Credit-equity models and High-Throughput Computing

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Summary

- ▶ Fundamental Theorem and the Meta-Theoretical Question of Decidability
- ▶ Mathematics and Technology
- ▶ Algebraization of Probability Theory
- ▶ System Architecture
- ▶ Applications

Co-authors of referenced papers

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Introduction: Stating a Decidable Question

- ▶ Questions in Finance are often hard. CDO modeling, counterparty risk and hybrid modeling are all hard questions.
- ▶ Sometimes questions are so hard that they are not decidable in a way that is consistent with basic theoretical principles.
- ▶ When questions are not decidable by honest means, one either arrives to a theoretically incosistent answer using analogy or one recognizes a limitation and gives up.
- ▶ It is better to give up than to come up with a patched up solution!

Fundamental Theorem

The basic principle behind valuation theory lies in the Fundamental Theorem of Finance invented by de Finetti in 1931.
The Fundamental Theorem rests upon three pillars:

- ▶ The principle of no-arbitrage according to which there are no riskless profits. In Medieval times this principle was cast in Law while free markets enforce it endogenously (supposedly);

- ▶ Aristotelian temporal modal logic, the language of financial contracts;

- ▶ The theory of duality of systems of linear inequalities initiated by Fourier and advanced by Farkas.

Fundamental Theorem

De Finetti establishes the fundamental duality relationship between prices and probabilities showing that the condition of absence of arbitrage can be expressed as a system of linear inequalities whose solutions are all given by a probabilistic representation of asset prices as discounted expectation from a **single** measure.
In formulas:

$$\frac{A_t}{g_t} = E_t^{Q(g)} \left[\frac{A_T}{g_T} \right] \tag{1}$$

where A_t is an asset price process and g_t is any numeraire process. The measure $Q(g)$ depends on g.

Local Valuation

The Fundamental Theorem is not only the most basic result in pricing theory, it is also the single most misrepresented and most ignored one!

Current Financial Engineering is predicated on the principle of **local valuation**: each instrument is assigned its own measure calibrated to a few instruments designated as hedging vehicles. Local valuation examples are:

- ▶ copula models for CDOs
- ▶ counterparty credit risk methodologies where scenarios are generated historically and portfolios revalued with instrument specific models in a big loop.
- ▶ hybrid models where interest rates are modeled by simple HW processes that are instrument specific and don't calibrate to the swaption cube, CMS spreads, etc..

Local Valuation

▶ Local valuation is a patched up methodology which is both ubiquitous and theoretically incosistent.

▶ It is not rooted in theory but is deceivingly presented using arguments by analogy.

▶ The net effect of lack of consistency is that it opens the way for creating a myriad of local bubbles which tend to burst in a highly correlated fashion.

▶ Hence model risk, i.e. the risk of breaking the Fundamental Theorem, is highly correlated.

Global Valuation

- ▶ Global valuation is based on the principle that one has to insist on the Fundamental Theorem.

- ▶ Each risk factor needs to be represented by one single model estimated against the broadest possible universe of assets sensitive to that factor alone.

- ▶ Models for risk management need to be related to models for pricing by a controllable and directly estimated risk adjustment.

- ▶ Each bank should only have one interest rate model for swaptions, CMSs, hybrids, risk management, etc..

- ▶ Each corporate name should have a single model for its equity and CDS processes.

Global Valuation

- ▶ Global valuation is computationally hard.
- ▶ Calibrating globally is only possible if one insists on having economically realistic processes.
- ▶ Economic realism can be achieved parsimoniously with few parameters but cannot be achieved within the class of analytically solvable models.

Decidability

- ▶ The bad news is that global valuation may lead one to posing questions which are not decidable. If they are not decidable, one should not force an incosistent answer by analogy but one should simply recognize that the question is computationally too hard.

- ▶ The good news is that global valuation is decidable nowadays.

Mathematics and Technology

- ▶ Decidability is a meta-theoretical problem:
- ▶ Given the current technology environment, can one build a mathematical framework within which one can produce inventions enabling global valuation?

Mathematics and Technology

▶ Mathematics depends on hardware, not the other way around!

▶ Analytic solutions are a result of technology that enable mankind to evaluate power series expansions since they were invented by the Kermala school and used by the first "computers" (i.e. graduate students) to handicraft tables.

▶ Early computer generations also aimed at evaluating special functions.

▶ In addition to that, computers enable the use of two other techniques: semi-implicit schemes based on sparse linear algebra and Monte Carlo simulation.

▶ Now we live in interesting times as the situation is changing again. as a further class of base algorithms is technologically available: High Throughput Computing microchips enable high performance matrix multiplication.

Mathematics and Technology

- ▶ Matrix multiplication enables one to find transition probability kernels for any process specified by a Markov generator.

- ▶ Analytic solvability is not any more a requirement for pricing models.

- ▶ Calibration can be executed by backward induction with generic model specifications against generic payoffs.

- ▶ Monte Carlo simulations can be carried out against transition probability kernels obtained numerically and persisted in memory as opposed to requiring algorithms leveraging on analytic closed form solvability.

Algebraization of Probability Theory

- ▶ To take advantage of the newly found ability to multiply matrices, Mathematics needs to adapt and change.
- ▶ Probability Theory needs to be algebraized.
- ▶ All useful constructs in stochastic calculus can be expressed by means of matrix manipulations such as multiplication and numerical differentiation.
- ▶ At the algebraic level, virtually all these constructs appear also in the Quantum Mechanics formalism, although applications are organized along very different usage patterns.

Algebraization of Probability Theory

▶ The numerical algorithm I find as being the one of most pivotal importance is fast exponentiation.

▶ Fast Exponentiation was known since the Greeks as the most effective way of valuing an exponential.

▶ In probability theory, Fast Exponentiation takes the premise from a transition probability kernel over a short time interval written as follows:

$$u_{\delta t} = \mathbb{I} + \delta t \mathcal{L}. \qquad (2)$$

where $\mathcal{L}$ is a Markov generator matrix.

▶ Assuming that $u_{\delta t}$ has positive elements, Fast Exponentiation then goes as follows;

$$u_{2\delta t} = u_{\delta t} \cdot u_{\delta t}, u_{4\delta t} = u_{2\delta t} \cdot u_{2\delta t}, \dots u_{2^n \delta t} = u_{2^{n-1}\delta t} \cdot u_{2^{n-1}\delta t}. \qquad (3)$$

Algebraization of Probability Theory

▶ In order for $u_{\delta t}$ to be a matrix with positive elements, the time step δt needs to satisfy the CFL (Courant-Friederichs-Levy) condition.

▶ The CFL condition is the single most misrepresented and most misunderstood condition in numerical analysis.

▶ It is called a "curse" because it often implies that δt is small. That was justified on early hardware where memory was at a premium, matrices could not be stored in full format and consequently FE was not available. But FE proceeds in a log number of steps, so the smallness of δt is not a problem any longer.

▶ Nevertheless, numerical analysis for PDEs is still defining itself as the discipline of techniques to get around the CFL condition.

Algebraization of Probability Theory

- ▶ What is often not recognized is that the CFL condition implies a strong form of stability, while breaking it comes at a cost of introducing marginal instabilities for high frequency modes.
- ▶ In particular, while FE is stable in single precision thanks to the CFL condition, semi-implicit methods are only stable in double precision.
- ▶ The empirical evidence is stunning. A Mathematical explanation is subtle and involves estimating how errors of different sign tend to cancel against each other because of the intrinsic smoothing properties of discretized diffusion processes obeying CFL. See these papers on my website for formal results in this direction:

C. Albanese: *Stochastic Integrals and Abelian Processes*

C. Albanese: *Kernel Convergence Estimates for Diffusions with*

System Architecture

Using fast exponentiation and matrix multiplication hardware, it is possible to build a valuation and simulation engine which

- ▶ Supports all model specifications represented as a Markov matrix with 512-1024 state variables;
- ▶ Supports all asset classes;
- ▶ Implements forward induction for term structure fitting;
- ▶ Implements (possibly multipass) backward induction for concurrent valuation of portfolios single factor derivatives;
- ▶ Implements a general purpose global calibration optimizer;
- ▶ Implements correlated scenario generation using dynamic Gaussian copulas.

System Architecture

High density GPU nodes as a matrix factory

High density CPU nodes for simulations

System Architecture

- ▶ The hardware entails a hybrid architecture combining 4-GPU boards (with Teslas) and 48-core Magnycour boards.
- ▶ The multi-GPU boards are used as a calibration engine and as a matrix factory.
- ▶ The Magnycour boards have a quarter terabyte of memory and are used for scenario simulation.
- ▶ Applications currently running: counterparty credit risk (covering FX, rates, equity and CDSs), CMS spreads, FX TARNs, CPPIs.
- ▶ Extensions to CDOs and commodity derivatives are in the works.
- ▶ **A single engine runs them all.**

System Architecture

The job of myself as a Mathematician as I interpret it involves

- ▶ Designing a small family of a handful of base calls that capture the numerical bottleneck and is worth optimizing in a hardware specific fashion using assembler;

- ▶ Orchestrating the entire world of valuation and risk management algorithms around the base interfaces by defining a framework for probability theory in the process.

System Architecture

▶ I released OPLib as a service to hardware manufacturers with my base calls. OPLib is on my website and is an open source distribution under GPL.

▶ I released various papers on probability theory and pricing theory using operator methods. A good place to start would be this one:

C. Albanese, H. Li: *Monte Carlo Pricing Using Operator Methods and Measure Changes*

System Architecture

▶ Global calibration at the bank level is not a small undertaking. It is a multi-exaflop exercise leveraging on GPU technology.

▶ Current Teslas offer sustained performance on matrix multiplcation of 360GF/sec while AMD-ATI Cypress class microchips offer 1.6 TF/sec.

▶ GPUs are often misunderstood as Monte Carlo engines. They are not as performance is seriously hampered by asynchronous thread branching.

▶ Monte Carlo simulation using look-up tables is heavily memory intensive. A good fraction of a terabyte of memory is recommendable. Performance is also sensitive to cache sizes.

▶ Current cutting edge boards include the AMD 48-core Magnycour and the Intel 12-core Westmere (with hyperthreading).

Organization of Labor

- ▶ Centering operations on a model agnostic valuation and risk management engine allows organizations to decouple the four poles of engineering, modeling, contract capture and trading.
- ▶ Engineers can maintain the engine without having to be aware of model and payoff specifications.
- ▶ Economists can maintain models and calibrate them on the engine.
- ▶ Lexical analyzers can turn term sheets into classes that capture payoffs generically.
- ▶ Traders can focus on business development.

Applications: CDOs and credit-equity derivatives

▶ It is possible to generate in the order of 50,000 dynamic scenarios per second, quarter by quarter to 10 years, with 100 risk factors.

▶ This implies that bottom up, structural credit-equity models for CDOs **are decidable** with Monte Carlo simulations.

▶ Work I published in the years 2004-2008 on CDO modeling with Alicia Vidler applies in this framework.

▶ Credit-equity calibration is possible through the use of the general purpose optimizer.

▶ The algebraic technique of dynamic conditioning that was needed in my earlier papers [and based on binomial copulas] can now be replaced by full fledged Monte Carlo simulation with Gaussian copulas thanks to the new high density CPU boards. Work in this direction is in progress.

Applications: CDOs and credit-equity derivatives

C. Albanese, A. Vidler: *Dynamic Conditioning and Credit Correlation Baskets*
C. Albanese, A. Vidler: *A Structural Model for Credit-Equity Derivatives and Bespoke CDOs*

Applications: Counterparty credit risk

- ▶ It is possible to develop on the very same engine a counter-party credit risk application going out to 30 years and of similar efficiency.

- ▶ It is possible to avoid the "big loop" requiring separate calibration-valuation of each instrument by pre-processing valuation tables.

- ▶ Valuation tables give prices of portfolios of single factor instruments as a function of state variables. They can be obtained concurrently in a pre-processing phase.

- ▶ Once faced the initial stumbling block of an investment into global calibration, just for the sake of avoiding the big loop performance is accelerated by a factor of about 10,000.

Applications: Hybrids

- ▶ I successfully used the very same engine and framework on exotics and long maturity hybrids such as complex CPPI products, CMS spreads, FX TARNs

- ▶ The marginal cost for adding a new instrument is reduced to the cost of implementing a full description of the term sheet as a class which embeds model attributes and cash flow generators.

- ▶ The engine is generic and marginal calibration to single factors has high quality.

Conclusions

▶ The advent of high throughput microchips finally allows to decide complex financial questions while respecting the Fundamental Theorem of Finance.

▶ The calibration stage is a computationally heavy data mining exercise across all derivatives of known prices. Nowadays, this task is feasible.

▶ Assuming global valuation, hard problems such as CDO and hybrid pricing and counterpart credit risk modeling become decidable respecting the basic theoretical principles.

General Papers

- ▶ Claudio Albanese, Guillaume Gimonet and Steve White: *An Introduction to Global Valuation*, to appear in the next issue (May 2010) of Risk Magazine
- ▶ Claudio Albanese, Guillaume Gimonet and Steve White: *Global Valuation and Dynamic Risk Management*, to appear in a forthcoming Risk book on the valuation of illiquid instruments