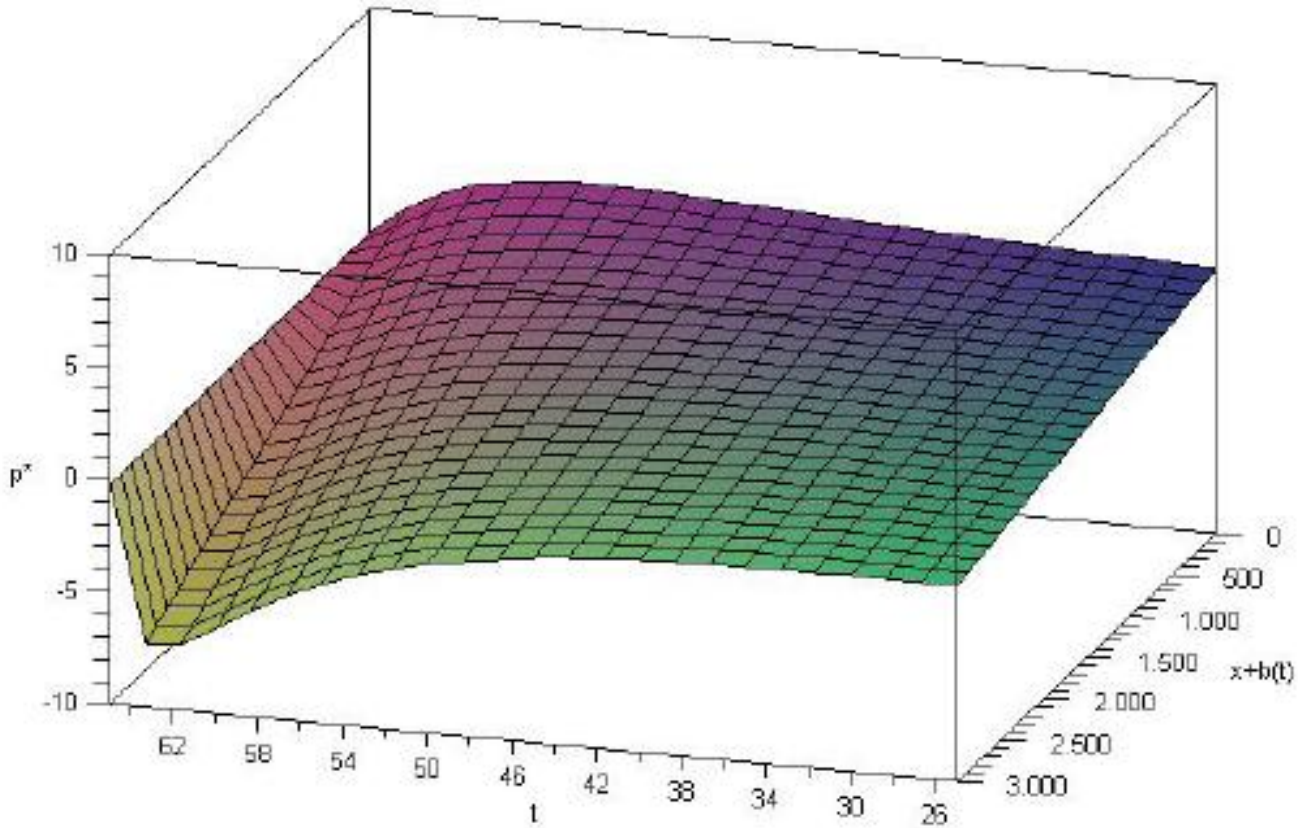

Comparison of Optimal Life Insurance Rules
eta=0.001125t+0.005
eta=0.003375t+0.015
Optimal Life Insurance(in Thousand Dollars)
t, Time(in Years)
20
15
10
5
0
−5
0
5
10
15
20
25
30
35
40



Comparison of Optimal Life Insurance Rules
eta=0.001125t+0.005
eta=0.0045t+0.02
Optimal Life Insurance(in Thousand Dollars)
t, Time(in Years)
15
10
5
0
−5
−10
0
5
10
15
20
25
30
35
40

Optimal Life Insurance Purchase (2 Risky Securities)



Some Conclusions About $p^*(t,x)$ for the CRRA Case (with some modest assumptions)

- The optimal insurance purchase strategy $p^*(t,x)$ is:
  - A decreasing function of the total wealth x
  - An increasing function of the human capital $b(t)$
  - For all small enough values of the wealth x it is a unimodal function of the age t
  - A function that can be negative for some x and t
  - A decreasing function of the interest rate r
  - A decreasing function of the risk aversion parameter γ
  - An increasing function of the utility discount rate ρ
  - An increasing function of the hazard rate λ

Some Conclusions About The Optimal Risky Proportions (With an Additional, Modest Assumption)

- For each risky security *n* the optimal proportion $\theta_n(t)$ is a decreasing function of the total wealth x
- For each risky security *n* the optimal proportion $\theta_n(t)$ is an increasing function of the human capital $b(t)$
- For each risky security *n* the optimal proportion $\theta_n(t)$ is a unimodal function of the wage earner's age t
- For small enough wealth it can be optimal to borrow money from the riskless bank and buy the risky securities on margin

Comparison With an Otherwise Identical Wage Earner Who Does Not Have the Opportunity To Purchase Life Insurance

The optimal portfolio of the wage earner who can buy life insurance is more conservative (that is, smaller values of $\theta_n(t)$ for $n = 1, \ldots, N$) than the optimal portfolio for the otherwise identical wage earner who is unable to purchase life insurance. In other words, the insurance buyer will have a bigger percentage of invested funds in the bank.

A "Mutual Fund" Result

Recall the optimal strategy:

$$\theta^*(t,x) \quad = \quad \frac{1}{x(1-\gamma)}(x+b(t))\xi\alpha(t)$$

Actually, this vector has only N components, corresponding to the N risky securities, so the optimal proportion for the bank is

$$\theta_0 = 1 - \theta_1 - \ldots - \theta_N$$

Here ξ is the covariance matrix and α is the column vector of excess returns describing the dynamics of the risky securities. Since the preceding factor is a scalar, we see that the <u>relative</u> proportions among the risky securities are independent of the insurance and mortality data (intuition: due to no correlation).

Now For Some Mathematics

- The wage earner's goal is to maximize his expected utility, i.e.

$$V(x) = \sup_{\nu \in \mathcal{A}(x)} E_{0,x}\left[\int_0^{T\wedge\tau} U(c(s),s)\,\mathrm{d}s + B(Z(\tau),\tau)I_{\{\tau\le T\}} + W(X(T))I_{\{\tau>T\}}\right],$$

  where

  - $\mathcal{A}(x)$ is the set of all admissible decision strategies;
  - $T \wedge \tau = \min\{T,\tau\}$;
  - $U(c,\cdot)$ is the utility function for consumption;
  - $B(Z,\cdot)$ is the utility function for the legacy;
  - $W(X)$ is the utility function for the terminal wealth.

The Dynamic Programming Principle

- Let $\mathcal{A}(t,x)$ be the set of admissible decision strategies $\nu = (c, p, \theta)$ for the dynamics of the wealth process with boundary condition $X(t) = x$.

- For any $\nu \in \mathcal{A}(t,x)$, define the functional

$$J(t,x;\nu) = E_{t,x}\left[\int_t^{T\wedge\tau} U(c(s),s)\,\mathrm{d}s + B(Z(\tau),\tau)I_{\{\tau\le T\}} + W(X(T))I_{\{\tau>T\}} \,\middle|\, \tau > t, \mathcal{F}_t\right].$$

- The optimal control problem can be restated in dynamic programming form as

$$V(t,x) = \sup_{\nu\in\mathcal{A}(t,x)} J(t,x;\nu).$$

The Hamilton-Jacobi-Bellman Equation

Suppose that the maximum expected utility V is of class C^2. Then V satisfies the Hamilton-Jacobi-Bellman equation

$$\begin{cases} V_t(t,x) - \lambda(t)V(t,x) + \sup\limits_{\nu \in \mathcal{A}(t,x)} \mathcal{H}(t,x;\nu) = 0 \\ V(T,x) = W(x) \end{cases},$$

where the Hamiltonian function $\mathcal{H}$ is given by

$$\mathcal{H}(t,x;\nu) = \left(i(t) - c - p + \left(r(t) + \sum_{n=1}^{N} \theta_n(\mu_n(t) - r(t)) \right) x \right) V_x(t,x)$$

$$+ \frac{x^2}{2} \sum_{m=1}^{M} \left(\sum_{n=1}^{N} \theta_n \sigma_{nm}(t) \right)^2 V_{xx}(t,x) + \lambda(t) B\left(x + \frac{p}{\eta(t)}, t \right) + U(c,t) \, .$$

Moreover, an admissible strategy $\nu^* = (c^*, p^*, \theta^*)$ whose corresponding wealth is X^* is optimal if and only if for a.e. $s \in [t,T]$ and P-a.s. we have

$$V_t(s, X^*(s)) - \lambda(s)V(s, X^*(s)) + \mathcal{H}(s, X^*(s); \nu^*) = 0 \, .$$

Optimal Strategies For the CRRA Case

Let ξ denote the non-singular square matrix given by $(\sigma\sigma^T)^{-1}$. The optimal strategies in the case of discounted constant relative risk aversion utility functions are given by

$$\begin{aligned} c^*(t,x) &= \frac{1}{e(t)}(x+b(t)) \\ p^*(t,x) &= \eta(t)\left((D(t)-1)\,x + D(t)b(t)\right) \\ \theta^*(t,x) &= \frac{1}{x(1-\gamma)}(x+b(t))\xi\alpha(t)\ , \end{aligned}$$

where

$$\begin{aligned} b(t) &= \int_t^T i(s)\exp\left(-\int_t^s r(v)+\eta(v)\,\mathrm{d}v\right)\mathrm{d}s \\ e(t) &= \exp\left(-\int_t^T H(v)\,\mathrm{d}v\right) + \int_t^T \exp\left(-\int_t^s H(v)\,\mathrm{d}v\right)K(s)\,\mathrm{d}s \\ H(t) &= \frac{\lambda(t)+\rho}{1-\gamma} - \gamma\frac{\Sigma(t)}{(1-\gamma)^2} - \frac{\gamma}{1-\gamma}(r(t)+\eta(t)) \\ D(t) &= \frac{1}{e(t)}\left(\frac{\lambda(t)}{\eta(t)}\right)^{1/(1-\gamma)},\ K(t) = \frac{(\lambda(t))^{1/(1-\gamma)}}{(\eta(t))^{\gamma/(1-\gamma)}}+1 \\ \Sigma(t) &= \alpha^T(t)\xi\alpha(t) - \frac{1}{2}\|\sigma^T\xi\alpha(t)\|^2\ . \end{aligned}$$

Final Remarks

- Our model provides a variety of conclusions about optimal decisions for financial planning
  - Some conclusions simply confirm what we always thought
  - Other conclusions provide new insight and intuition

- There are considerable opportunities to develop various extensions, such as:
  - Other utility functions
  - Uncertain future income
  - A constraint ruling out selling one's own insurance policy

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