On the pricing of game options (including convertible bonds)

A review of some general concepts

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Outline

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Options

Notation

- European option:
  defined by random payoff H at time T
- American option:
  defined by exercise process $X = (X_t)_{t \in [0,T]}$.
  This includes the European option for $X_t = H1_{\{t=T\}}$.
- Game option (Kifer 2000):
  defined by exercise process $L = (L_t)_{t \in [0,T]}$
  and cancellation process $U = (U_t)_{t \in [0,T]}$.
  This includes the American option for $L = X$, $U = \infty$.

What are fair prices for such products?

Option pricing

Folklore

Reasonable prices π are of the following form (for some EMM Q):

- European option: expectation $\pi = E_Q(H)$
- American option: Snell envelope

$$\pi = \sup_{\tau \text{ stopping time}} E_Q(X_\tau)$$

- Game option: Dynkin game

$$\pi = \inf_{\sigma \text{ st.t.}} \sup_{\tau \text{ st.t.}} E_Q(R(\sigma, \tau)) = \sup_{\tau \text{ st.t.}} \inf_{\sigma \text{ st.t.}} E_Q(R(\sigma, \tau))$$

with $R(\sigma, \tau) = L_\sigma 1_{\sigma \leq \tau} + U_\tau 1_{\sigma > \tau}$

But why?

Option pricing
General concepts

Distinguish between

- static (OTC) prices vs. dynamic (liquidly traded) price processes,
- arbitrage vs. utility-based approaches.

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Static prices in complete markets
European option

H can be replicated for

$$\pi = E_Q(H)$$

$\rightsquigarrow$ only this price is compatible with absence of arbitrage
(up to technical issues due to admissibility)

Static prices in complete markets

American option

- Absence of arbitrage $\rightsquigarrow$ price must be at least

$$\sup_{\tau \text{ stopping time}} E_Q(X_\tau).$$

- Moreover,

$$\pi = \sup_{\tau \text{ stopping time}} E_Q(X_\tau)$$

  allows to buy portfolio with value $\geq X$.
- Together: π is the only reasonable price.

Static prices in complete markets

Game option

- $$\pi = \inf_{\sigma \text{ st.t.}} \sup_{\tau \text{ st.t.}} E_Q(R(\sigma, \tau)) = \sup_{\tau \text{ st.t.}} \inf_{\sigma \text{ st.t.}} E_Q(R(\sigma, \tau))$$

  allows to superhedge $R(\sigma, t)$ for optimal stopping time σ and any t
  $\leadsto \pi$ is upper limit for no-arbitrage price
- Symmetry: $\leadsto \pi$ is also lower limit for no-arbitrage price

Price processes in complete markets

European option

Accordingly, the only possible intermediate prices are:

- European option: conditional expectation $\pi_t = E_Q(H|\mathcal{F}_t)$
- American option: Snell envelope

$$\pi_t = \operatorname{esssup}_{\tau \in \mathcal{T}_{[t,T]}} E_Q(X_\tau|\mathcal{F}_t)$$

  where $\mathcal{T}_{[t,T]}$ contains the $[t, T]$-valued stopping times.

- Game option: Dynkin game

$$\begin{aligned} \pi_t &= \operatorname{essinf}_{\sigma \in \mathcal{T}_{[t,T]}} \operatorname{esssup}_{\tau \in \mathcal{T}_{[t,T]}} E_Q(R(\sigma,\tau)|\mathcal{F}_t) \\ &= \operatorname{esssup}_{\tau \in \mathcal{T}_{[t,T]}} \operatorname{essinf}_{\sigma \in \mathcal{T}_{[t,T]}} E_Q(R(\sigma,\tau)|\mathcal{F}_t) \end{aligned}$$

  with $R(\sigma,\tau) = L_\sigma 1_{\sigma \le \tau} + U_\tau 1_{\sigma > \tau}$.

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Static no-arbitrage prices

European option

- Fundamental theorem of asset pricing
  $\rightsquigarrow \pi_t = E_Q(H|\mathcal{F}_t)$ leads to no-arbitrage price process for EMM's Q
  $\rightsquigarrow \pi = E_Q(H)$ does not lead to arbitrage.
- Superhedging theorem
  $\rightsquigarrow \sup_{Q \text{ EMM}} E_Q(H)$ allows to superreplicate H
- Together + symmetry + convexity $\rightsquigarrow$ Prices of the form $\pi = E_Q(H)$ with EMM Q constitute no-arbitrage interval.

Static no-arbitrage prices

Game option (including the American case)

- Prices above

$$\overline{\pi} = \inf_{\sigma \text{ st.t.}} \sup_{\tau \text{ st.t.}} \sup_{Q \text{ EMM}} E_Q(R(\sigma,\tau)) = \sup_{\tau \text{ st.t.}} \sup_{Q \text{ EMM}} \inf_{\sigma \text{ st.t.}} E_Q(R(\sigma,\tau))$$

  lead to seller-arbitrage.

- Prices below

$$\underline{\pi} = \inf_{\sigma \text{ st.t.}} \inf_{Q \text{ EMM}} \sup_{\tau \text{ st.t.}} E_Q(R(\sigma,\tau)) = \sup_{\tau \text{ st.t.}} \inf_{\sigma \text{ st.t.}} \inf_{Q \text{ EMM}} E_Q(R(\sigma,\tau))$$

  lead to buyer-arbitrage.

- Prices within these bounds do not lead to either buyer- or seller-arbitrage.

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No-arbitrage price processes

European option

- Fundamental theorem of asset pricing $\rightsquigarrow$ $(\pi_t)_{t\in[0,T]}$ no-arbitrage price process iff $\pi_t = E_Q(H|\mathcal{F}_t)$ for some EMM Q
- Initial prices coincide essentially with the static approach.

No-arbitrage price processes

Game option (including the American case)

- Key ideas:
  - $L_t \leq \pi_t \leq U_t$
  - Trading American or game options = trading under constraints: Negative option positions only possible as long as $L_{t-} < \pi_{t-}$, positive option positions only possible as long as $\pi_{t-} < U_{t-}$.
- Need version of no arbitrage (NFLVR) and the FTAP under trading constraints.
- Deduce: No-arbitrage option price processes are those of the form

$$\begin{aligned}\pi_t &= \operatorname{essinf}_{\sigma \in \mathcal{T}_{[t,T]}} \operatorname{esssup}_{\tau \in \mathcal{T}_{[t,T]}} E_Q(R(\sigma,\tau)|\mathcal{F}_t) \\ &= \operatorname{esssup}_{\tau \in \mathcal{T}_{[t,T]}} \operatorname{essinf}_{\sigma \in \mathcal{T}_{[t,T]}} E_Q(R(\sigma,\tau)|\mathcal{F}_t)\end{aligned}$$

  for some EMM Q.
- Initial prices essentially as in the static case.

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Neutral price processes

European option

- Key assumptions:
  - Options are liquidly traded.
  - "Representative" agent is expected utility maximizer with given utility function u.
  - Options are in zero net supply, i.e. the optimal portfolio contains no options.
- There exists a unique neutral option price process, namely

$$\pi_t = E_{Q^\star}(H|\mathcal{F}_t),$$

where the EMM $Q^\star$ is the dual minimizer corresponding to the utility maximization problem without options, e.g. the minimal entropy martingale measure for exponential utility $u(x) = 1 - \exp(-x)$.

Neutral price processes

Game option (including American)

- Key assumptions:
  - as for European options
  - Trading American or game options means trading under positivity resp. negativity constraints (as above).
- There exists a unique neutral option price process, namely

$$
\begin{aligned}
\pi_t &= \operatorname{essinf}_{\sigma\in\mathcal{T}_{[t,T]}}\operatorname{esssup}_{\tau\in\mathcal{T}_{[t,T]}} E_{Q^\star}(R(\sigma,\tau)|\mathcal{F}_t)\\
&= \operatorname{esssup}_{\tau\in\mathcal{T}_{[t,T]}}\operatorname{essinf}_{\sigma\in\mathcal{T}_{[t,T]}} E_{Q^\star}(R(\sigma,\tau)|\mathcal{F}_t)
\end{aligned}
$$

where the EMM $Q^\star$ is the dual minimizer corresponding to the utility maximization problem without options (as before).

Outline

Utility-indifference prices for exponential utility

European option

- Key idea:
  - Asymmetric OTC situation.
  - Potential buyer wants to buy γ options.
  - Seller maximizes (here:) exponential utility of terminal wealth.
  - Here threshold is the utility-indifference price π:

$$\sup_{\varphi} E(u(v_0 + \varphi \bullet S_T + \gamma(\pi - H))) = \sup_{\varphi} E(u(v_0 + \varphi \bullet S_T)).$$

  - The normalized difference of the optimizers $(\varphi^0 - \varphi^\gamma)/\gamma$ is called utility-based hedging strategy.
- Utility-indifference price for $u(x) = 1 - \exp(-x)$:

$$\pi = E_{Q_\gamma}(H) + \frac{1}{\gamma}\left(H(Q_0, P) - H(Q_\gamma, P)\right),$$

where $\frac{dP_\gamma}{dP} := \frac{e^{\gamma H}}{E(e^{\gamma H})}$ and Q_γ minimal entropy martingale measure relative to P_γ.

Utility-indifference prices for exponential utility

American option (tentative)

- Key idea:
  - Situation as in the European case.
  - Problem: seller does not know exercise time τ of the buyer $\rightsquigarrow$ consider worst case approach
  - For exponential utility (and only then) this leads to the utility-indifference price π:

$$\inf_{\tau} \sup_{\varphi} E(u(v_0 + \varphi^0 \bullet S_T + \varphi \bullet S_\tau + \gamma(\pi - X_\tau))) = E(u(v_0 + \varphi^0 \bullet S_T)).$$

- Utility-indifference price for $u(x) = 1 - \exp(-x)$:

$$\pi = \sup_{\tau} \left(E_{Q_\gamma}(X_\tau) + \frac{1}{\gamma}(H(Q_0, P) - H(Q_\gamma, P)) \right),$$

where $\frac{dP_\gamma}{dP} := \frac{e^{\gamma X_\tau}}{E(e^{\gamma X_\tau})}$ and Q_γ minimal entropy martingale measure relative to P_γ.

Utility-indifference prices for exponential utility

Game option (tentative)

- Key idea:
  - Situation as in the American case.
  - For exponential utility (and only then) this leads to the utility-indifference price π:

$$\inf_{\tau} \sup_{\varphi,\sigma} E(u(v_0+\varphi^0 \bullet S_T+\varphi \bullet S_{\sigma\wedge\tau}+\gamma(\pi-R(\sigma,\tau))) = E(u(v_0+\varphi^0 \bullet S_T))$$

- Utility-indifference price for $u(x) = 1 - \exp(-x)$:

$$\pi = \sup_{\tau} \inf_{\sigma} \left(E_{Q_\gamma}(X_{\sigma\wedge\tau}) + \frac{1}{\gamma}(H(Q_0, P) - H(Q_\gamma, P)) \right),$$

where $\frac{dP_\gamma}{dP} := \frac{e^{\gamma X_{\sigma\wedge\tau}}}{E(e^{\gamma X_{\sigma\wedge\tau}})}$ and Q_γ minimal entropy martingale measure relative to P_γ.

Outline

Asymptotics for small numbers of claims

European options

- Key idea:
  - Problem: utility-indifference price size-dependent and hard to compute
  - Consider first-order approximation for small γ:

$$\pi(\gamma) \approx \pi^0 + \gamma\delta, \quad \varphi^\gamma \approx \varphi^0 + \gamma\eta.$$

- For exponential utility $u(x) = 1 - \exp(-x)$ this leads to:

$$\pi^0 = E_{Q_0}(H), \quad \delta = \frac{1}{2} \inf_\eta E_{Q_0}((\pi_0 + \eta \bullet S_T - H)^2),$$

and η as minimizer of the quadratic hedging problem leading to δ.

Asymptotics for small numbers of claims

Game options (tentative)

- Key idea: Consider as before first-order approximation for small γ:

$$\pi(\gamma) \approx \pi^0 + \gamma\delta, \quad \varphi^\gamma \approx \varphi^0 + \gamma\eta.$$

- For exponential utility $u(x) = 1 - \exp(-x)$ this leads to:

$$\begin{aligned} \pi^0 &= \sup_\tau \inf_\sigma E_{Q_0}(R(\sigma, \tau)), \\ \delta &= \frac{1}{2} \inf_\eta E_{Q_0}((\pi_0 + \eta \bullet S_T - R(\sigma^\star, \tau^\star))^2), \end{aligned}$$

and η as minimizer of the quadratic hedging problem leading to δ.

Outline

Some references

very incomplete and somewhat random list

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- Lepeltier & Maingueneau (1984)
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