Perfect and Partial Hedging for Multiple Exercise (Swing) Game Options in Discrete and Continuous Time

Yuri Kifer

Hebrew University of Jerusalem

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Joint work with **Yan Dolinsky** and **Yonathan Iron**

Multiple Exercise Options: Introduction

Multiple exercise options were suggested to model swing contracts emerging in energy and commodity markets where several exercises are allowed in a prescribed order and conditions of contracts are allowed to change several times during their duration. Their valuation in the American options framework were considered by Carmona and Touzi (Math. Finance, 2008) and by others as the value of an optimal multiple stopping problem. We suggest the first approach to value multiple exercise options even in the more general game options framework by classical hedging arguments. We deal also with the problem of minimizing the shortfall risk for such options which was not considered before, at all.

Multiple Exercise Options: Examples

1) A university in a warm country wants to protect itself from fluctuations in the heating oil price and buys a swing option with 10 exercises within January guarantying its buying price with a restriction of no more than one exercise per day. This swing option is cheaper than 10 single American options and can be a sufficient protection based on the history of cold days in January. As hedging securities the seller of such option may use futures on the heating oil or a linked to the oil price basket security.

2) Volkswagen delivers cars to US in several shipments within a year and wants to ensure the euro-dollar exchange rate at these times. It buys an appropriate (multi exercise) swing option with no more than one exercise per month and the seller of such option may use currencies as hedging securities for his portfolio. Because of restrictions (which could be acceptable for VW) this type of option is cheaper than several American options.

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Multiple Exercise Options: Definitions in Discrete Time

- A discrete time multi exercise (swing) game option is a contract between its seller and the buyer allowing to the seller to cancel (or terminate) and to the buyer to exercise L specific claims or rights in a particular order while a positive delay between exercises/cancellations is required.
- The contract is determined given $2L$ payoff processes $X_i(n) \geq Y_i(n) \geq 0$, $n = 0, 1, ...,$ $i = 1, 2, ..., L$ adapted to a filtration $\mathcal{F}_n$, $n \geq 0$ generated by the evolution of an underlying risky security S_n, $n \geq 0$. If the buyer exercises k-th claim $k \leq L$ at the time n then the seller pays to him the amount $Y_k(n)$ but if the latter cancels k-th claim at the time n before the buyer he pays to the buyer the amount $X_k(n)$ with $\delta_k(n) = X_k(n) - Y_k(n)$ being the cancellation penalty.

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Remarks and goals

- Without restrictions such as an order of exercises and a delay between them the problem reduces to dealing with just L separate game options and nothing new arises then.
- 1st goal: to provide a rigorous pricing theory of such multiple exercise options based on an appropriate notion of (perfect) hedging (which nobody did before).
- 2nd goal: to study hedging with risk for multiple exercise options where an investor (option's writer) starts with a portfolio whose value is less than needed for the perfect hedging (i.e. less than the fair price of the option) and he wants to minimize additional funds (shortfall risk) he may need to add to fulfill his contract obligation. We can fulfill this goal only in the discrete time case since hedging with risk in continuous time faces problems even for usual 1-exercise game options.

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Game multi exercise options in CRR (binomial) market

- For $p \in (0,1)$ the probability space is:

  (Ω, P), $\Omega = \{\omega = (\omega_1, ..., \omega_N), \omega_i \in \{1, -1\}\}$, $P = \{p, 1-p\}^N$

  and the market consists of two securities:

  a zero interest bank account: $B_n = B_0 > 0$, $\forall n \geq 0$

  and a risky security (stock) $S_n > 0$, $\forall n \geq 0$ given by

$$S_n = S_0 \prod_{i=1}^{n} (1+\rho_i), \ \rho_i = \frac{1}{2}(a+b+(b-a)\omega_i), \ -1 < a < 0 < b.$$

- We consider game multi exercise (swing) option with the ith payoff if the seller cancels at m and the buyer at n given by:

$$H^{(i)}(m,n) = X_i(m)\mathbb{I}_{m<n} + Y_i(n)\mathbb{I}_{n\leq m}$$

  where $X_i(n), Y_i(n)$ are $\mathcal{F}_n = \sigma\{\rho_k, k \leq n\}$-adapted.

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Stopping strategies for multi exercise game options

- For any $1 \leq i \leq L-1$ and $a_{j+1} \geq N \wedge (a_j+1),\ \forall j < i$ introduce
$$C_i = \{((a_1, ..., a_i), (d_1, ..., d_i)) \in \{0, ..., N\}^i \times \{0, 1\}^i\}$$
which represent the history of payoffs up to the i-th one.
- Namely, $a_j = k$ and $d_j = 1$ ($d_j = 0$) sais that the seller canceled (the buyer (or both) exercised) the j-th claim at the moment k.
- Let Γ_n, $n \geq 1$ be the set of all stopping times with values from n to N and $\Gamma = \Gamma_0$.

Definition

A stopping strategy is a sequence $s = (s_1, ..., s_L)$ such that $s_1 \in \Gamma$ is a stopping time and for $i > 1$, $s_i : C_{i-1} \to \Gamma$ is given by $s_i((a_1, ..., a_{i-1}), (d_1, ..., d_{i-1})) \in \Gamma_{N \wedge (1+a_{i-1})}$.

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Portfolio strategies for multi exercise game options

- For multi exercise options the notion of a self financing portfolio involves not only allocation of capital between stocks and the bank account but also payoffs at exercise times. The writer's decision how much money to invest in stocks (while depositing the remaining money into a bank account) depends not only on his present portfolio value but also on the current claim. Denote by Ξ the set of functions on the (finite) probability space Ω.

Definition

A portfolio strategy with an initial capital $x > 0$ is a pair $\pi = (x, \gamma)$ where $\gamma : \{0, ..., N-1\} \times \{1, ..., L\} \times \mathbb{R} \to \Xi$ and $\gamma(k, i, y)$ is the number of stocks the seller buy at the moment k provided that the current claim has the number i and the present portfolio value is y. At the same time the sum $y - \gamma(k, i, y)S_k$ is deposited to the bank account of the portfolio.

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Portfolio strategies and hedges

- We allow only portfolio strategies $\pi = (x, \gamma)$ without borrowing, called admissible, which means that $y + \gamma(k, i, y)(S_{k+1} - S_k) \geq 0$ for any $y \geq 0$.
- Let $\pi = (x, \gamma)$ be a portfolio strategy, s, b be stopping strategies and $c_k = c_k(s, b)$ be the number of payoffs until k. The portfolio value at k after the payoffs is given by $V_0^{(\pi,s,b)} = x - H^{(1)}(\sigma_1, \tau_1)$ if $\sigma_1 \wedge \tau_1 = 0$ and for $k > 0$,

$$V_k^{(\pi,s,b)} = V_{k-1}^{(\pi,s,b)} + \mathbb{I}_{c_{k-1}<L}\big(\gamma(k-1, c_{k-1}+1, V_{k-1}^{(\pi,s,b)})(S_k - S_{k-1}) - \sum_{i=1}^{L} H^{(i)}(\sigma_i, \tau_i)\mathbb{I}_{\sigma_i \wedge \tau_i = k}\big).$$

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Fair option price

Fair option price V^*: infinum of initial capitals $V \geq 0$ which allow a perfect hedge.

Theorem

Denote by $\tilde{E}$ the expectation with respect to the martingale measure $\tilde{P} = \{\tilde{p}, 1 - \tilde{p}\}^N$, $\tilde{p} = \frac{a}{a-b}$. Then

$$V^* = \min_s \max_b G(s, b)$$

where $s = (s_1, ..., s_L)$ and $b = (b_1, ..., b_L)$ are stopping strategies, $G(s, b) = \tilde{E} \sum_{i=1}^{L} H^{(i)}(\sigma_i, \tau_i)$ and σ_i, τ_i are corresponding images of s_i, b_i in the space of stopping times. Furthermore, there exist rational (optimal) stopping strategies $s^* = (s_1^*, ..., s_L^*) \in S$ and $b = (b_1^*, ..., b_L^*)$ and a portfolio strategy π^* with the initial capital V^* such that (π^*, s^*) is a perfect hedge.

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Dynamical programming algorithm 1

- The fair price and rational stopping strategies can be obtained by means of the following dynamical programming algorithm.
- For any $n \leq N$ set

$$X_n^{(1)} = X_L(n),\ Y_n^{(1)} = Y_L(n),\ V_n^{(1)} = \min_{\sigma\in\Gamma_n}\max_{\tau\in\Gamma_n} \tilde{E}\big(H^{(L)}(\sigma,\tau)|\mathcal{F}_n\big)$$

and for $1 < k \leq L$,

$$X_n^{(k)} = X_{L-k+1}(n) + \tilde{E}(V^{(k-1)}_{(n+1)\wedge N}|\mathcal{F}_n),$$

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- Then $V^* = V_0^{(L)}$.

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Dynamical programming algorithm 2

- The stopping strategies $s^* = (s_1^*, ..., s_L^*) \in S$ and $b = (b_1^*, ..., b_L^*)$ given by
- $$s_1^* = N \wedge \min\{k | X_k^{(L)} = V_k^{(L)}\},\ \ b_1^* = \min\{k | Y_k^{(L)} = V_k^{(L)}\},$$
$$s_i^*((a_1, ..., a_{i-1}), (d_1, ..., d_{i-1})) = N \wedge \min\{k > a_{i-1}|$$
$$X_k^{(L-i+1)} = V_k^{(L-i+1)}\},\ \ b_i^*((a_1, ..., a_{i-1}), (d_1, ..., d_{i-1}))$$
$$= N \wedge \min\{k > a_{i-1} | Y_k^{(L-i+1)} = V_k^{(L-i+1)}\},\ \ i > 1$$
satisfy
- $$G(s^*, b) \le G(s^*, b^*) \le G(s, b^*) \ \text{ for all } s, b$$
- and there exists a portfolio strategy $\pi^* \in \mathcal{A}(V_0^{(L)})$ such that (π^*, s^*) is a perfect hedge.

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$$s_i^*((a_1, ..., a_{i-1}), (d_1, ..., d_{i-1})) = N \wedge \min\{k > a_{i-1}|$$
$$X_k^{(L-i+1)} = V_k^{(L-i+1)}\}, \;\; b_i^*((a_1, ..., a_{i-1}), (d_1, ..., d_{i-1}))$$
$$= N \wedge \min\{k > a_{i-1} | Y_k^{(L-i+1)} = V_k^{(L-i+1)}\}, \;\; i > 1$$

  satisfy
- $$G(s^*, b) \le G(s^*, b^*) \le G(s, b^*) \;\text{ for all } s, b$$
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Dynamical programming algorithm 2

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Infusion of capital and hedging

- Consider an option seller whose initial capital x is less than the option price, i.e. $x < V^*$. Then, the seller (in order to fulfill his obligation to the buyer) must add money to his portfolio from other sources and we allow him to do this only at times when the contract is exercised requiring that after addition of money the portfolio value must be positive.
- Let $I(k, j, y)$ be the amount, called infusion of capital, that the seller adds to his portfolio immediately after the j-th payoff payed at the moment k while the portfolio value after this payment is y. A hedge with an initial capital $x < V^*$ is a triple (π, I, s) consisting of an admissible portfolio strategy, infusion of capital and a stopping strategy.

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Shortfall risk

Definition

Let $C(\pi, I, s, b)$ denotes the total infusion of capital under an admissible portfolio strategy π and stopping strategies s and b of the seller and the buyer, respectively. The shortfall risk for a hedge (π, I, s) is defined by

$$R(\pi, I, s) = \max_b EC(\pi, I, s, b)$$

where E is the expectation with respect to the market probability. The shortfall risk for an initial capital $x > 0$ is defined by

$$R(x) = \inf_{(\pi, I, s)} R(\pi, I, s)$$

for portfolio strategies π having the initial capital x.

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Hedge minimizing the shortfall risk

Theorem

The shortfall risk $R(x)$ with an initial capital x can be obtained by a dynamical programming algorithm (constructed for the value of a multiple stopping Dynkin's game with appropriately modified payoffs). There exists a hedge $(\tilde{\pi} = (x, \tilde{\gamma}), \tilde{I}, \tilde{s})$ (which also can be produced by a dynamical programming algorithm) such that $R(\tilde{\pi}, \tilde{I}, \tilde{s}) = R(x)$.

-

Remarks on proofs

- Valuation of swing options proof: 1) First, prove the existence of the saddle point in finite discrete multiple stopping Dynkin's game;
  2) Then, construct perfect hedges by induction on each stage of the option depending on the initial time of the stage and the capital at that time.
- Minimizing the shortfall risk: 1) Consider a modified Dynkin's game where the 1st player tries to minimize his shortfall risk against worst possible reply of the 2nd one i.e. as if the latter tries to maximize this risk. This is not straightforward since the 1st player not only chooses his stopping time but also manages his portfolio and infusion of capital makes this more complicated than in the one stopping case.
  2) Dynamical programming (backward induction) construction of a (partial hedging) portfolio minimizing the shortfall risk.

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Definitions in the Continuous Time Case

Continuous time swing (multiple exercise) game option is a contract between its seller and the buyer allowing to the seller to cancel (terminate) and to the buyer to exercise L specific claims or rights in a particular order. Such contract is determined given $2l$ payoff processes $X_i(t) \geq Y_i(t) \geq 0$, $t \geq 0$, $i = 1, 2, ..., l$ adapted to a filtration $\mathcal{F}_t$, $t \geq 0$ generated by the stock (risky security) S_t, $t \geq 0$ evolution governed by a geometric Brownian motion. If the buyer exercises the k-th claim $k \leq l$ at the time t then the seller pays to him the amount $Y_k(t)$ but if the latter cancels the claim k at the time t before the buyer he has to pay to the buyer the amount $X_k(t)$ and the difference $\delta_k(t) = X_k(t) - Y_k(t)$ is the cancelation penalty. A time delay $\delta > 0$ between successive exercises and cancellations is required. Payoffs may depend on the exercise number so, for instance, our options may change from call to put and vice versa after different exercises.

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Stopping and Portfolio Strategies

Stopping strategy: (σ, τ), $\sigma = (\sigma_1, ..., \sigma_l), \tau = (\tau_1, ..., \tau_l)$ where $\sigma_1, \tau_1 \in \mathcal{T}_{0T}$ with $\mathcal{T}_{st} = \{\text{stopping times}\, s \le \tau \le t\}$ and for $1 \le i \le l$,

$$\sigma_i \in \mathcal{T}_{\min(T, \sigma_{i-1} \wedge \tau_{i-1} + \delta), T}.$$

Portfolio strategy: A function π on the set $\mathcal{L}$ of all sequences $(t_1, ..., t_i), 1 \le i \le l-1$ with $(t_i + \delta) \wedge T \le t_{i+1} \le T$ (and the empty sequence ϕ) such that

$$\pi(t_1, ..., t_i) = \left(\beta_s^\pi(t_1, ..., t_i), \gamma_{1,s}^\pi(t_1, ..., t_i), ..., \gamma_{m,s}^\pi(t_1, ..., t_i)\right)$$

with progressively measurable processes $\beta_s^\pi, \gamma_{i,s}^\pi$ in $s \ge t_i$ satisfying usual integrability conditions in s. It is a self-financing portfolio strategy starting at t_i with the portfolio value at $s \ge t_i$,

$$Z_s^\pi(t_1, ..., t_i) = \beta_s^\pi(t_1, ..., t_i) B_s + \gamma_s^\pi(t_1, ..., t_i) \cdot S_s$$

where B_s and S_s are the bond and stock prices at time s and $\cdot$ is the inner product.

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Hedge and fair price

Requirement:

$$G^{\pi}(t_1, ..., t_i) = Z^{\pi}_{t_i}(t_1, ..., t_{i-1}) - Z^{\pi}_{t_i}(t_1, ..., t_i) \geq 0$$

–no infusion of capital but possible payment at exercise/cancellation times.

Hedge: a pair (π, σ) of portfolio and stopping strategies such that for any stopping strategy τ each $\pi(\sigma_1 \wedge \tau_1, ..., \sigma_i \wedge \tau_i)$ is self-financing on the time interval $[\sigma_i \wedge \tau_i, T]$. A hedge is called perfect if for any τ and i,

$$G^{\pi}(\sigma_1 \wedge \tau_1, ..., \sigma_i \wedge \tau_i) \geq R_i(\sigma_i, \tau_i).$$

Fair price of the swing option $(X_i, Y_i, \delta), 1 \leq i \leq l$ is infinum of initial capitals x for which there exists a perfect hedge

Main theorem

For $\sigma = (\sigma_1, ..., \sigma_l)$ and $\tau = (\tau_1, ..., \tau_l)$ set

$$H(\sigma, \tau) = \tilde{E}\Big(\sum_{i=1}^{l} e^{-r\sigma_i \wedge \tau_i} R_i(\sigma_i, \tau_i)\Big)$$

where $\tilde{E}$ is the expectation with respect to the martingale measure.

Theorem

The fair price V^* of the swing option is given by

$$V^* = inf_\sigma sup_\tau H(\sigma, \tau)$$

and there exist stopping strategies σ^*, τ^* and a portfolio strategy π^* such that $(V_0^{(l)}, \pi^*, \sigma^*)$ is a perfect hedge and for all σ, τ,

$$H(\sigma^*, \tau) \le H(\sigma^*, \tau^*) \le H(\sigma, \tau^*).$$

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Main theorem: details 1

For every $0 \le t \le T$ set

$$X_t^{(1)} = e^{-rt} X_l(t),\ \ Y_t^{(1)} = e^{-rt} Y_l(t),$$

$$V_t^{(1)} = esssup_{\tau \in \mathcal{T}_{t,T}} essinf_{\sigma \in \mathcal{T}_{t,T}} \tilde{E}\big(R^{(1)}(\sigma,\tau)|\mathcal{F}_t\big),$$

$$X_t^{(i)} = e^{-rt} X_{l-i+1}(t) + \tilde{E}\big(V^{(i-1)}_{(t+\delta)\wedge T}|\mathcal{F}_t\big),\ \ Y_t^{(i)} = e^{-rt} Y_{l-i+1}(t) + \tilde{E}\big(V^{(i-1)}_{(t+\delta)\wedge T}$$

$$\text{and} \quad V_t^{(i)} = esssup_{\tau \in \mathcal{T}_{t,T}} essinf_{\sigma \in \mathcal{T}_{t,T}} \tilde{E}\big(R^{(i)}(\sigma,\tau)|\mathcal{F}_t\big)$$

for $1 < i \le l$ where by the definition

$$R^{(i)}(\sigma,\tau) = X_\sigma^{(i)} \mathbf{1}_{\{\sigma<\tau\}} + Y_\tau^{(i)} \mathbf{1}_{\{\sigma \ge \tau\}}.$$

Then the fair price V^* for the swing game option is given by

$$V^* = V_0^{(l)} = inf_\sigma sup_\tau H(\sigma,\tau).$$

Main theorem: details 2

The stopping strategies σ^*, τ^* given by

$$\sigma_1^* = \inf\{0 \le t : V_t^{(l)} = X_t^{(l)}\} \wedge T, \ \tau_1^* = \inf\{0 \le t : V_t^{(l)} = Y_t^{(l)}\}$$

and for $1 < i \le l$,

$$\sigma_i^* = \inf\{t \ge \sigma_{i-1} \wedge \tau_{i-1} + \delta : V_t^{(l-i+1)} = X_t^{(l-i+1)}\} \wedge T,$$

$$\tau_i^* = \inf\{t \ge \sigma_{i-1} \wedge \tau_{i-1} + \delta : V_t^{(l-i+1)} = Y_t^{(l-i+1)}\}$$

satisfy conditions of the main theorem.

Steps of the proof

- Proof of existence of a saddle point (optimal strategies) for the multi stopping Dynkin's game with continuous time and cádlág payoffs.
- Construction of perfect hedging for multi exercise game options.

In both cases we have to work on time intervals starting at an arbitrary stopping time and then glue the results beginning at the last time interval (one before last exercise/cancellation) and proceeding backward.