

Pricing game option using reflected BSDEs

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BSDEs as a tool for pricing and hedging

BSDEs

Discretization of the BSDE

Convergence results

American option and Game option

Reflected BSDEs

Approximation

Game Option with call protection

Definition

Representation

Approximation

Definition

Given a forward SDE X (b, σ Lipschitz):

$$X_t = X_0 + \int_0^t b(u, X_u) du + \int_0^t \sigma(u, X_u) dW_u$$

A Backward SDE (Y, Z) satisfies

$$Y_t = g(X_T) + \int_t^T f(X_u, Y_u, Z_u) du - \int_t^T (Z_u)' dW_u$$

f, g are Lipschitz. Bismut (73), Pardoux-Peng (90)

In finance

$$\text{BSDE} : Y_t = g(X_T) + \int_t^T f(X_u, Y_u, Z_u) du - \int_t^T (Z_u)' dW_u$$

- ▶ g payoff of the *European* option, X the assets price
- ▶ Y is the price of the option
- ▶ $Z = \Delta\sigma$ almost the Delta of the option
- ▶ Example for f :
 - $f = -rY$ interest rate = r , pricing under RN probability
 - $f = -rY - \theta Z$, pricing under historical probability
 - f non-linear : market imperfection

Link with PDE

$Y_t = u(t, X_t)$ where u is the solution of

$$-\partial_t u - \mathcal{L}u - f = 0$$

thus non-linear Feynman-Kac representation

$$u(t, x) = \mathbb{E} \left[g(X_T^{t,x}) + \int_t^T f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds \right]$$

Conclusion : alternative way to compute u (the price) and even the Delta... need for approximation scheme.

SDE approximation

- ▶ For $(b, \sigma) : \mathbb{R}^d \rightarrow \mathbb{R}^d \times M^d$ Lipschitz :

$$X_t = X_0 + \int_0^t b(X_u)du + \int_0^t \sigma(X_u)dW_u$$

- ▶ Euler scheme on X with $\pi = \{0 = t_0 < \dots < t_i < \dots < t_n = T\}$:

$$\begin{cases} X_0^\pi &= X_0 \\ X_t^\pi &= X_{t_i}^\pi + b(X_{t_i}^\pi)(t - t_i) + \sigma(X_{t_i}^\pi)(W_t - W_{t_i}), \quad t \in (t_i, t_{i+1}] \end{cases}$$

- ▶ Error (b, σ Lipschitz)

$$\mathcal{E}rr(X, X^\pi) := \mathbb{E} \left[\sup_{t \in [0, T]} |X_t - X_t^\pi|^2 \right]^{\frac{1}{2}} \leq \frac{C}{\sqrt{n}}$$

$$(\max_i |t_{i+1} - t_i| \leq \frac{C}{n})$$

Approximation scheme for the BSDE

- ▶ Key idea

$$Y_{t_i} = Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(X_u, Y_u, Z_u) du - \int_{t_i}^{t_{i+1}} (Z_u)' dW_u$$

$$Y_{t_i}^\pi \simeq Y_{t_{i+1}}^\pi + (t_{i+1} - t_i) f(X_{t_i}^\pi, Y_{t_i}^\pi, \bar{Z}_{t_i}^\pi) - (\bar{Z}_{t_i}^\pi)' (W_{t_{i+1}} - W_{t_i})$$

- ▶ A backward scheme

$$Y_{t_i}^\pi := \mathbb{E} \left[Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i} \right] + (t_{i+1} - t_i) f(X_{t_i}^\pi, Y_{t_i}^\pi, \bar{Z}_{t_i}^\pi)$$

$$\bar{Z}_{t_i}^\pi := (t_{i+1} - t_i)^{-1} \mathbb{E} \left[(W_{t_{i+1}} - W_{t_i}) (Y_{t_{i+1}}^\pi)' \mid \mathcal{F}_{t_i} \right]$$

↪ terminal condition $Y_T^\pi := g(X_T^\pi)$.

Error to control

- continuous version of the scheme (Y^π, Z^π) :

$$Y_{t_{i+1}}^\pi = \mathbb{E} \left[Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i} \right] + \int_{t_i}^{t_{i+1}} (Z_u^\pi)' dW_u$$

$$Y_t^\pi = Y_{t_{i+1}}^\pi + (t_{i+1} - t) f(X_{t_i}^\pi, Y_{t_i}^\pi, \bar{Z}_{t_i}^\pi) - \int_t^T (Z_u^\pi)' dW_u.$$

- Error:

$$\mathcal{E}rr(Y, Y^\pi) := \sup_{t \in [0, T]} \mathbb{E}[|Y_t - Y_t^\pi|^2]^{\frac{1}{2}}$$

$$\mathcal{E}rr(Z, \bar{Z}^\pi) := \mathbb{E} \left[\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z_t - \bar{Z}_{t_i}^\pi|^2 dt \right]^{\frac{1}{2}} = \|Z - \bar{Z}^\pi\|_{\mathcal{H}^2}$$

Regularity and convergence

- ▶ (Bouchard-Touzi 2004):

$$\mathcal{E}rr(Y, Y^\pi) + \mathcal{E}rr(Z, \bar{Z}^\pi) \leq C(\mathcal{E}rr(X, X^\pi) + \mathcal{R}eg(Y) + \mathcal{R}eg(Z))$$

- ▶ Need "regularity" on Z . (Zhang 2001)

$$\mathcal{R}eg(Y) \leq \mathcal{R}eg(Z) \leq \frac{C}{\sqrt{n}}$$

- ▶ Convergence rate

$$\mathcal{E}rr(Y, Y^\pi) + \mathcal{E}rr(Z, \bar{Z}^\pi) \leq \frac{C}{\sqrt{n}}.$$

simply reflected BSDE

Reflection on a lower boundary $\ell(X)$:

$$\begin{aligned} Y_t &= g(X_T) + \int_t^T f(X_u, Y_u, Z_u) du - \int_t^T Z_u dW_u + \int_t^T dK_u^+ \\ Y_t &\geq \ell(X_t) , \quad t \leq T \text{ and } \int_0^T (Y_u - \ell(X_u)) dK_u^+ = 0 , \end{aligned}$$

Linked to Optimal Stopping problem (take $f = 0$), American option pricing (el Karoui et al. 97)

doubly reflected BSDEs

$$\begin{aligned}
 Y_t &= g(X_T) + \int_t^T f(X_u, Y_u, Z_u) du - \int_t^T Z_u dW_u + \int_t^T dK_u^+ - \int_t^T dK_u^- \\
 Y_t &\geq \ell(X_t) , \quad t \leq T \quad \text{and} \quad \int_0^T (Y_u - \ell(X_u)) dK_u^+ = 0 , \\
 Y_t &\leq h(X_t) , \quad t \leq T \quad \text{and} \quad \int_0^T (Y_u - h(X_u)) dK_u^- = 0 .
 \end{aligned}$$

Linked to game option pricing.

(Cvitanic-Karatzas 96, Cvitanic - Ma 01, Crepey-Matoussi 09)

Link with PDEs

Variational Inequalities: ($Y_t = u(t, X_t)$)

- ▶ Simply reflected case: (terminal condition g)

$$\min(-\partial_t u - \mathcal{L}u - f, u - \ell) = 0$$

- ▶ Doubly reflected case: (terminal condition g)

$$\max(\min(-\partial_t u - \mathcal{L}u - f, u - \ell), u - h) = 0$$

- ▶ Feynman-Kac representation

$$u(t, x) = \mathbb{E}\left[g(X_T^{t,x}) + \int_t^T f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds + \int_t^T dK_T^{t,x}\right]$$

Approximation

Example for the simply reflected case: say $f = 0$, $\ell = g$

$$\text{RBSDE: } Y_t = \ell(X_T) - \int_t^T (Z_u)' dW_u + \int_t^T dK_u^+, \quad Y_t \geq \ell(X_t)$$

- ▶ $X \rightarrow X^\pi$
- ▶ Y^π : Discrete Snell envelope of $\ell(X_{t_i}^\pi)$

$$Y_{t_i}^\pi = \mathbb{E}_{t_i} \left[Y_{t_{i+1}}^\pi \right] \vee \ell(X_{t_i}^\pi) =: \tilde{Y}_{t_i}^\pi \vee \ell(X_{t_i}^\pi)$$

Note: one can use of different grid for discrete-time approximation π and for discretization of reflection $\mathfrak{R} \subset \pi$.

Euler scheme for simply and doubly RBSDEs

Given the grids $\mathfrak{R} \subset \pi$

- ▶ Starting from the terminal condition $\tilde{Y}_T^\pi = Y_T^\pi := g(X_T^\pi)$
- ▶ compute at each step

$$\begin{cases} \bar{Z}_{t_i}^\pi &= (t_{i+1} - t_i)^{-1} \mathbb{E} \left[Y_{t_{i+1}}^\pi (W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i} \right] \\ \tilde{Y}_{t_i}^\pi &= \mathbb{E} \left[Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i} \right] + (t_{i+1} - t_i) f(X_{t_i}^\pi, Y_{t_i}^\pi, \bar{Z}_{t_i}^\pi) \\ Y_{t_i}^\pi &= \tilde{Y}_{t_i}^\pi \mathbf{1}_{\{t_i \notin \mathfrak{R}\}} + \mathcal{P}(X_{t_i}^\pi, \tilde{Y}_{t_i}^\pi) \mathbf{1}_{\{t_i \in \mathfrak{R}\}} \end{cases}$$

↪ simply reflected case: $\mathcal{P}(x, y) := y \vee \ell(x)$

↪ doubly reflected case: $\mathcal{P}(x, y) := (y \vee \ell(x)) \wedge h(x)$

Some results

- ▶ Simply reflected case
 - Bally and Pages (2002): f does not depend on Z, ℓ, b, σ lipschitz.
 \hookrightarrow bound $\frac{C}{\sqrt{n}}$.
 - Ma and Zhang (2005): f depends on Z, b, σ is C_b^1 and σ elliptic,
 ℓ is C_b^2 \hookrightarrow bound $\frac{C}{n^{\frac{1}{4}}}$
 - Bouchard-Chassagneux (08) : $\frac{C}{n^{\frac{1}{4}}}$ with less regularity or $\frac{C}{\sqrt{n}}$ with more regularity (on σ and ℓ)
- ▶ Doubly reflected case
 - Chassagneux (09) : $\ell, -h$ semi-convex \hookrightarrow bound $\frac{C}{n^{\frac{1}{4}}}$

A word on the proof

- ▶ Discretely reflected BSDEs (discretization of the reflection only): a good “proxy” for the continuously reflected BSDEs
- ▶ Representation of the Z of the **discretely** reflected BSDEs ($f = 0$)

$$(Z_t^d)' = \mathbb{E} [\nabla \ell(X_{\tau_j})(\Lambda^t D_t X)_{\tau_j} \mid \mathcal{F}_t], \quad t \in [r_j, r_{j+1}).$$

- ▶ allows to obtain regularity results
- ▶ gives also an alternative way to approximate Z

Definition and framework

- ▶ American option: Option which can be exercised at any time between 0 and T
- ▶ Game option: American option which can be called back by the seller of the option at any time between 0 and T
- ▶ Game option with call protection: The seller can call back the option only on random intervals $[\vartheta_{2l-1}, \vartheta_{2l})$
 - ↪ lower boundary still $\ell(X)$, upper boundary now given by

$$U_t = \sum_{l \geq 0} \infty \mathbf{1}_{\{[\vartheta_{2l}, \vartheta_{2l+1})\}} + \sum_{l \geq 0} h(X_t) \mathbf{1}_{\{[\vartheta_{2l-1}, \vartheta_{2l})\}}$$

- ▶ American option = no call, Game option = no call protection .

monitoring switching times

- ▶ Discrete grid \mathcal{T} to monitor the activation/deactivation of call protection
 - ↪ for this talk $\mathcal{T} = \{T_0 = 0 < T_1 < T_2 = T\}$
- ▶ Call is possible if a certain marker process $H(X_T, T \in \mathcal{T})$ belongs to a certain discrete set K .
 - ↪ for this talk $H(x) = \mathbf{1}_{\{x \geq \bar{s}\}}$, $K = \{1\}$.
- ▶ Allow to take into account highly path dependent call protection : “I out of d”
 - Call is possible if the asset price has been $\geq \bar{s}$ on at least l of the last d monitoring dates.

Associated BSDEs

- ▶ Price is given by a Doubly reflected BSDE (Y, Z) with Intermittent upper boundary
- ▶ Concatenation of Doubly Reflected BSDEs / Simply reflected BSDEs depending on the protection regime (call allowed / no call allowed)
- ▶ Y may be discontinuous ! backward: us option then game option...

Link with PDEs

$Y = u(t, X_t, H_t)$, u solution of a system of VI.

- ▶ For $t \in [T_1, T_2]$: $u(t, x, 1)$ solution of VI for Game option, $u(t, x, 0)$ solution of VI for American option, with terminal condition g at T_2 .
- ▶ idem for $t \in [T_0, T_1)$ but with terminal condition
$$(u(T_1, x, 1) \vee \ell(x)) \wedge h(x)\mathbf{1}_{\{x \geq \bar{s}\}} + u(T_1, x, 0) \vee \ell(x)\mathbf{1}_{\{x < \bar{s}\}}$$
- ▶ may become very complex and practically untractable...

Scheme for the BSDEs

- ▶ Approximation of switching time (ϑ_I^π): $\mathbf{1}_{\{X_{T_I} \geq \bar{s}\}} \rightarrow \mathbf{1}_{\{X_{T_I}^\pi \geq \bar{s}\}}$
- ▶ Starting from the terminal condition $\tilde{Y}_T^\pi = Y_T^\pi := g(X_T^\pi)$ then compute at each step

$$\begin{cases} \bar{Z}_{t_i}^\pi &= (t_{i+1} - t_i)^{-1} \mathbb{E} \left[Y_{t_{i+1}}^\pi (W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i} \right] \\ \tilde{Y}_{t_i}^\pi &= \mathbb{E} \left[Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i} \right] + (t_{i+1} - t_i) f(X_{t_i}^\pi, Y_{t_i}^\pi, \bar{Z}_{t_i}^\pi) \\ Y_{t_i}^\pi &= \tilde{Y}_{t_i}^\pi \mathbf{1}_{\{t_i \notin \mathfrak{R}\}} + \mathcal{P}(t_i, X_{t_i}^\pi, \tilde{Y}_{t_i}^\pi) \mathbf{1}_{\{t_i \in \mathfrak{R}\}} \end{cases}$$

- ▶ now: $\mathcal{P}(t, x, y) := y + [\ell(x) - y]^+ - [y - h(x)]^+ \sum_{I \geq 1} \mathbf{1}_{\{\vartheta_{2I-1}^\pi \leq t \leq \vartheta_{2I}^\pi\}}$

Convergence results

Crépey-Chassagneux (10)

- ▶ first need to control $\mathbb{E}|\vartheta - \vartheta^\pi|$ (regularity on σ around \bar{s})
- ▶ to obtain (semi-convex boundaries)

$$\mathcal{E}rr(Y, Y^\pi) \leq \frac{C}{n^{\frac{1}{4}}}$$

- ▶ but... f does not depend on Z : because upper boundary is irregular.