Pricing Game Options with Call Protection: Doubly Reflected Intermittent BSDEs and their Approximation

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Convertible bond with underlying stock S

- Coupons from time 0 onwards
- Terminal payoff at $\zeta = \tau \wedge \theta$

$$\mathbf{1}_{\zeta=\tau<T}\ell(\tau, S_\tau) + \mathbf{1}_{\vartheta<\tau}h(\vartheta, S_\vartheta) + \mathbf{1}_{\zeta=T}g(S_T)$$

  - $[0, T]$-valued bond holder put time τ and bond issuer call time θ
  - Cancelable American claim, or game option
- Call protections preventing the issuer from calling the bond on certain random time intervals
  - Typically monitored at discrete monitoring times
  - In a possibly very path-dependent way

Agenda

Mathematical issues

- Doubly reflected backward stochastic differential equations with an intermittent upper barrier, only active on random time intervals (RIBSDE)
- Related variational inequality approach (VI)
    - Highly-dimensional pricing problems (path dependence)
    - Deterministic pricing schemes ruled out by the curse of dimensionality
- → Simulation methods

Contributions

- A convergence rate for a discrete time approximation scheme by simulation to an RIBSDE
- VI approach
- Practical value of this approach on the benchmark problem of pricing by simulation highly path-dependent convertible bonds
- A demonstration of the real abilities of simulation/regression numerical schemes in high dimension (up to $d = 30$ in this work)

Markovian RIBSDE
Approximation Results
Application:Pricing convertible Bonds with call protection

Diffusion Set-Up with Marker Process
Markovian RIBSDE
Connection with Finance
Solution of the RIBSDE

Outline

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Diffusion Set-Up with Marker Process

- Diffusion with Lipschitz coefficients in $\mathbb{R}^q$

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t$$

- Call protection monitoring times $\mathfrak{T} = \{0 = T_0 < \ldots < T_N = T\}$
- Marker process H keeping track of the path-dependence, in view of 'markovianizing' the model
- $\mathbb{R}^q \times \mathcal{K}$-valued factor process $\mathcal{X} = (X, H)$ (finite set $\mathcal{K}$)
  - $u = u(t, x, k) = u^k(t, x)$
- $\mathcal{K}$-valued pure jump marker process H supposed to be constant except for deterministic jumps at the T_Is

$$H_{T_I} = \kappa_I(X_{T_I}, H_{T_I-})$$

  - Jump functions κ_I^k continuous in x outside $\partial\mathcal{O}$ (constant on $\mathcal{O}$ and on ${}^c\mathcal{O}$) for an open, 'regular' domain $\mathcal{O} \subseteq \mathbb{R}^q$

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Call Protection

- Subset K of $\mathcal{K}$
- Call forbidden/possible whenever $H_t \in K$ / $\notin K$
- $\mathfrak{T}$-valued stopping times given as successive times of exit from and entrance to K, so $\vartheta_0 = 0$ and then

$$\vartheta_{2l+1} = \inf\{t > \vartheta_{2l}\,;\, H_t \notin K\} \wedge T\,,\ \vartheta_{2l+2} = \inf\{t > \vartheta_{2l+1}\,;\, H_t \in K\} \wedge T$$

- Call forbidden/possible on the 'even'/'odd' intervals $[\vartheta_l, \vartheta_{l+1})$
  - $H_t \in K$ / $\notin K$

Starting from $H_0 = k \notin K$ ('Call at the beginning')

$0 = \vartheta_0 = \vartheta_1 < \vartheta_2 \leq \ldots \leq \vartheta_{N+1} = T$
Call possible on the first non-void time interval $[\vartheta_1 = 0 = \vartheta_0, \vartheta_2 > 0)$

Starting from $H_0 = k \in K$ ('No Call at the beginning')

$0 = \vartheta_0 < \vartheta_1 \leq \ldots \leq \vartheta_{N+1} = T$
Call forbidden on the first non-void time interval $[\vartheta_0 = 0, \vartheta_1 > 0)$

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Markovian RIBSDE

Reflected BSDE $(\mathcal{S})$ with data

$$f(t, X_t, y, z),\ \xi = g(X_T),\ \ell(t, X_t),\ h(t, X_t),\ \vartheta$$

- 'Standard Lipschitz and L^2-integrability assumptions' (if not for ϑ)
- Mokobodski condition
  - Existence of a square-integrable quasimartingale Q between L and U
- Doubly reflected BSDE with lower barrier $L_t = \ell(t, X_t)$ and intermittent (the 'I' in RIBSDE) upper barrier given by, for $t \in [0, T]$

$$U_t = \sum_{l=0}^{[N/2]} \mathbf{1}_{[\vartheta_{2l}, \vartheta_{2l+1})} \infty + \sum_{l=1}^{[(N+1)/2]} \mathbf{1}_{[\vartheta_{2l-1}, \vartheta_{2l})} h(t, X_t)$$

  - 'Nominal' upper obstacle $h(t, X_t)$ only active on the 'odd' random time intervals $[\vartheta_{2l-1}, \vartheta_{2l})$
  - Call protection on the 'even' random time intervals $[\vartheta_{2l}, \vartheta_{2l+1})$

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Risk-neutral pricing problems in finance

Driver coefficient function f typically given as

$$f = f(t, x, y) = c(t, x) - \mu(t, x)y$$

- Dividend and interest-rate related functions c and μ
    - Single-name credit risk (counterparty risk)
        - Recovery-adjusted dividend-yields c
        - Credit-spread adjusted interest-rates μ
        - Pre-default factor process X
- Affine in y, does not depend on z
    - Historical rather than RN modeling $\rightarrow$ 'z-dependent' f
    - Market imperfections $\rightarrow$ nonlinear f

Terminal cost functions typically given by

$$\ell(t, x) = \bar{P} \vee S, \ h(t, x) = \bar{C} \vee S, \ g(x) = \bar{N} \vee S$$

$\bar{P} \leq \bar{N} \leq \bar{C}$ Constants

$\quad S = x_1$ first component of x

- Mokobodski condition satisfied with $Q = S$ provided S is a square-integrable Itï¿½$\frac{1}{2}$ process

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Highly path dependent call protection

Example ('l out of d')

Given a constant trigger level $\bar{S}$ and constants $l \leq d \leq N$, call possible iff S has been $\geq \bar{S}$ on at least l of the last d monitoring times

- $\mathcal{K} = \{0,1\}^d$, $\kappa_l^k(x) = (\mathbf{1}_{S \geq \bar{S}}, k_1, \ldots, k_{d-1})$

- H_t vector of the indicator functions of the events $S_{T_l} \geq \bar{S}$ at the last d monitoring dates preceding time t

Call possible iff $|H_t| \geq l \Leftrightarrow H_t \notin K$ with $|k| = \sum_{1 \leq p \leq d} k_p$ and $K = \{k \in \mathcal{K} \,;\, |k| < l\}$

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Solution of the RIBSDE

Definition

A solution $\mathcal{Y}$ to $(\mathcal{S})$ is a triple $\mathcal{Y} = (Y, Z, A)$ such that:

(i) $Y \in \mathcal{S}^2, Z \in \mathcal{H}_q^2, A \in \mathcal{A}^2$

(ii) $Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)ds + A_T - A_t - \int_t^T Z_s dW_s \ t \in [0, T]$

(iii) $L_t \leq Y_t$ on $[0, T]$, $Y_t \leq U_t$ on $[0, T]$

and $\int_0^T (Y_t - L_t)dA_t^+ = \int_0^T (U_{t-} - Y_{t-})dA_t^- = 0$

(iv) A^+ is continuous, and

$\{(\omega, t) ; \Delta Y \neq 0\} = \{(\omega, t) ; \Delta A^- \neq 0\} \subseteq \bigcup_{l=0}^{[N/2]} \llbracket \vartheta_{2l} \rrbracket$

$\Delta Y = \Delta A^-$ on $\bigcup_{l=0}^{[N/2]} \llbracket \vartheta_{2l} \rrbracket$

- $\mathcal{S}^2, \mathcal{H}_q^2$ and $\mathcal{A}^2$ 'usual L^2 spaces'
- $A^{\pm}$ Jordan component of A
- Convention that $0 \times \pm\infty = 0$ in (iii)
- Obvious extension to a random terminal time θ

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- For l decreasing from N to 0, let us define $\mathcal{Y}^l = (Y^l, Z^l, A^l)$ on $[\vartheta_l, \vartheta_{l+1}]$ as the solution, with A^l continuous, to the stopped RBSDE (for l even) or R2BSDE (for l odd) with data (with $Y^{N+1}_{\vartheta_{N+1}} \equiv g(X_T)$)

$$
\begin{cases}
f(t, X_t, y, z), \ Y^{l+1}_{\vartheta_{l+1}}, \ \ell(t, X_t) & (l \text{ even}) \\
f(t, X_t, y, z), \ \min(Y^{l+1}_{\vartheta_{l+1}}, h(\vartheta_{l+1}, X_{\vartheta_{l+1}})), \ \ell(t, X_{st}), \ h(t, X_t) & (l \text{ odd})
\end{cases}
$$

- Let us define $\mathcal{Y} = (Y, Z, A)$ on $[0, T]$ by, for every $l = 0, \ldots, N$:
  - $(Y, Z) = (Y^l, Z^l)$ on $[\vartheta_l, \vartheta_{l+1})$, and also at $\vartheta_{N+1} = T$ in case $l = N$. So in particular

$$
Y_0 = \begin{cases} Y^0_0, & k \in K \\ Y^1_0, & k \notin K \end{cases}
$$

  where k is the initial condition of the marker process H.
  - $dA = dA^l$ on $(\vartheta_l, \vartheta_{l+1})$,

$$
\Delta A_{\vartheta_l} = \Delta A^-_{\vartheta_l} = \left(Y^l_{\vartheta_l} - h(\vartheta_l, X_{\vartheta_l}) \right)^+ = \Delta Y_{\vartheta_l} (= 0 \text{ for l odd })
$$

  and $\Delta A_T = \Delta Y_T = 0$.

Proposition

$\mathcal{Y} = (Y, Z, A)$ is the unique solution to $(\mathcal{S})$

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Verification principle

Risk-neutral pricing problems in finance

Financial interpretation of a solution $\mathcal{Y}$ to $(\mathcal{S})$

Y_0 'NFLVR' Arbitrage price at time 0 for the game option
with payoff functions c, l, h, g and call protection ϑ
Bilateral super-hedging price and infimal issuer
super-hedging price

up to a local martingale cost process

Z Hedging strategy

Outline

Approximation of the Forward Process

- Time-grid $\mathfrak{t} = \{0 = t_0 < t_1 < \ldots < t_n = T\} \supseteq \mathfrak{T}$
- Euler scheme approximation of $\widehat{X}$

$$\widehat{X}_{t_{i+1}} = \widehat{X}_{t_i} + b(t_i, \widehat{X}_{t_i})(t_{i+1} - t_i) + \sigma(t_i, \widehat{X}_{t_i})(W_{t_{i+1}} - W_{t_i})$$

- Approximation of the marker process H

$$\widehat{H}_{T_I} = \kappa_I(\widehat{X}_{T_I}, \widehat{H}_{T_I-})$$

Approximation of the Call Protection Switching Times

Approximation $\widehat{\vartheta}$ of ϑ obtained by using $\widehat{\mathcal{X}} = (\widehat{X}, \widehat{H})$ instead of $\mathcal{X}$ in the definition of ϑ

Proposition (Assuming σ non-degenerate and 'some regularity of σ and b around $\partial\mathcal{O}$)

For every $l \leq N + 1$

$$\mathbb{E}\Big[|\vartheta_l - \widehat{\vartheta}_l|\Big] \leq C_\varepsilon |\mathsf{t}|^{\frac{1}{2} - \varepsilon}$$

$|\mathsf{t}| = \max_{i \leq n-1}(t_{i+1} - t_i)$

Approximation of the RIBSDE

- Projection operator $\widehat{\mathcal{P}}$ defined by

$$\widehat{\mathcal{P}}(t,x,y) = y + [\ell(t,x) - y]^+ - [y - h(t,x)]^+ \sum_{l=1}^{[(N+1)/2]} \mathbf{1}_{\{\widehat{\vartheta}_{2l-1} \leq t \leq \widehat{\vartheta}_{2l}\}}$$

- Reflection operating only on a subset $\mathfrak{r}$ of $\mathfrak{t}$ in the approximation scheme for $\mathcal{Y}$

$$\mathfrak{r} = \{0 = r_0 < r_1 < \cdots < r_\nu = T\} \text{ with } \mathfrak{T} \subseteq \mathfrak{r} \subseteq \mathfrak{t}$$

Components Y and Z of a solution $\mathcal{Y} = (Y, Z, A)$ to $(\mathcal{S})$ approximated by a triplet of processes $(\widehat{Y}, \widetilde{Y}, \bar{Z})$ defined on $\mathfrak{t}$

Terminal condition

$$\widehat{Y}_T = \widetilde{Y}_T = g(\widehat{X}_T)$$

and then for i decreasing from $n-1$ to 0

$$\begin{cases} \bar{Z}_{t_i} &= \mathbb{E}\left[\widehat{Y}_{t_{i+1}}\left(\frac{W_{t_{i+1}} - W_{t_i}}{t_{i+1} - t_i}\right) \mid \mathcal{F}_{t_i}\right] \\ \widetilde{Y}_{t_i} &= \mathbb{E}\left[\widehat{Y}_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + (t_{i+1} - t_i)f(t_i, \widehat{X}_{t_i}, \widetilde{Y}_{t_i}, \bar{Z}_{t_i}) \\ \widehat{Y}_{t_i} &= \widetilde{Y}_{t_i}\mathbf{1}_{\{t_i \notin \mathfrak{r}\}} + \widehat{\mathcal{P}}(t_i, \widehat{X}_{t_i}, \widetilde{Y}_{t_i})\mathbf{1}_{\{t_i \in \mathfrak{r}\}} \end{cases}$$

Continuous-time extension of the scheme still denoted by $(\widehat{Y}, \widetilde{Y}, \bar{Z})$

$\widehat{Z}$ Integrand in a stochastic integral representation of $\widehat{Y}$

Theorem (No call or no call protection, Chassagneux 08)

In case of Lipschitz barriers and for $|\mathfrak{r}| \sim |\mathfrak{t}|^{\frac{2}{3}}$ (resp. semi-convex barriers and for $|\mathfrak{r}| \sim |\mathfrak{t}|^{\frac{1}{2}}$), one has

$$
\max_{i \leq n-1} \sup_{t \in [t_i, t_{i+1})} \mathbb{E}\Big[|Y_t - \widetilde{Y}_{t_i}|^2\Big] + \max_{i \leq n-1} \sup_{t \in [t_i, t_{i+1})} \mathbb{E}\Big[|Y_{t-} - \widehat{Y}_{t_i}|^2\Big] \leq C|\mathfrak{t}|^{\alpha}
$$

with $\alpha = \frac{1}{3}$ (resp. $\frac{1}{2}$).

Theorem (Call protection, this work, assuming f does not depend on z)

In case of Lipschitz barriers and for $|\mathfrak{r}| \sim |\mathfrak{t}|^{\frac{1}{2}}$ (resp. semi-convex barriers and for $|\mathfrak{r}| \sim |\mathfrak{t}|$), one has

$$
\max_{i \leq n-1} \sup_{t \in [t_i, t_{i+1})} \mathbb{E}\Big[|Y_t - \widetilde{Y}_{t_i}|^2\Big] + \max_{i \leq n-1} \sup_{t \in [t_i, t_{i+1})} \mathbb{E}\Big[|Y_{t-} - \widehat{Y}_{t_i}|^2\Big] \leq C_{\varepsilon}|\mathfrak{t}|^{\alpha-\varepsilon}
$$

with $\alpha = \frac{1}{4}$ (resp. $\frac{1}{2}$).

- Proof of the theorem based on a suitable concept of time-continuous discretely reflected BSDEs
    - Bermudan options
- Possible extension to the case where f depends on z
- Representations of $\widetilde{Y}$ and $\widehat{Z}$ using approximated optimal policies
    - Cf. 'MC Backward versus Forward' in the numerical part

Variational Inequality Approach

- Comparing the simulation results them with those of an alternative, deterministic numerical scheme
- Deterministic scheme for $(\mathcal{S})$ based on an analytic characterization of $(\mathcal{S})$
- Let $\mathcal{E} = [0, T] \times \mathbb{R}^q \times \mathcal{K}$ and for $I = 1, \ldots, N$

$$\mathcal{E}_I = [T_{I-1}, T_I] \times \mathbb{R}^q \times \mathcal{K}, \ \mathcal{E}_I^* = [T_{I-1}, T_I) \times \mathbb{R}^q \times \mathcal{K}$$

  - The $\mathcal{E}_I^*$s and $\{T\} \times \mathbb{R}^q \times \mathcal{K}$ partition $\mathcal{E}$

Continuity of ϑ with respect to(t, x, i)

- Continuous outside $\mathfrak{T} \times \mathbb{R}^q \times \mathcal{K}$
- Cadlag on $(\mathfrak{T} \times \mathbb{R}^q \times \mathcal{K}) \setminus (\mathfrak{T} \times \partial\mathcal{O} \times \mathcal{K})$
- Cad but not'lag' on $\mathfrak{T} \times \partial\mathcal{O} \times \mathcal{K}$

Cauchy cascade

Definition

(i) Cauchy cascade (g, ν) on $\mathcal{E}$

- Terminal condition g at T
- Sequence $\nu = (u_I)_{1 \le I \le N}$ of functions u_Is on the $\mathcal{E}_I$s
- Jump condition for $x \notin \partial\mathcal{O}$ (with $u_{N+1} \equiv g$):

$$u_I^k(T_I, x) = \begin{cases} \min(u_{I+1}(T_I, x, \kappa_I^k(x)), h(T_I, x)) & \text{if } k \notin K \text{ and } \kappa_I^k(x) \in K \\ u_{I+1}(T_I, x, \kappa_I^k(x)) & \text{else} \end{cases}$$

(ii) Continuous Cauchy cascade

- Cauchy cascade with continuous ingredients g at T and u_Is on the $\mathcal{E}_I$s, except maybe for discontinuities of the u_I^ks on $\mathfrak{T} \times \partial\mathcal{O}$

(iii) Function on $\mathcal{E}$ defined by a Cauchy cascade

- Concatenation on the $\mathcal{E}_I^*$s of the u_Is + terminal condition g at T

Cascade Characterization of $\mathcal{Y}$

Proposition

$Y_t = u(t, \mathcal{X}_t)$, $t \in [0, T]$, for a deterministic *pricing function* u, defined by a continuous Cauchy cascade $(g, \nu = (u_I)_{1 \leq I \leq N})$ on $\mathcal{E}$

Analytic characterization of u?

Generator of X

$\mathcal{G}\phi(t, x) = \partial_t \phi(t, x) + \partial \phi(t, x) b(t, x) + \frac{1}{2} \text{Tr}[a(t, x)\mathcal{H}\phi(t, x)]$

$a(t, x)$ $\sigma(t, x)\sigma(t, x)^{\mathsf{T}}$

$\partial\phi$, $\mathcal{H}\phi$ Row-gradient and Hessian of ϕ with respect to x

Cauchy cascade ($\mathcal{VI}$)

For l decreasing from N to 1,

- At $t = T_l$ for every $k \in \mathcal{K}$ and $x \in \mathbb{R}^q$

$$u_l^k(T_l, x) = \begin{cases} \min(u_{l+1}(T_l, x, \kappa_l^k(x)), h(T_l, x)), & k \notin K \text{ and } \kappa_l^k(x) \in K \\ u_{l+1}(T_l, x, \kappa_l^k(x)), & \text{else} \end{cases}$$

with $u_{N+1} \equiv g$

- On the time interval $[T_{l-1}, T_l)$ for every $k \in \mathcal{K}$,

$$\begin{cases} \min\left(-\mathcal{G}u_l^k - f^{u_l^k}, u_l^k - \ell\right) = 0, \ k \in K \\ \max\left(\min\left(-\mathcal{G}u_l^k - f^{u_l^k}, u_l^k - \ell\right), u_l^k - h\right) = 0, \ k \notin K \end{cases}$$

with for any function $\phi = \phi(t, x)$

$$f^\phi = f^\phi(t, x) = f(t, x, \phi(t, x))$$

- Technical difficulty due to the potential discontinuity in x of the functions u_l^ks on $\partial\mathcal{O}$
  - Characterizing ν in terms of a suitable notion of discontinuous viscosity solution of $(\mathcal{VI})$?
    - Convergence results? for deterministic approximation schemes to u
- Curse of dimensionality
  - $(\mathcal{VI}) = \mathrm{Card}(\mathcal{K})$ equations in the u^ks
  - $\sim (q + d)$ – dimensional pricing problem with $d = \log(\mathrm{Card}(\mathcal{K}))$
  - Simulation schemes the only viable alternative for d greater than few units

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Regressions

regression approaches for computing by regression functions (conditional expectations)

$$x \mapsto \rho(x) = \mathbb{E}(\xi|X = x)$$

ξ, X Real- and $\mathbb{R}^q$-valued square integrable random variables

Pairs $(X^j, \xi^j)^{1 \leq j \leq m}$ simulated independently according to the law of $(X, \xi) \rightarrow$ Estimate the conditional expectation $\mathbb{E}(\xi|X)$

Parametric regression of the ξ^js against the $(\varphi^l(X^j))_{1 \leq l \leq p}^{1 \leq j \leq m}$, where (φ^l) is a well chosen 'basis' of functions from $\mathbb{R}^q$ to $\mathbb{R}$

Regression basis

- parametric vs non-parametric

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Parametric versus Non parametric estimation

parametric regression

$m(x^1, ..., x^d) = a_0 + \sum_{i=1}^{d} a_i x^{(i)}$ and the coefficients $a_0...., a_d$ are estimated by LSM

Nonparametric regression

$Y_i = m(X_i) + \varepsilon$
Estimation of $m(x)$ by the average of those Y_i where X_i is close to x
$m_n(x) = \sum_{i=1}^{n} W_{n,i}(x).Y_i$
$m_n(x) = \frac{\sum_{i=1}^{n} I_{x_i \in A_{n,j}} Y_i}{\sum_{i=1}^{n} I_{x_i \in A_{n,j}}}$

Conclusion

For multivariate X,it is not clear how to choose a proper form of a parametric estimate,and a wrong form will lead to a bad estimate .This inflexibility concerning the structure of the regression function is avoided by so called non parametric regression estimate

Convertible bond with underlying stock S

- Coupons from time 0 onwards
- Terminal payoff at $\zeta = \tau \wedge \vartheta$

$$\mathbf{1}_{\zeta=\tau<T}\ell(\tau, S_\tau) + \mathbf{1}_{\vartheta<\tau}h(\vartheta, S_\vartheta) + \mathbf{1}_{\zeta=T}g(S_T)$$

  - $\ell(t, S_t) = \overline{P} \vee S_t; h(t, S_t) = \overline{C} \vee S_t; g(S_T) = \overline{N} \vee S_T$
  - $[0, T]$-valued bond holder put time τ and bond issuer call time ϑ
  - Cancelable American claim, or game option
- Call protections preventing the issuer from calling the bond on certain random time intervals
  - Typically monitored at discrete monitoring times
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Benchmark Model

Local drift and volatility pre-default model for a stock $X = S$

$$\frac{dS_t}{S_t} = b(t, S_t)dt + \sigma(t, S_t)dW_t$$

$$b(t, S) = r(t) - q(t) + \eta\gamma(t, S), \ \gamma(t, S) = \gamma_0(S_0/S)^\alpha, \ \sigma(t, S) = \sigma$$

$r(t)$ Riskless short interest rate

$q(t)$ Dividend yield

$\gamma(t, S)$ Local default intensity ($\gamma_0, \ \alpha \geq 0$)

$0 \leq \eta \leq 100\%$ Loss Given Default of the firm issuing the bond

Coupon rate function

$$c(t, S) = \bar{c}(t) + \gamma(t, S)\left((1 - \eta)S \vee \bar{R}\right)$$

$\bar{c}$ Nominal coupon rate function

$\bar{R}$ Nominal recovery on the bond upon default

Discounting

$\mu(t, S) = r(t) + \gamma(t, S)$ Credit-risk adjusted interest rate

$\beta_t = e^{-\int_0^t \mu(s, S_s)ds}$ Risk-neutral credit-risk adjusted discount factor

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General Conditions for the Numerical Experiments

General Data

$\overline{P}$	$\overline{N}$	$\overline{C}$	η	σ	r	q	γ_0	α	m
0	100	103	1	0.2	0.05	0	0.02	1.2	10^4

m number of Monte Carlo trajectories

Time-step $t_{i+1} - t_i = h$

six hours (four time steps per day) in the case of simulation methods

one day in the case of deterministic schemes

Space-steps in the S variable

$S^{j+1} - S^j = 0.5$ in the case of the (fully implicit) deterministic schemes

Cells of diameter one (segments of length one) in the case of
simulation/regression methods involving a method of cells
in S

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Monte Carlo different methods(American Option)

Monte Carlo Backward Method

$$\left\{ \begin{array}{l} Q_N := \varphi\left(S_{t_N}\right) \\ Q_{j-1} := \max\left(\varphi\left(S_{t_{j-1}}\right), \mathbb{E}\left(B\left(t_{j-1}, t_j\right) Q_j | \mathcal{F}_{t_{j-1}}\right)\right), \quad 1 \leq j \leq N. \end{array} \right.$$

Monte Carlo Forward Method

$$\left\{ \begin{array}{l} Q_0 = \mathbb{E}\left(B\left(0, \tau^*\right) \varphi\left(S_{\tau^*}\right)\right) \text{ with} \\ \tau^* := \min\left\{t_j; \varphi\left(S_{t_j}\right) = Q_j\right\} \end{array} \right.$$

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No Call Protection
Call Protection

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Standard Game Option
$\vartheta_1 = 0, \vartheta_2 = T$
call possible on $[\vartheta_1, T]$,forbidden before

> Simulated mesh $(S_i^j)_{0 \leq i \leq n}^{1 \leq j \leq m} \rightarrow$ Estimate $(u_i^j) = u(t_i, S_i^j)_{0 \leq i \leq n}^{1 \leq j \leq m}$
>
> $u_n = g$, then for $i = n - 1 \ldots 0$, for $j = 1 \ldots m$,
>
> $$u_i^j = \min\left(h_i(S_i^j), \ \max\left(\ell_i(S_i^j), \ e^{-\mu_i^j h} \mathbb{E}_i^j\left(u_{i+1} + hc_{i+1}\right) \right) \right)$$

$\mathbb{E}_i^j(u_{i+1} + hc_{i+1})$ Conditional expectation given $t = t_i, S_i = S_i^j$
- Computed by parametric regression of $(u_{i+1} + hc_{i+1})_{1 \leq j \leq m}$ against $(S_i)_{1 \leq j \leq m}$, using a global parametric regression basis $1, S, S^2$ in S

Regression estimate of the delta

$$\delta_i^j = \frac{\mathbb{E}_i^j\{u_{i+1}(S_{i+1} - S_i)\}}{(\sigma_i S_i^j)^2 h}$$

Alternative MC forward estimates of price and delta at time 0

Markovian RIBSDE
Approximation Results
Application:Pricing convertible Bonds with call protection

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Backward vs Forward MC

Maturity $T = 125$ days, Nominal coupon rate $\bar{c} = 0$

MC Fd less volatile than MC Bd (Deviations over 50 trials, $S_0 = 100.55$)

	Value VI	Dev MC Bd	Dev MC Fd
Price	102.049	0.821	0.010
Delta	0.416	0.071	0.019

MC Fd more accurate than MC Bd (%Err=1 $\leftrightarrow$ relative difference of 1% between MC and VI)

S_0	VI Price	%Err Bd	%Err Fd	VI delta	%Err Bd	%Err Fd
98.55	101.246	1.90	0.04	0.376	1.07	0.07
99.55	101.637	1.92	0.01	0.396	0.95	0.50
100.55	102.049	1.99	0.01	0.416	2.77	0.67
101.55	102.479	1.65	0.07	0.435	3.97	3.47

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Non-decreasing sequence of $[0, T]$-valued stopping times
Effective call payoff process

$$U_t = \Omega_t^c \infty + \Omega_t h(t, X_t) = U(t, S_t, H_t)$$

$\Omega_t = \mathbf{1}_{\{H_t \notin K\}}$

Simulated mesh $(S_i^j, H_i^j)_{0 \leq i \leq n}^{1 \leq j \leq m} \rightarrow$ Estimate $(u_i^j) = u(t_i, S_i^j, H_i^j)_{0 \leq i \leq n}^{1 \leq j \leq m}$

$u_n = g$, then for $i = n - 1 \ldots 0$, for $j = 1 \ldots m$

$$u_i^j = \min \left(U_i \left(S_i^j, H_i^j \right) , \max \left(\ell_i \left(S_i^j \right) , e^{-rh} \mathbb{E}_i^j (u_{i+1} + hc_{i+1}) \right) \right)$$

min plays no role outside the support of U_i, where $U_i (S, H)$ is equal to
$+\infty$

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Highly path dependent call protection

Example ('l out of d')

Given a constant trigger level $\bar{S}$ and constants $l \leq d \leq N$, call possible iff S has been $\geq \bar{S}$ on at least l of the last d monitoring times

- $\mathcal{K} = \{0,1\}^d$, $\kappa_l^k(x) = (\mathbf{1}_{S \geq \bar{S}}, k_1, \ldots, k_{d-1})$

- H_t vector of the indicator functions of the events $S_{T_l} \geq \bar{S}$ at the last d monitoring dates preceding time t

Call possible iff $|H_t| \geq l \Leftrightarrow H_t \notin K$ with $|k| = \sum_{1 \leq p \leq d} k_p$ and $K = \{k \in \mathcal{K} \, ; \, |k| < l\}$

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$\mathbb{E}_i^j(u_{i+1} + hc_{i+1})$ Conditional expectation given $t = t_i, S_i = S_i^j, H_i = H_i^j$

- computed by non-linear regression of $(u_{i+1} + hc_{i+1})_{1 \leq j \leq m}$ against $(S_i, H_i)_{1 \leq j \leq m}$, using for example a method of cells in (S, H)

Numerical Data

'l out of d' with $\bar{S} = 103$
Maturity $T = 180$ days, Nominal coupon rate $\bar{c} = 1.2/\text{month}$
Other data unchanged

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Reducible Case

In case $l = d$ one can reduce the problem to two space dimensions instead of $d + 1$

- S and the number N of consecutive monitoring dates T_ls with $S_{T_l} \geq \bar{S}$ from time t backwards (capped at l)

Two simulation schemes

MC_d a method of cells in (S, H)

MC^1 a method of cells in (S, N)

MC_d more accurate then MC^1 ($S_0 = 100$),Intermittent Path

l	1	5	10	20	30
VI^1 price	103.91	105.10	106.03	107.22	108.01
MC^1 %Err	0.04	0.16	0.47	0.88	1.34
MC_d %Err	0.04	0.15	0.03	0.04	0.24

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General Case

Computation Times

d	1	5	10	20	30
VI_d	332s	5332s	44h	—	—
MC_d	154s	212s	313s	474s	628s
Rel Err	range 1 bp—1%			—	—

Will use two methods for the computation of the conditional expectations in MC_d:

$\qquad MC_d$ a method of cells in (S, H),

$\qquad MC_d^\sharp$ a method of cells in $(S, |H|^\sharp)$

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Approximate $MC_d^\sharp$ Algorithm

$|H|^\sharp$ number of ones in H starting from the $(l - |H|)^{th}$ zero

- $|H|^\sharp = N$ in case $l = d$

Example ($d = 10$, $l = 8$)

- $H = (1, 1, 1, 1, \mathbf{0}, 1, 1, 1, 0, 0)$
  $l - |H| = 8 - 7 = 1$, $|H|^\sharp = 3$ (number of ones on the right of the first zero, in bold in H),

- $H = (1, 1, 1, 0, 1, 1, 1, \mathbf{0}, 0, 0)$ $l - |H| = 8 - 6 = 2$, $|H|^\sharp = 0$ (number of ones on the right of the second zero, in bold in H)

Rationale Entries of H preceding its $(l - |H|)^{th}$ zero irrelevant to the price

- Necessarily superseded by new ones before the bond may become callable

- Approximate algorithm $\sim$ reducible case based on the 'good regressor' $|H|^\sharp$ for estimating highly path-dependent conditional expectations

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MC_d good , $MC_d^{\sharp}$ 'rather good' ($d = 5, S_0 = 100$),HP Intermittent

l	2	3	5
VI_d price	104.07	104.43	105.10
MC_d %Err	0.21	0.15	0.15
$MC_d^{\sharp}$ %Err	0.19	0.23	0.18

MC_d good , $MC_d^{\sharp}$ 'OK' ($d = 10, S_0 = 100$),HP Intermittent

l	2	5	10
VI_d price	104.27	104.87	106.03
MC_d %Err	0.01	0.15	0.03
$MC_d^{\sharp}$ %Err	0.04	0.26	0.38

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Deviations over 50 trials and relative difference
$(d = 30, S_0 = 102.55)$,HP Intermittent

l	5	10	20	30
Dev MC_d	0.056	0.061	0.086	0.152
Dev $MC_d^\sharp$	0.060	0.069	0.092	0.175
% Err	0.09	0.24	0.72	1.06

'Good regressor' algorithm $MC_d^\sharp$ rather accurate in practice
Ability to work with a 'good' (as opposed to exact), low-dimensional
regressor

- An interesting feature of simulation as opposed to deterministic
  numerical schemes