VaR vs CVaR in Risk Management and Optimization

Stan Uryasev

Joint presentation with

Sergey Sarykalin, Gaia Serraino and Konstantin Kalinchenko

Risk Management and Financial Engineering Lab, University of Florida

and

American Optimal Decisions

Agenda

- Compare Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR)
  - definitions of VaR and CVaR
  - basic properties of VaR and CVaR
  - axiomatic definition of Risk and Deviation Measures
  - reasons affecting the choice between VaR and CVaR
  - risk management/optimization case studies conducted with Portfolio Safeguard package by AORDA.com

Risk Management

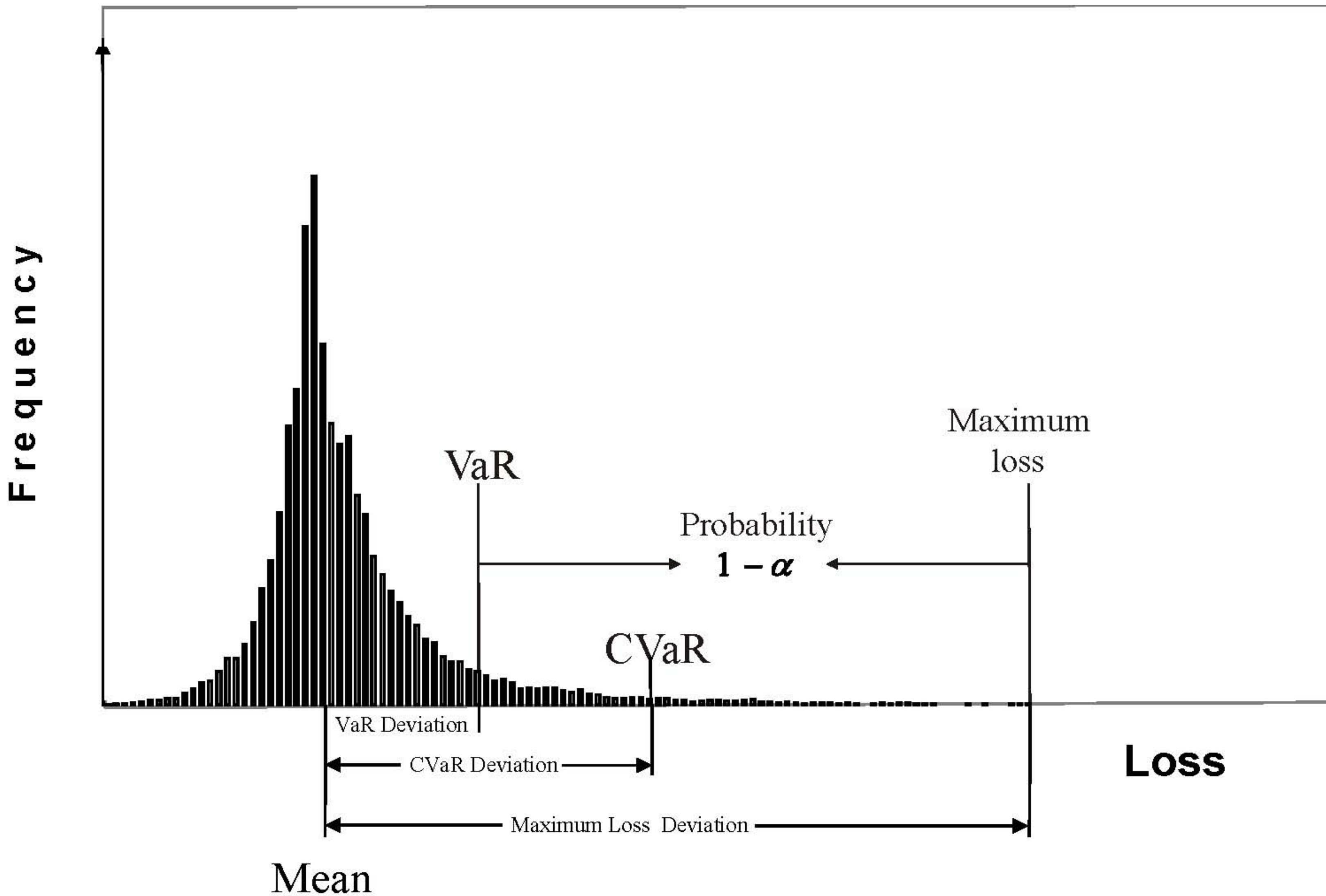
- Risk Management is a procedure for shaping a loss distribution
- Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) are popular function for measuring risk
- The choice between VaR and CVaR is affected by:
- differences in *mathematical properties*,
- *stability* of statistical estimation,
- *simplicity of optimization* procedures,
- *acceptance by regulators*

- *Conclusions* from these properties are *contradictive*

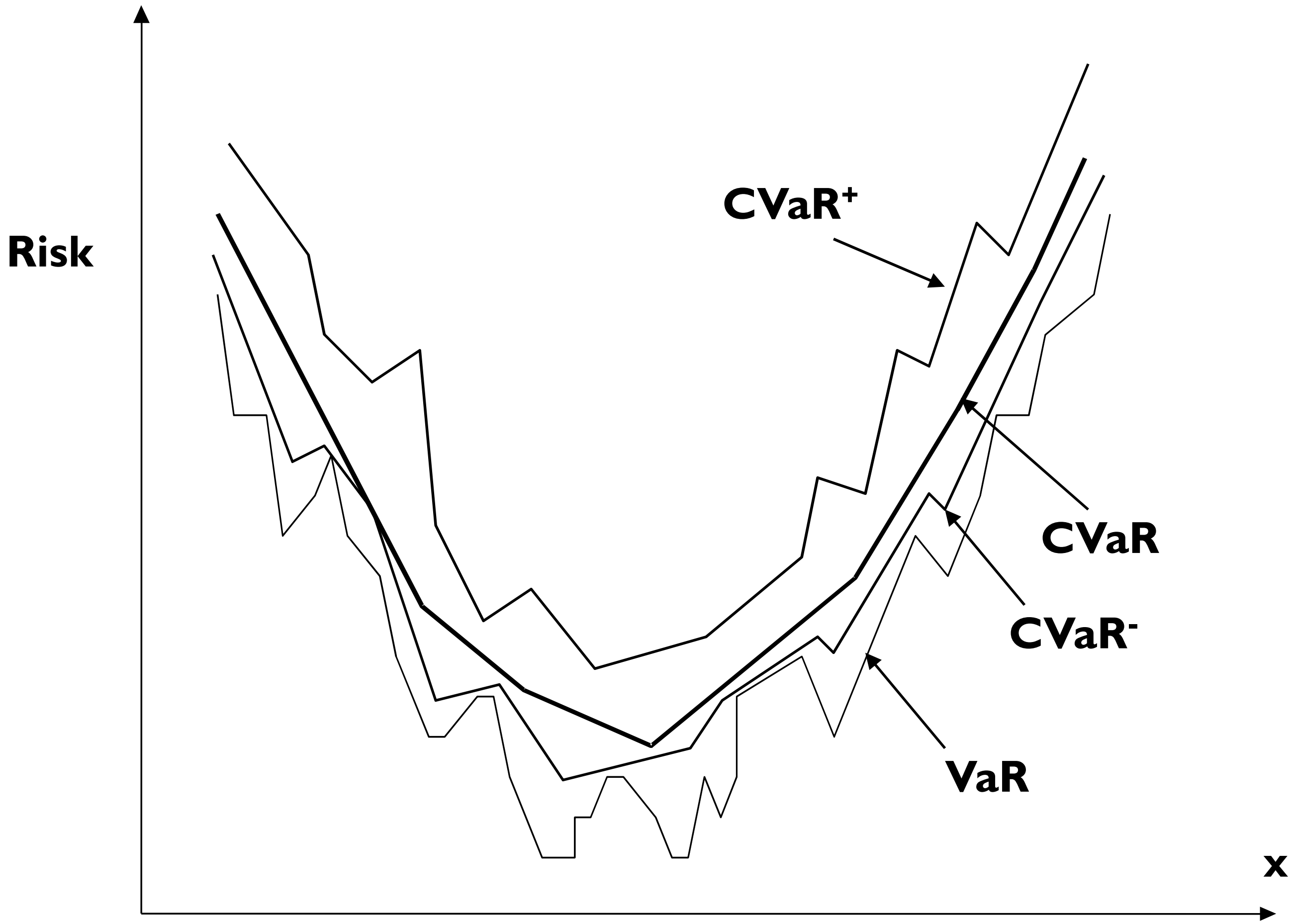
Risk Management

- Key observations:
  - CVaR has *superior mathematical properties* versus VaR
  - *Risk management* with *CVaR* functions can be done very *efficiently*
  - VaR *does not control* scenarios exceeding VaR
  - CVaR *accounts* for losses exceeding VaR
  - *Deviation* and *Risk* are different risk management concepts
  - *CVaR Deviation* is a strong *competitor to* the *Standard Deviation*

VaR and CVaR Representation



VaR, CVaR, CVaR$^+$ and CVaR$^-$



Value-at-Risk

X a *loss* random variable

$$VaR_\alpha(X) = \min\{z \mid F_X(z) \geq \alpha\} \quad \text{for} \quad \alpha \in]0,1[$$

- $VaR_\alpha(X)$ is non convex and discontinuous function of the confidence level α for discrete distributions
- $VaR_\alpha(X)$ is non-sub-additive
- difficult to control/optimize for non-normal distributions: VaR has many extremums for discrete distributions

Conditional Value-at-Risk

- Rockafellar and Uryasev, "*Optimization of Conditional Value-at-Risk*", Journal of Risk, 2000 introduced the term Conditional Value-at-Risk

- For $\alpha \in]0,1[$

$$CVaR_\alpha(X) = \int_{-\infty}^{+\infty} z dF_X^\alpha(z)$$

where

$$F_X^\alpha(z) = \begin{cases} 0 & \text{when } z < VaR_\alpha(X) \\ \dfrac{F_X(z) - \alpha}{1-\alpha} & \text{when } z \geq VaR_\alpha(X) \end{cases}$$

Conditional Value-at-Risk

- **CVaR⁺ (Upper CVaR):** expected value of X strictly exceeding VaR (also called Mean Excess Loss and Expected Shortfall)

$$CVaR_\alpha^+(X) = E[X \mid X > VaR_\alpha(X)]$$

- **CVaR⁻ (Lower CVaR):** expected value of X weakly exceeding VaR (also called Tail VaR)

$$CVaR_\alpha^-(X) = E[X \mid X \geq VaR_\alpha(X)]$$

Property: $CVaR_\alpha(X)$ is weighted average of $CVaR_\alpha^+(X)$ and $VaR_\alpha(X)$

$$CVaR_\alpha(X) = \begin{cases} \lambda_\alpha(X)\, VaR_\alpha(X) + (1-\lambda_\alpha(X))\, CVaR_\alpha^+(X) & \text{if } F_X(VaR_\alpha(X)) < 1 \\ VaR_\alpha(X) & \text{if } F_X(VaR_\alpha(X)) = 1 \end{cases}$$

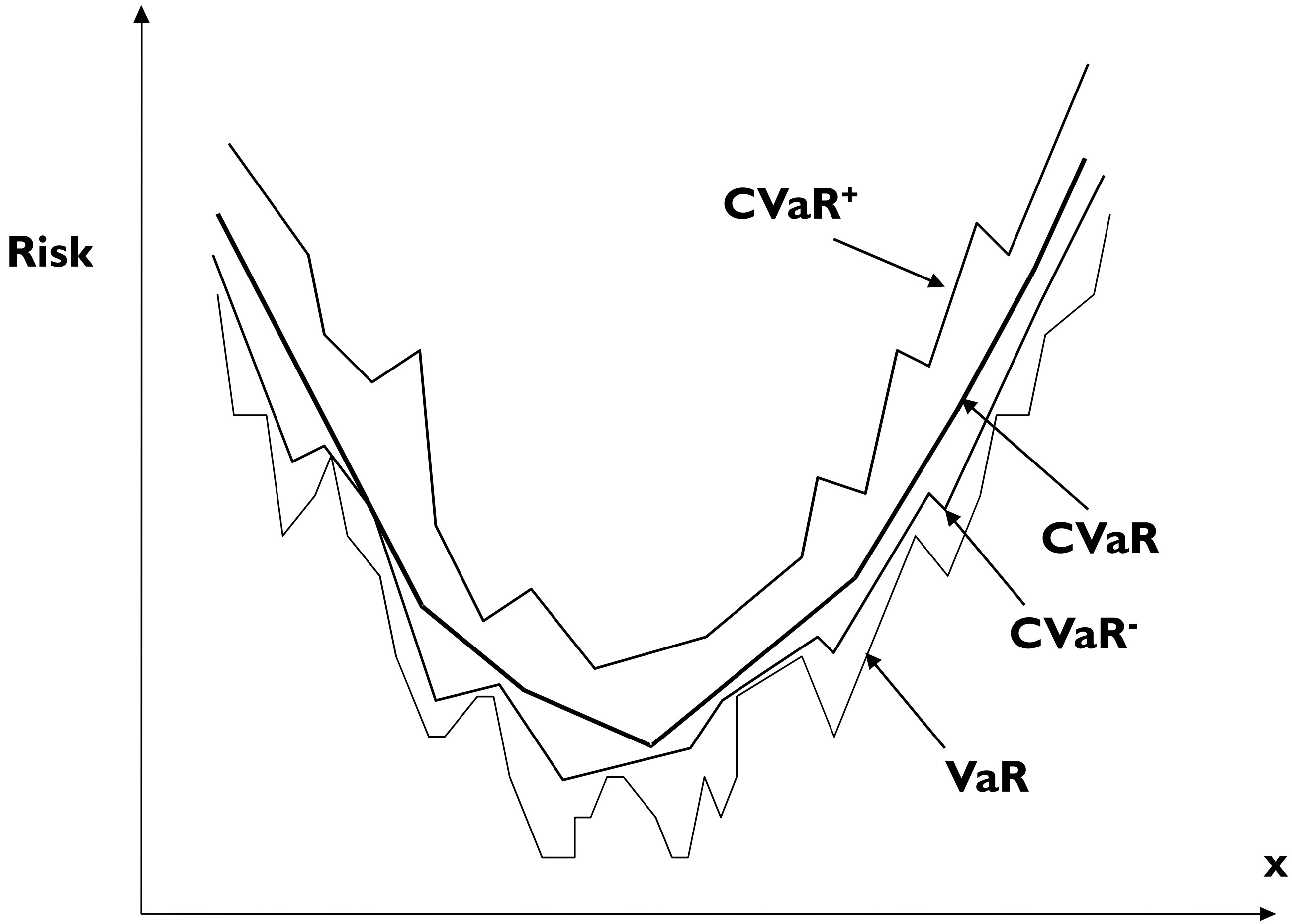
$$\lambda_\alpha(X) = \frac{F_X(VaR_\alpha(X)) - \alpha}{1-\alpha}$$

zero for continuous distributions!!!

Conditional Value-at-Risk

- Definition on previous page is a major innovation
- $CVaR_{\alpha}^{+}(X)$ and $VaR_{\alpha}(X)$ for general loss distributions are discontinuous functions
- CVaR is continuous with respect to α
- CVaR is convex in X
- VaR, $CVaR^{-}$,$CVaR^{+}$ may be non-convex
- $VaR \leq CVaR^{-} \leq CVaR \leq CVaR^{+}$

VaR, CVaR, CVaR$^+$ and CVaR$^-$

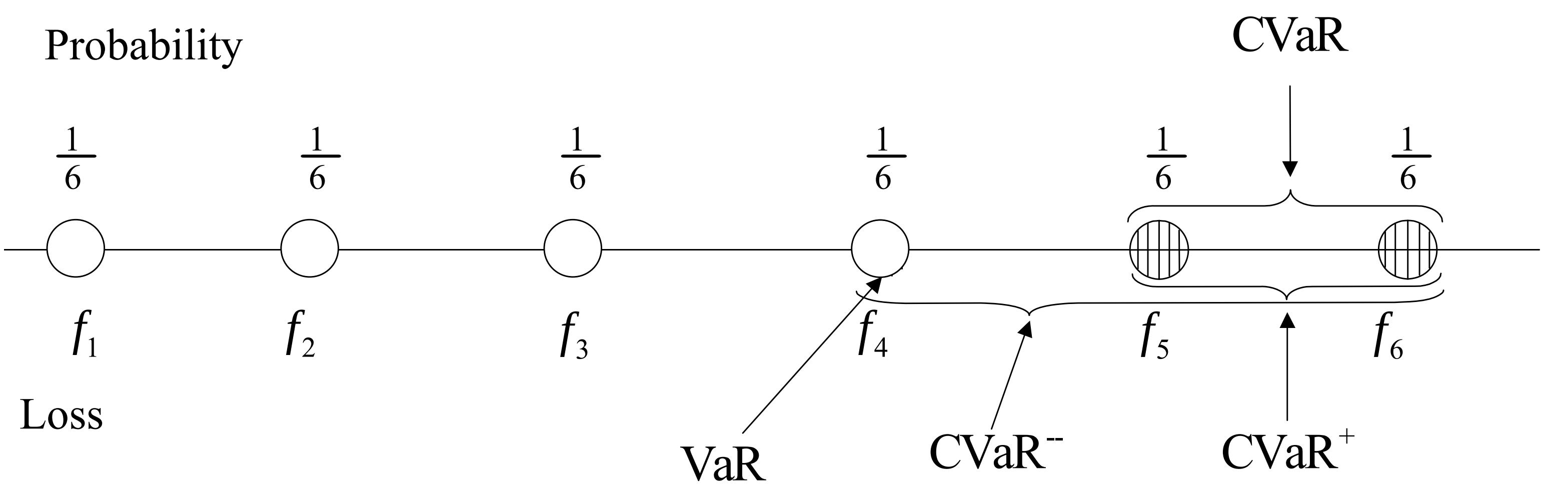


CVaR: Discrete Distributions

- α does not "split" atoms: VaR < CVaR$^-$ < CVaR = CVaR$^+$, $\lambda = (\Psi - \alpha)/(1-\alpha) = 0$

$$\text{Six scenarios,} \quad p_1 = p_2 = \cdots = p_6 = \tfrac{1}{6}, \;\; \alpha = \tfrac{2}{3} = \tfrac{4}{6}$$

$$\text{CVaR} = \text{CVaR}^+ = \tfrac{1}{2} f_5 + \tfrac{1}{2} f_6$$

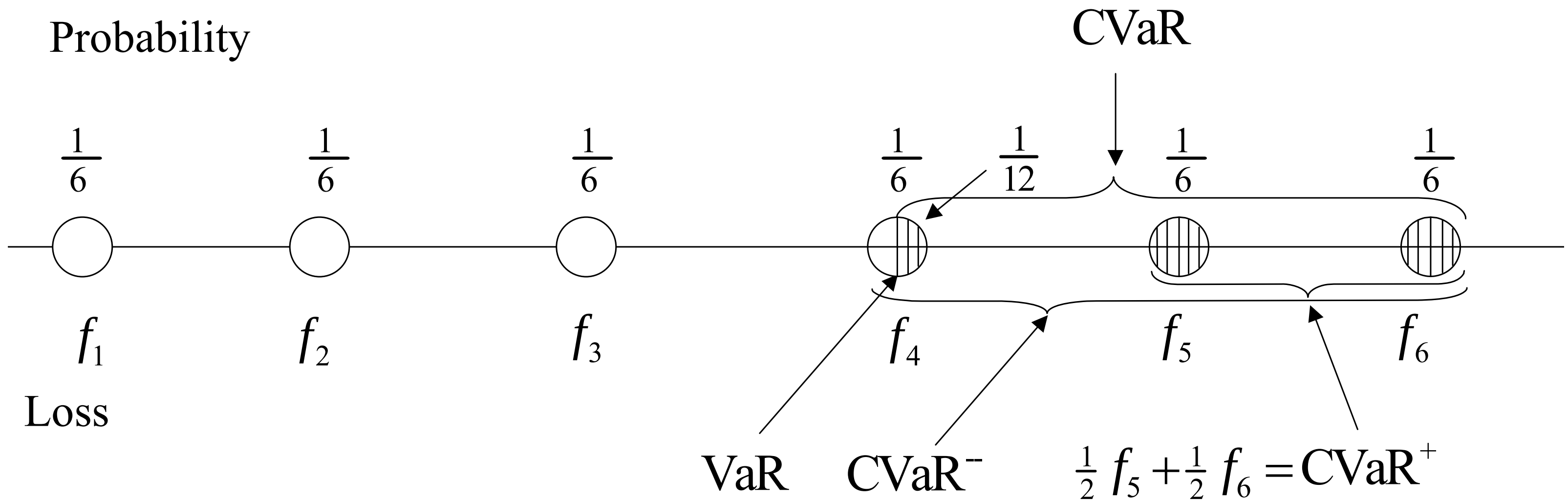


CVaR: Discrete Distributions

- α "splits" the atom: VaR < CVaR$^-$ < CVaR < CVaR$^+$,
  $\lambda = (\Psi - \alpha)/(1 - \alpha) > 0$

$$\text{Six scenarios,} \quad p_1 = p_2 = \cdots = p_6 = \tfrac{1}{6}, \;\; \alpha = \tfrac{7}{12}$$

$$\text{CVaR} = \tfrac{1}{5}\text{VaR} + \tfrac{4}{5}\text{CVaR}^+ = \tfrac{1}{5} f_4 + \tfrac{2}{5} f_5 + \tfrac{2}{5} f_6$$

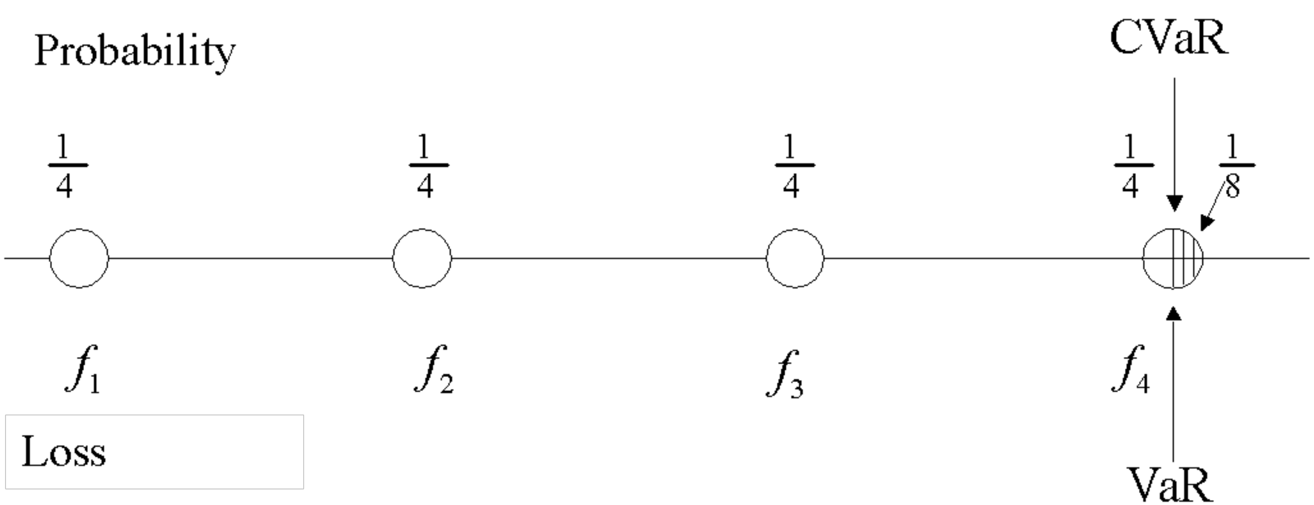


CVaR: Discrete Distributions

- α "splits" the last atom: VaR = CVaR$^-$ = CVaR , CVaR$^+$ is not defined, $\lambda = (\Psi - \alpha)/(1 - \alpha) > 0$

$$\text{Four scenarios, } \quad p_1 = p_2 = p_3 = p_4 = \tfrac{1}{4}, \ \alpha = \tfrac{7}{8}$$

$$\text{CVaR} = \text{VaR} = f_4$$



CVaR: Equivalent Definitions

- Pflug defines CVaR via an optimization problem, as in Rockafellar and Uryasev (2000)

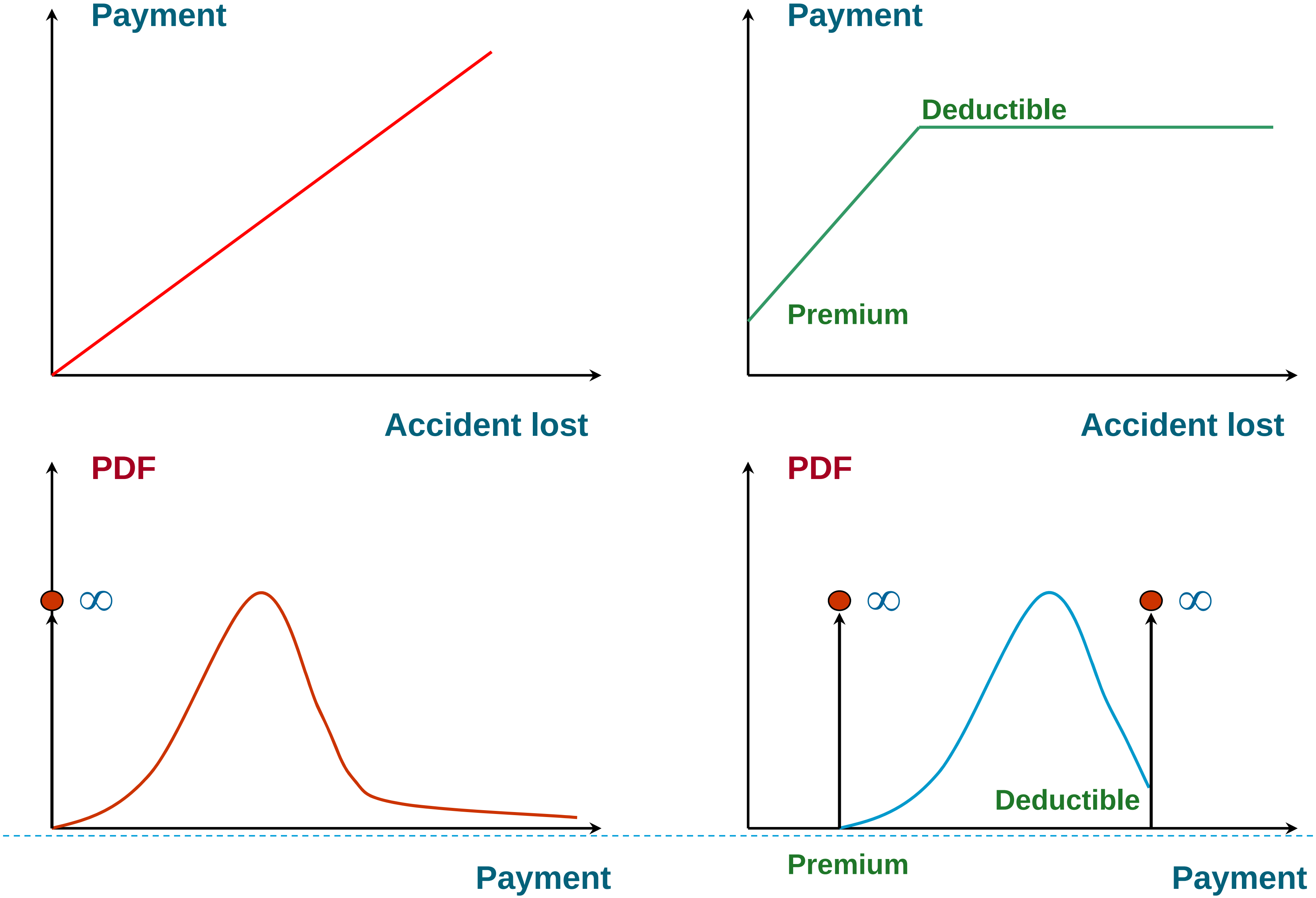
$$CVaR_\alpha(X) = \min_C \left\{ C + \frac{1}{1-\alpha} E[X - C]^+ \right\}$$

- Acerbi showed that CVaR is equivalent to Expected Shortfall defined by

$$CVaR_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha VaR_\beta(X)\, d\beta$$

Pflug, G.C., "Some Remarks on the Value-at-Risk and on the Conditional Value-at-Risk", Probabilistic Constrained Optimization: Methodology and Applications, (Uryasev ed), Kluwer, 2000
Acerbi, C., "Spectral Measures of Risk: a coherent representation of subjective risk aversion", JBF, 2002

RISK MANAGEMENT: INSURANCE



- Risk as a possible loss

Minimum amount of cash to be added to make a portfolio (or project) sufficiently safe

Example 1. MaxLoss

- Three equally probable outcomes, { -4, 2, 5 };
MaxLoss = -4; Risk = 4
- Three equally probable outcomes, { 0, 6, 9 };
MaxLoss = 0; Risk = 0

Risk as an uncertainty in outcomes

Some measure of deviation in outcomes

Example 2. Standard Deviation

- Three equally probable outcomes, { 0, 6, 9 }; Standard Deviation > 0

Risk Measures: axiomatic definition

- A functional $\mathcal{R}: \mathcal{L}^2 \to]-\infty,\infty]$ is a *coherent risk measure in the extended sense* if:

R1: $\mathcal{R}(C) = C$ for all constant C

R2: $\mathcal{R}\big((1-\lambda)X + \lambda X'\big) \le (1-\lambda)\mathcal{R}(X) + \lambda\mathcal{R}(X')$ for $\lambda \in]0,1[$ (convexity)

R3: $\mathcal{R}(X) \le \mathcal{R}(X')$ when $X \le X'$ (monotonicity)

R4: $\mathcal{R}(X) \le 0$ when $||X^k - X||_2 \to 0$ with $\mathcal{R}(X^k) \le 0$ (closedness)

- A functional $\mathcal{R}: \mathcal{L}^2 \to]-\infty,\infty]$ is a *coherent risk measure in the basic sense* if it satisfies axioms R1, R2, R3, R4 and R5:

R5: $\mathcal{R}(\lambda X) = \lambda\mathcal{R}(X)$ for $\lambda > 0$ (positive homogeneity)

Risk Measures: axiomatic definition

- A functional $\mathcal{R}: \mathcal{L}^2 \to]-\infty, \infty]$ is an *averse risk measure in the extended sense* if it satisfies axioms R1, R2, R4 and R6:

R6: $\mathcal{R}(X) > EX$ for all nonconstant X (aversity)

- A functional $\mathcal{R}: \mathcal{L}^2 \to]-\infty, \infty]$ is an *averse risk measure in the basic sense* if it satisfies axioms R1, R2, R4, R6 and R5

- Aversity has the interpretation that the risk of loss in a nonconstant random variable X *cannot be acceptable* unless EX<0

- R2 + R5 $\longrightarrow \mathcal{R}(X + X') \leq \mathcal{R}(X) + \mathcal{R}(X')$ (subadditivity)

Risk Measures: axiomatic definition

- Examples of coherent risk measures:
  - $\mathcal{R}(X) = E[X]$
  - $\mathcal{R}(X) = \sup X$
- Examples of risk measures not coherent:
  - $\mathcal{R}(X) = E[X] + \lambda\sigma(X)$, $\lambda>0$, violates R3 (monotonicity)
  - $\mathcal{R}(X) = VaR_\alpha(X)$ violates subadditivity

- $\mathcal{R}(X) = CVaR_\alpha(X)$ for $\alpha\in]0,1]$ is a coherent measure of risk in the basic sense and it is an averse measure of risk !!!

- Averse measure of risk might not be coherent, a coherent measure might not be averse

Deviation Measures: axiomatic definition

A functional $\mathcal{D}: \mathcal{L}^2 \to [0,\infty]$ is called a *deviation measure in the extended sense* if it satisfies:

D1: $\mathcal{D}(C) = 0$ for constant *C*, but $\mathcal{D}(X) > 0$ for nonconstant *X*

D2: $\mathcal{D}((1-\lambda)X + \lambda X') \leq (1-\lambda)\mathcal{D}(X) + \lambda\mathcal{D}(X')$ for $\lambda \in]0,1[$ (convexity)

D3: $\mathcal{D}(X) \leq d$ when $||X^k - X||_2 \to 0$ with $\mathcal{D}(X^k) \leq d$ (closedness)

A functional $\mathcal{D}: \mathcal{L}^2 \to [0,\infty]$ is called a *deviation measure in the basic sense* if it satisfies axioms D1,D2, D3 and D4:

D4: $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ (positive homogeneity)

A deviation measure in extended or basic sense is also *coherent* if it additionally satisfies D5:

D5: $\mathcal{D}(X) \leq \sup X - E[X]$ (upper range boundedness)

Deviation Measures: axiomatic definition

- Examples of deviation measures in the basic sense:
  - Standard Deviation
  - Standard Semideviations
  - Mean Absolute Deviation
- *α-Value-at-Risk Deviation measure*:

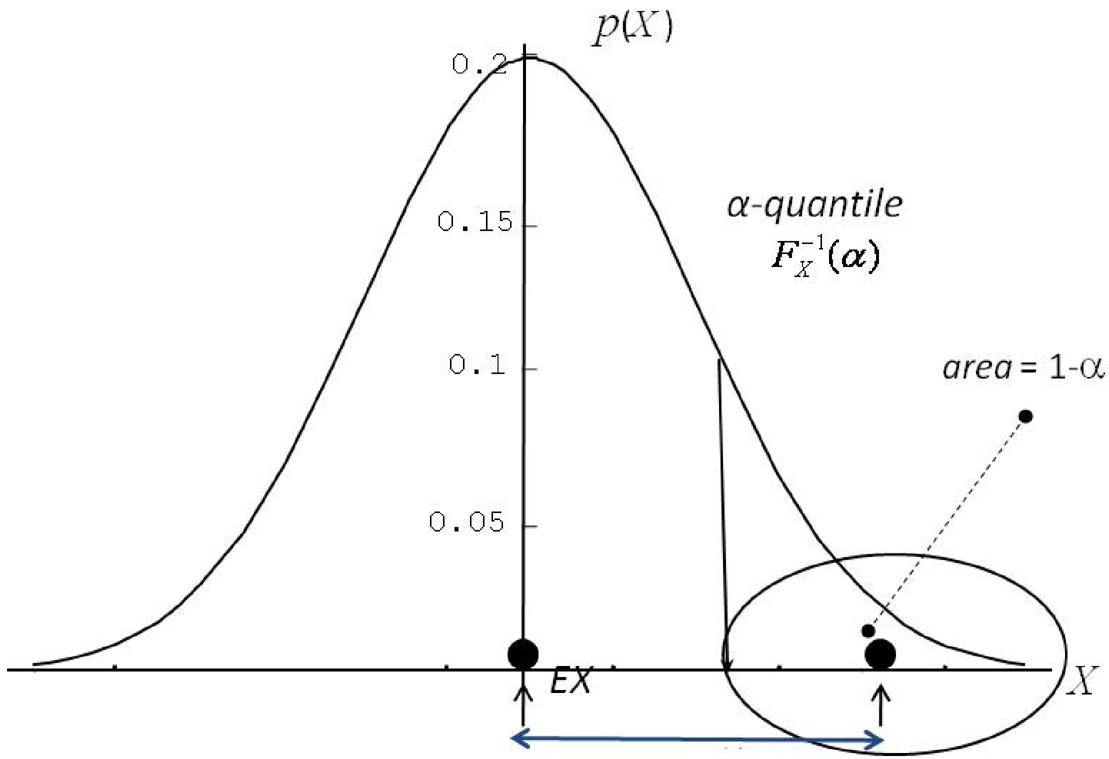
$$VaR_{\alpha}^{\Delta}(X) = VaR_{\alpha}(X - EX)$$

- α-VaR Dev does not satisfy convexity axiom D2 $\longrightarrow$ it is not a deviation measure
- *α-Conditional Value-at-Risk Deviation measure:*

$$CVaR_{\alpha}^{\Delta}(X) = CVaR_{\alpha}(X - EX)$$

Coherent deviation measure in basic sense !!!

Deviation Measures: axiomatic definition



- *CVaR Deviation Measure is a coherent deviation measure in the basic sense*

Risk vs Deviation Measures

- Rockafellar et al. (2006) showed the existence of a *one-to-one correspondence* between deviation measures in the extended sense and averse risk measures in the extended sense:

$$\mathcal{D}(X) = \mathcal{R}(X - EX)$$

$$\mathcal{R}(X) = \mathcal{D}(X) + EX$$

$$\mathcal{R} \text{ is coherent} \leftrightarrow \mathcal{D} \text{ is coherent}$$

$$\mathcal{R} \text{ is positive homogeneous} \leftrightarrow \mathcal{D} \text{ is positive homogeneous}$$

Rockafellar, R.T., Uryasev, S., Zabarankin, M., "Optimality conditions in portfolio analysis with general deviation measures", Mathematical Programming, 2006

Risk vs Deviation Measures

Deviation Measure	Counterpart Risk Measure
$\sigma(X)$	$EX + \sigma(X)$
$CVaR_\alpha^\Delta(X)$	$CVaR_\alpha(X)$
$Mixed - CVaR_\alpha^\Delta(X)$	$Mixed - CVaR_\alpha(X)$

where

$$Mixed - CVaR_\alpha^\Delta(X) = \sum_{k=1}^{K} \lambda_k \, CVaR_{\alpha_k}^\Delta(X)$$

$$Mixed - CVaR_\alpha(X) = \sum_{k=1}^{K} \lambda_k \, CVaR_{\alpha_k}(X)$$

$\lambda_k \geq 0, \sum_{k=1}^{K} \lambda_k = 1$ and α_k in $]0,1[$

Chance and VaR Constraints

- Let $f_i(x,\omega),\ i=1,..m$ be some random loss function.
- By definition: $VaR_\alpha(x)=\min\{\varepsilon : \Pr\{f(x,\omega)\le\varepsilon\}\ge\alpha\}$
- Then the following holds:

$$\Pr\{f(x,\omega)\le\varepsilon\}\ge\alpha \leftrightarrow \mathrm{VaR}_\alpha(X)\le\varepsilon$$

- In general $VaR_\alpha(x)$ is *nonconvex w.r.t. x*, (e.g., discrete distributions)
- $\mathrm{VaR}_\alpha(X)\le\varepsilon$ and $\Pr\{f(x,\omega)\le\varepsilon\}\ge\alpha$ may be *nonconvex constraints*

VaR vs CVaR in optimization

- VaR is difficult to optimize numerically when losses are not normally distributed
- PSG package allows VaR optimization
- In optimization modeling, CVaR is superior to VaR:
  - For elliptical distribution minimizing VaR, CVaR or Variance is equivalent
  - CVaR can be *expressed as a minimization formula* (Rockafellar and Uryasev, 2000)
  - CVaR preserve convexity

CVaR optimization

$$F_\alpha(x,\zeta) = \zeta + \frac{1}{1-\alpha} E\{[f(x,\xi) - \zeta]^+\}$$

Theorem 1:

1. $F_\alpha(x,\zeta)$ is *convex* w.r.t. α

2. α -VaR is a *minimizer* of F with respect to ζ :

$$\mathrm{VaR}_\alpha\left(f(x,\xi)\right) = \zeta_\alpha\left(f(x,\xi)\right) = \arg\min_\zeta F_\alpha(x,\zeta)$$

3. α - CVaR *equals minimal value* (w.r.t. ζ) of function F :

$$\mathrm{CVaR}_\alpha\left(f(x,\xi)\right) = \min_\zeta F_\alpha(x,\zeta)$$

CVaR optimization

- Preservation of convexity: if $f(x,\xi)$ is convex in x then $CVaR_\alpha(X)$ is convex in x
- If $f(x,\xi)$ is convex in x then $F_\alpha(x,\zeta)$ is convex in x and ζ
- $\min_{x} CVaR_\alpha(x) = \min_{(x,\zeta)} F_\alpha(x,\zeta)$
- If $f(x^*,\zeta^*)$ minimizes F_α over $X \times \Re$ then

  $CVaR_\alpha(x^*) = F_\alpha(x^*,\zeta^*)$
- $\min_{x\in X} g(x)$ $s.t.$ $CVaR_\alpha(x) \leq \omega_i, i = 1,..,I$ is equivalent to

$$\min_{x,\zeta_1,\dots,\zeta_I \in X\times\Re\times\dots\times\Re} g(x)$$

$$s.t. \; F_{\alpha_i}(x,\zeta_i) \leq \omega_i, i = 1,..,I$$

CVaR optimization

- In the case of discrete distributions:

$$F_\alpha(x,\zeta) = \zeta + (1-\alpha)^{-1} \sum_{k=1}^{N} p_k \; [f(x,\xi^k) - \zeta]^+$$

$$\mathrm{z}^+ = \max \; \{\mathrm{z}, \; 0\}$$

- The constraint $F_\alpha(x,\zeta) \le \omega$ can be replaced by a system of inequalities introducing additional variables η_k :

$$\eta_k \ge 0, \quad f(x, y_k) - \zeta - \eta_k \le 0, \mathrm{k} = 1,.., \mathrm{N}$$

$$\zeta + \frac{1}{1-\alpha} \sum_{k=1}^{N} p_k \eta_k \le \omega$$

Generalized Regression Problem

- Approximate random variable Y by random variables $X_1, X_2, \ldots, X_n$.

$$\min \mathcal{E}(Y - [c_0 + c_1 X_1 + \cdots + c_n X_n])$$

- Error measure = satisfies axioms

(E1) $\mathcal{E}(0) = 0,\ \varepsilon(X) > 0, \mathit{for}\ X \neq 0$

(E2) $\mathcal{E}(\lambda X) = \lambda\, \mathcal{E}(X)$ for $\lambda \geq 0$

(E3) $\mathcal{E}(X + X') \leq \mathcal{E}(X) + \mathcal{E}(X')$

(E4) Lower semicontinuity

Rockafellar, R.T., Uryasev, S. and M. Zabarankin:
"*Risk Tuning with Generalized Linear Regression*", accepted for publication in Mathematics of Operations Research, 2008

Error, Deviation, Statistic

- For an error measure =:
- the corresponding deviation measure <   is

$$D(X) = \min_{C} \mathcal{E}(X - C)$$

- the corresponding statistic K is

$$S(X) = \operatorname*{argmin}_{C} \mathcal{E}(X - C)$$

Theorem: Separation Principle

- General regression problem

$$\min_{c_0, c_1, \ldots, c_n} \mathcal{E}(Y - [c_0 + c_1 X_1 + \cdots + c_n X_n])$$

is equivalent to

$$\min_{c_1, \ldots, c\_n} D\,(Y - [\,c_1 X_1 + \cdots + c_n X_n])$$
$$\text{s.t.} \quad c_0 \in S(Y - [\,c_1 X_1 + \cdots + c_n X_n])$$

Percentile Regression and CVaR Deviation

Error : $\mathcal{E}_{KB}^{\alpha} = E[X_+] + (\alpha^{-1} - 1)E[X_-]$

Deviation : $D(X) = CVaR_\alpha(X - EX)$

Risk: $R(X) = CVaR_\alpha(X)$

Statistic: $S(X) = -VaR_\alpha(X)$

Koenker, R., Bassett, G. Regression quantiles. Econometrica 46, 33–50 (1978)

$$\min_{C \in R}\left(E[X-C]_+ + (\alpha^{-1}-1)E[X-C]_-\right) = CVaR_\alpha(X-EX)$$

$$\arg\min_{C \in \Re}(E[X-C]_+ + (\alpha^{-1}-1)E[X-C]_- = VaR_\alpha(X)$$

Stability of Estimation

- VaR and CVaR with same confidence level measure "different parts" of the distribution
- For a specific distribution the confidence levels α_1 and α_2 for comparison of VaR and CVaR should be found from the equation

$$VaR_{\alpha_1}(X) = CVaR_{\alpha_2}(X)$$

- Yamai and Yoshiba (2002), *for the same confidence level:*
  - VaR estimators are more stable than CVaR estimators
  - The difference is *more prominent for fat-tailed distributions*
  - *Larger sample sizes* increase accuracy of CVaR estimation
  - *More research needed* to compare stability of estimators for the same part of the distribution.

Decomposition According to Risk Factors Contributions

- For continuous distributions the following decompositions of VaR and CVaR hold:

$$VaR_\alpha(X) = \sum_{i=1}^{n} \frac{\partial VaR_\alpha(X)}{\partial z_i} z_i = E[X_i \mid X = VaR_\alpha(X)] z_i$$

$$CVaR_\alpha(X) = \sum_{i=1}^{n} \frac{\partial CVaR_\alpha(X)}{\partial z_i} z_i = E[X_i \mid X \geq VaR_\alpha(X)] z_i$$

- When a distribution is modeled by scenarios it is easier to estimate $E[X_i \mid X \geq VaR_\alpha(X)]$ than $E[X_i \mid X = VaR_\alpha(X)]$
- Estimators of $\frac{\partial CVaR_\alpha(X)}{\partial z_i}$ are more stable than estimators of $\frac{\partial VaR_\alpha(X)}{\partial z_i}$

Generalized Master Fund Theorem and CAPM

- Assumptions:
- Several groups of investors each with $U_j(ER_{j,}\mathcal{D}_j(R_j))$ utility function
- Utility functions are <u>concave</u> w.r.t. mean and deviation

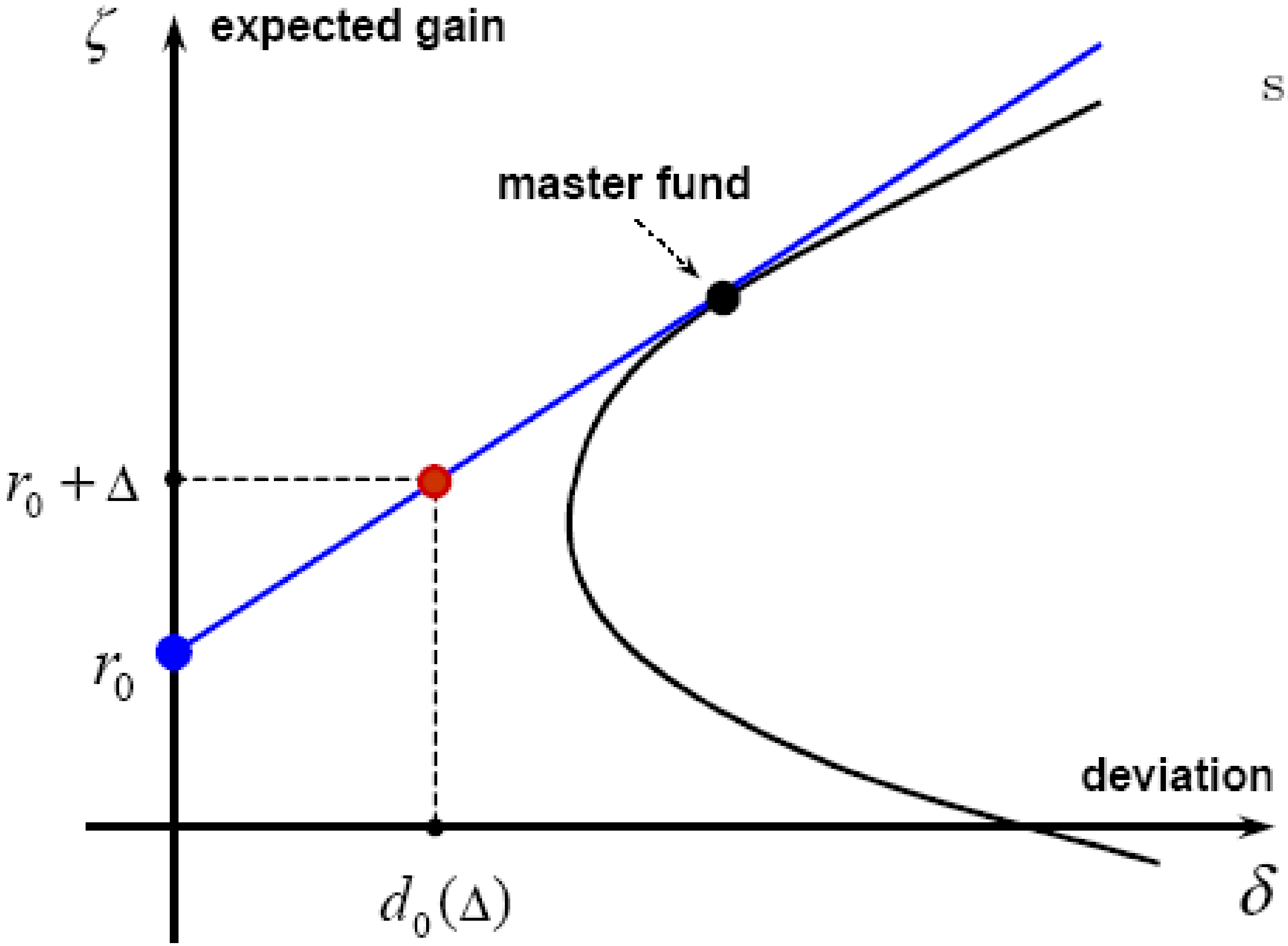
  <u>increasing</u> w.r.t. mean

  <u>decreasing</u> w.r.t. deviation
- Investors maximize utility functions s.t. budget constraint.

Rockafellar, R.T., Uryasev, S., Zabarankin, M. "Master Funds in Portfolio Analysis with General Deviation Measures", JBF, 2005
Rockafellar, R.T., Uryasev, S., Zabarankin, M. "Equilibrium with Investors using a Diversity of Deviation Measures", JBF, 2007

Efficient Set: Classical Theory

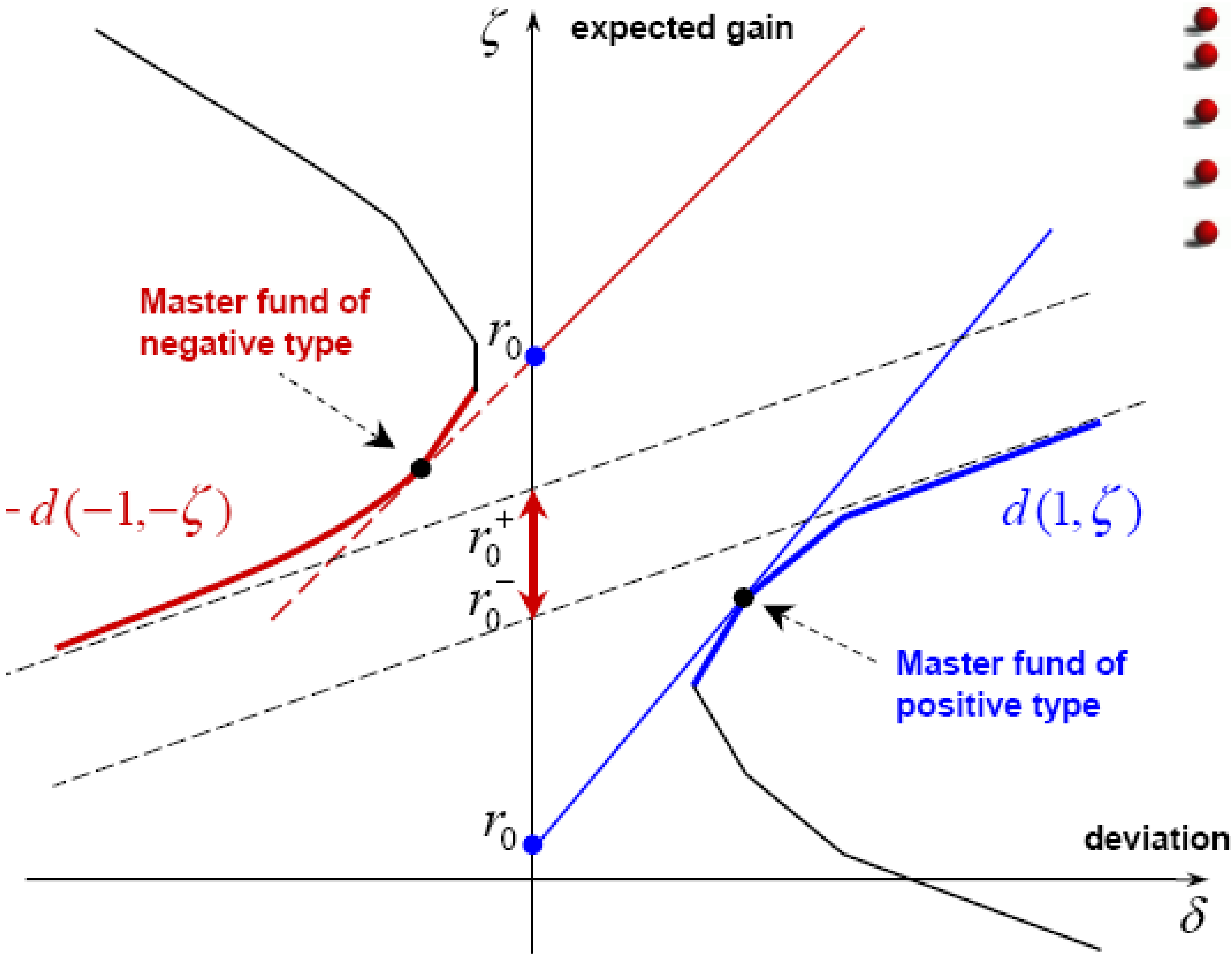
One Fund Theorem



Auxiliary Problem

$$\begin{aligned} \min_{y} \quad & \sigma(Y_p) \\ \text{s. t.} \quad & EY_p = \zeta \\ & y^\top e = 1 \end{aligned}$$

Efficient Sets: General Deviation



- convex
- slopes
- non-smooth
- non-symmetric
- threshold interval

Generalized Master Fund Theorem and CAPM

- Equilibrium exists w.r.t. $\mathcal{D}_j$
- Each investor has *its own master fund* and invests in its own master fund and in the risk-free asset
- Generalized CAPM holds:

$$\bar{r}_{ij} - r_0 = \beta_{ij}(\bar{r}_{iM} - r_0)$$

$$\beta_{ij} = \frac{Covar(G_j, r_{ij})}{\mathcal{D}(-r_{jM})} \qquad \mathrm{Covar}(\mathrm{G}_j, \mathrm{r}_{ij}) = E[(G_j - EG_j)(r_{ij} - Er_{ij})]$$

$\bar{r}_{ij}$ is expected return of asset *i* in group *j*
r_0 is risk-free rate
$\bar{r}_{jM}$ is expected return of market fund for investor group *j*
G_j is the risk identifier for the market fund *j*

Generalized Master Fund Theorem and CAPM

When $\mathcal{D}(X) = \sigma(X)$ then $\beta_i = \dfrac{Covar(r_i, r_M)}{\sigma^2(r_M)}$

When $\mathcal{D}(X) = \sigma_-(X)$ then $\beta_i = \dfrac{Covar(r_i, r_M)}{\sigma_-^2(r_M)}$

When $\mathcal{D}(X) = CVaR_\alpha^\Delta(X)$ then

$$\beta_i = \frac{E[r_i - \bar{r}_M \mid -r_M \geq VaR_\alpha(-r_M)]}{CVaR_\alpha^\Delta(-r_M)}$$

Classical CAPM => Discounting Formula

All investors have the same risk preferences: standard deviation

Discounting by risk-free rate with adjustment for uncertainties (derived in PhD dissertation of Sarykalin)

$$\pi_i = \frac{E\zeta_i}{1 + r_0 + \beta_i(r_M - r_0)}$$

$$\pi_i = \frac{1}{1 + r_0}\left(E\zeta_i - \beta_i(r_M - r_0)\right)$$

$$\beta_i = \frac{\operatorname{cov}(r_i, r_M)}{\sigma_M^2}$$

Generalized CAPM

- There are different groups of investors $\boldsymbol{k=1,\ldots,K}$
- Risk attitude of each group of investors can be expressed through its deviation measure $\boldsymbol{D_k}$
- Consequently:

Each group of investors invests its own Master Fund $\boldsymbol{M}$

Generalized CAPM

$$\beta_i = \frac{\operatorname{cov}\left(-r_i, Q_M^D\right)}{D\left(r_M\right)}$$

$$\pi_i = \frac{E\zeta_i}{1 + r_0 + \beta_i (r_M - r_0)}$$

$$\pi_i = \frac{1}{1 + r_0}\left(E\zeta_i - \beta_i (r_M - r_0)\right)$$

D = deviation measure

Q_M^D = risk identifier for *D* corresponding to *M*

Investors Buying Out-of-the-money S&P500 Put Options

- Group of investors buys S&P500 options
- Risk preferences are described by mixed CVaR deviation

$$D(X) = \sum_{i=1}^{n} \lambda_i CVaR_{\alpha_i}^{\Delta}(X)$$

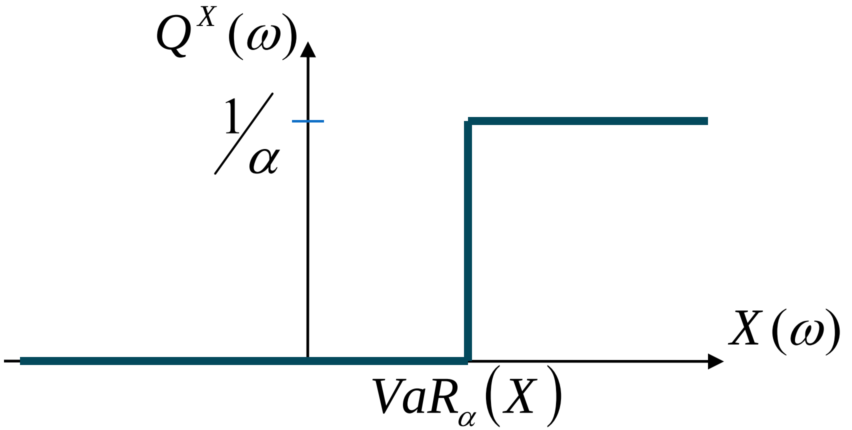
- Assume that S&P500 is their Master Fund
- Out-of-the-money put option is an investments in low tail of price distribution. CVaR deviations can capture the tail.

Mixed CVaR Deviation Risk Envelope

$$D(X) = \sup_{Q \in \mathbf{Q}} XQ - EX$$

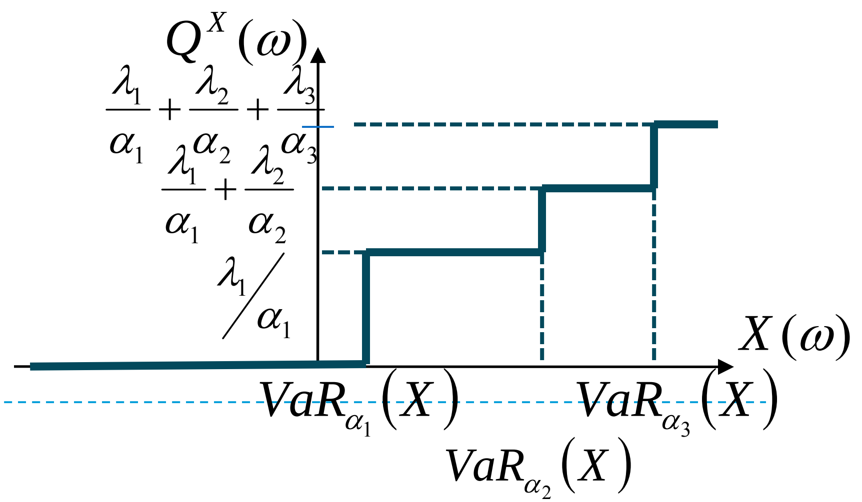
$$D(X) = CVaR_{\alpha}^{\Delta}(X)$$

$$Q_X(\omega) = \mathbf{1}\{X(\omega) \geq VaR_{\alpha}(X)\}$$



$$D(X) = \sum_{i=1}^{n} \lambda_i CVaR_{\alpha_i}^{\Delta}(X)$$

$$Q_X(\omega) = \sum_{i=1}^{n} \lambda_i \mathbf{1}\{X(\omega) \geq VaR_{\alpha_i}(X)\}$$



Data

- Put options prices on Oct,20 2009 maturing on Nov, 20 2009
- S&P500 daily prices for 2000-2009
- 2490 monthly returns (overlapping daily):

  every trading day ***t:*** $\boldsymbol{r_t = \ln S_t - \ln S_{t-21}}$
- Mean return adjusted to 6.4% annually.
- 2490 scenarios of S&P500 option payoffs on Nov, 20 2009

CVaR Deviation

- Risk preferences of Put Option buyers:

$$D(X) = \sum_{j=1}^{5} \lambda_j CVaR_{\alpha_j}^{\Delta}\left(X\right)$$

$$\alpha_1 = 99\% \quad \alpha_2 = 95\% \quad \alpha_3 = 85\% \quad \alpha_4 = 75\% \quad \alpha_5 = 50\%$$

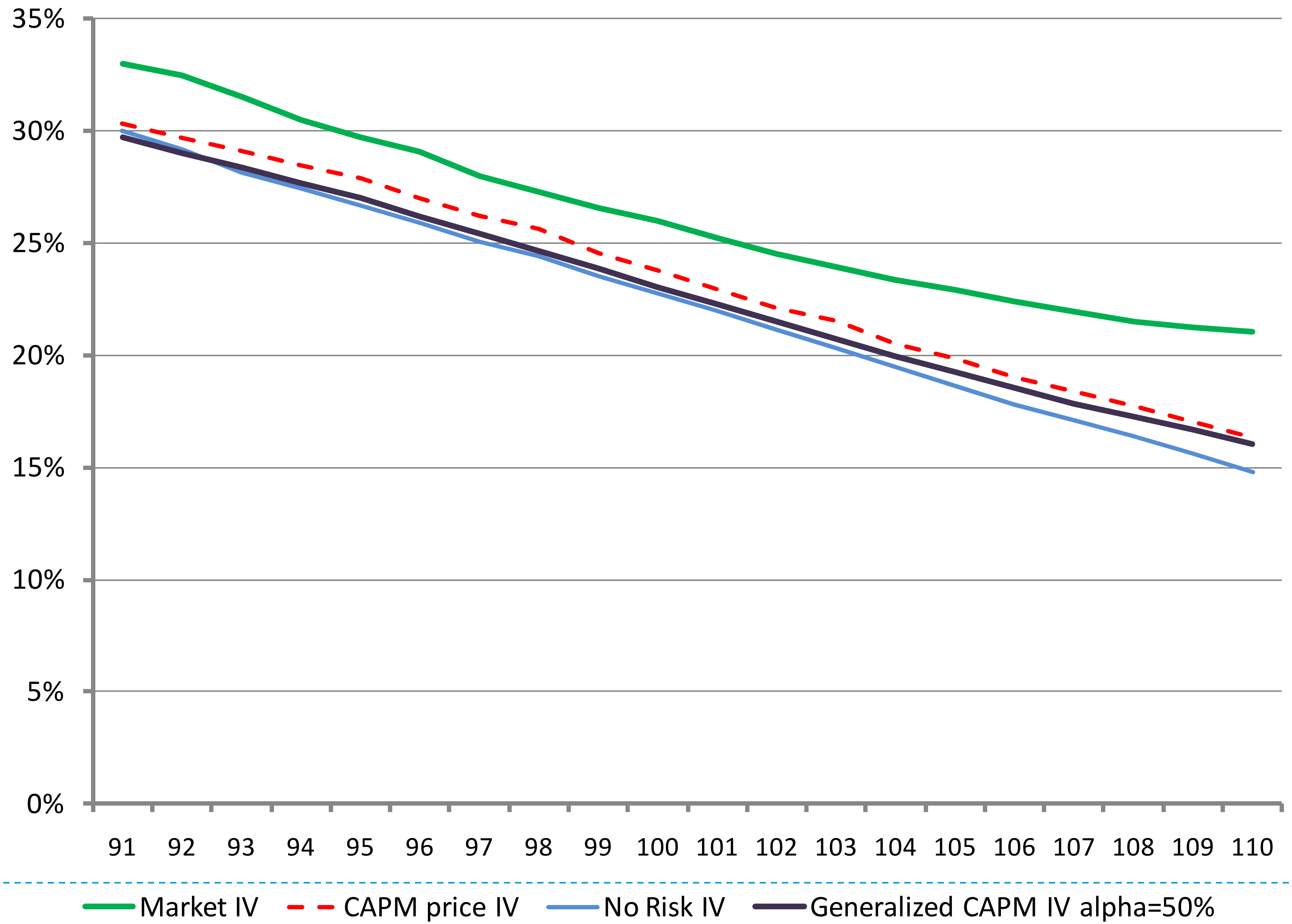
- We want to estimate values for coefficients λ_j

Prices <=> Implied Volatilities

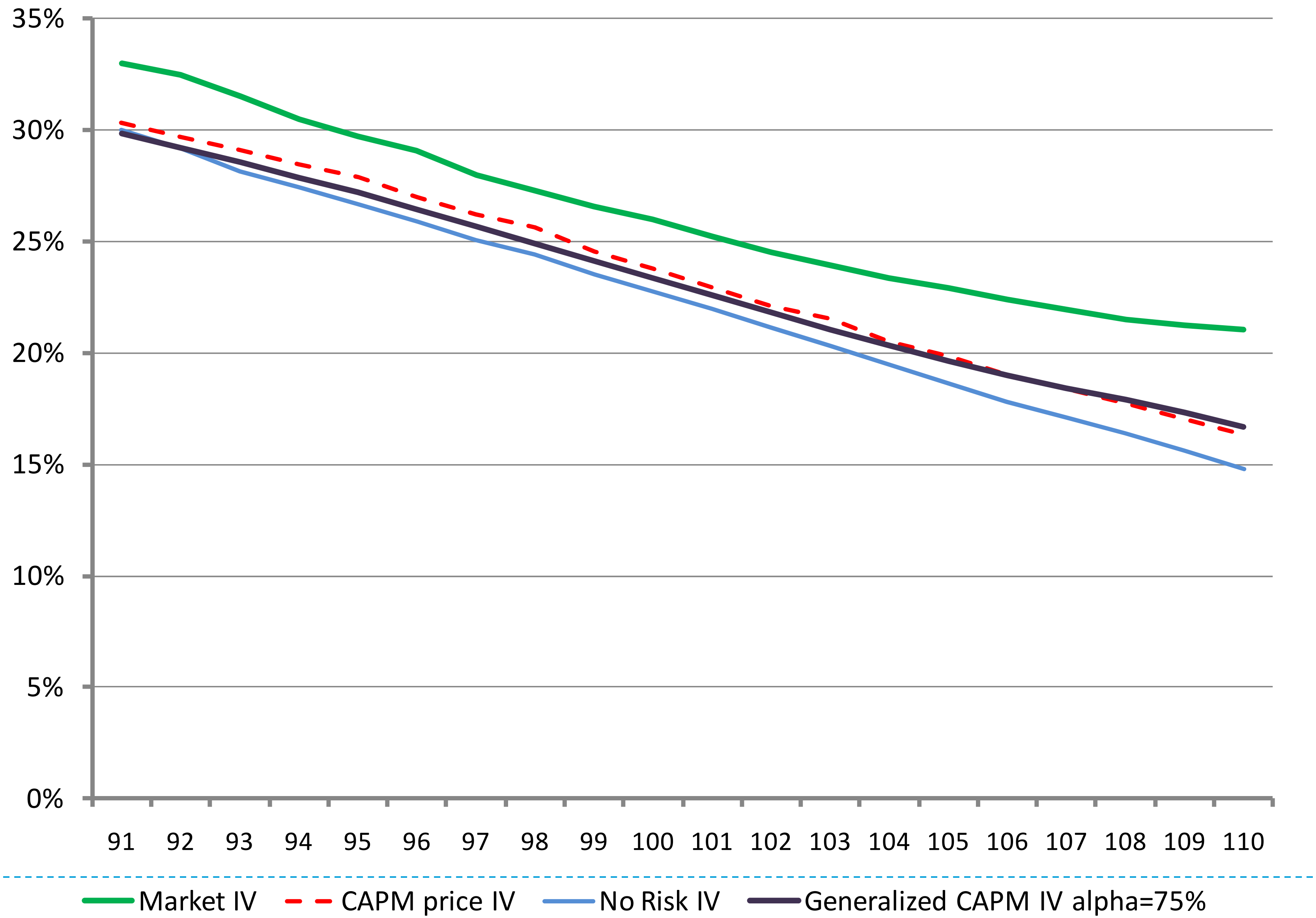
- Option prices with different strike prices vary very significantly

- Black-Scholes formula: prices ⇔ implied volatilities

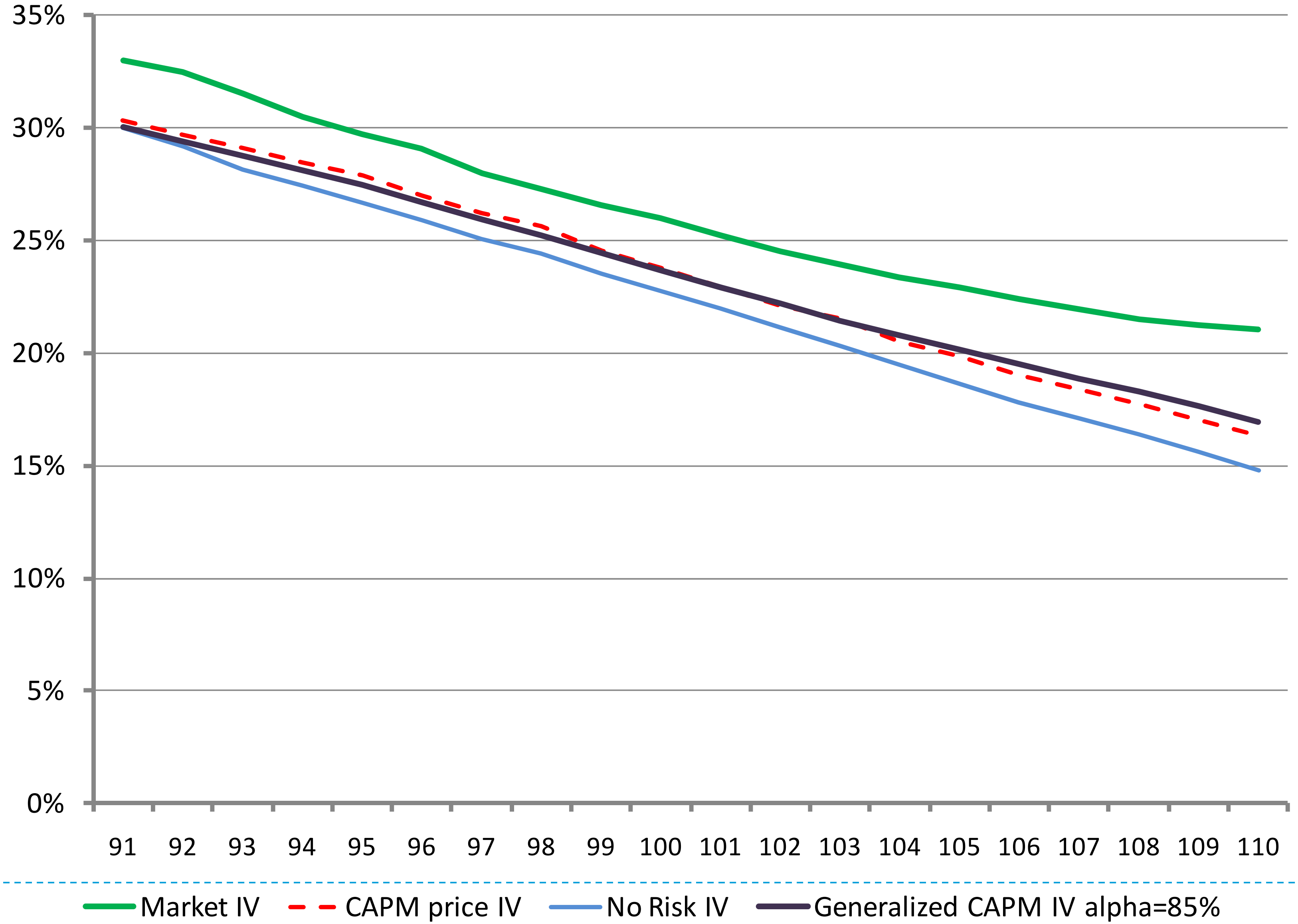
Graphs: $D(X) = CVaR^{\Delta}_{50\%}(X)$



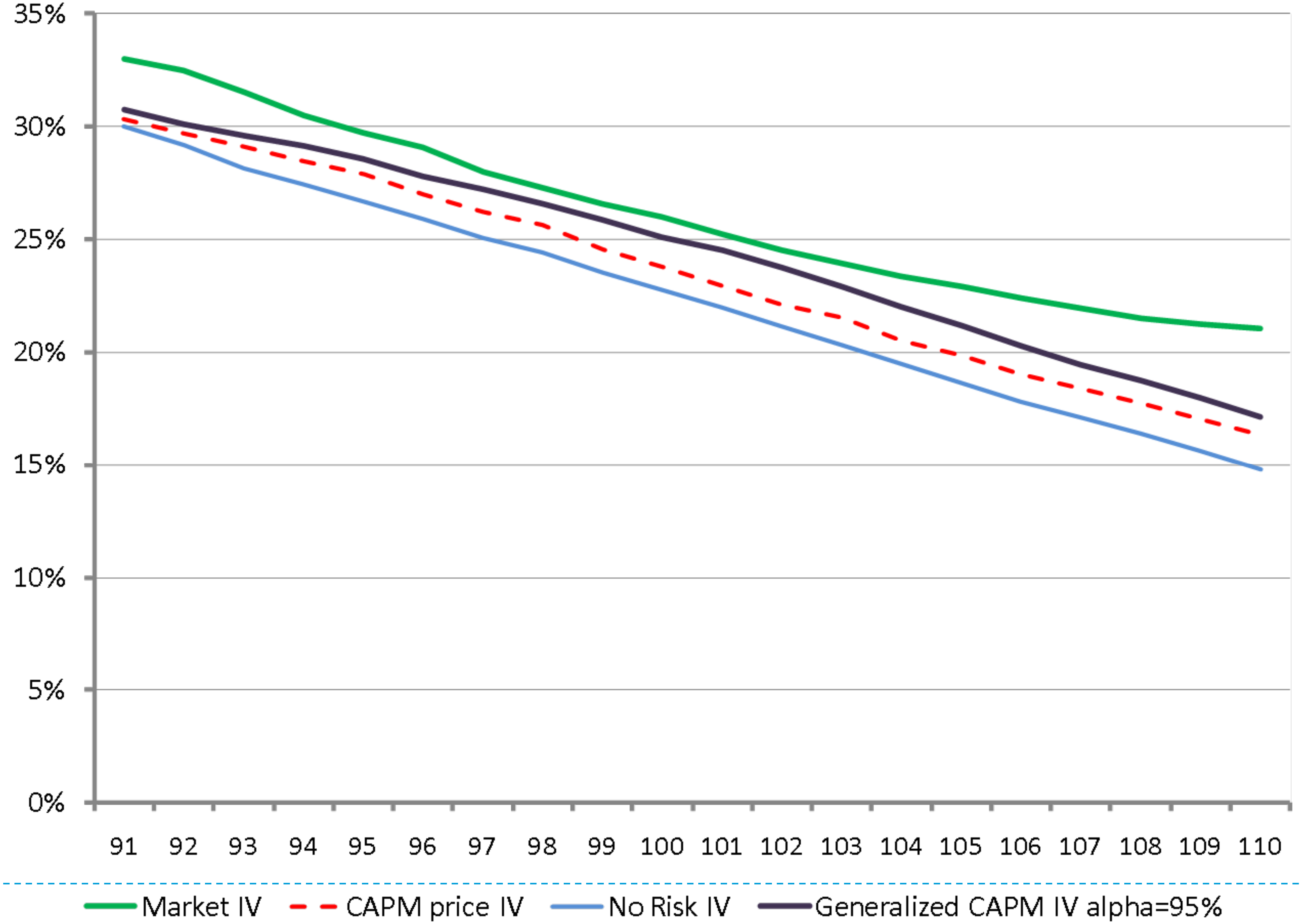
Graphs: $D(X) = CVaR^{\Delta}_{75\%}(X)$



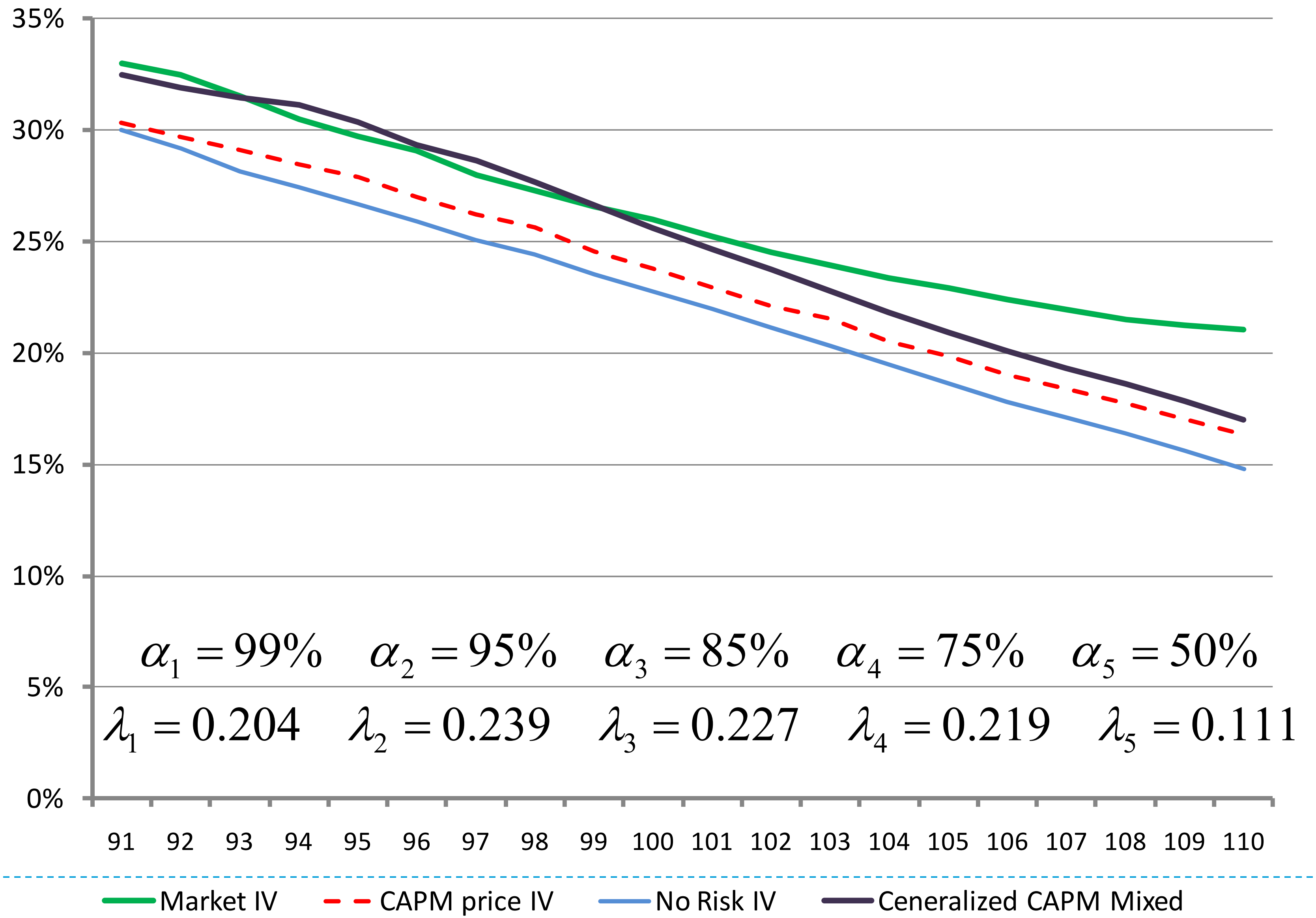
Graphs: $D(X) = CVaR^{\Delta}_{85\%}(X)$



Graphs: $D(X) = CVaR^{\Delta}_{95\%}(X)$



Graphs: $D(X)=\sum_{j=1}^{5}\lambda_j CVaR^{\Delta}_{\alpha_j}(X)$



VaR: Pros

1. VaR is a relatively simple risk management concept and has a clear interpretation
2. Specifying VaR for all confidence levels completely defines the distribution (superior to σ)
3. VaR focuses on the part of the distribution specified by the confidence level
4. Estimation procedures are stable
5. VaR can be estimated with parametric models

VaR: Cons

1. VaR does not account for properties of the distribution beyond the confidence level
2. Risk control using VaR may lead to undesirable results for skewed distributions
3. VaR is a *non-convex* and *discontinuous* function for discrete distributions

CVaR: Pros

1. VaR has a clear engineering interpretation
2. Specifying CVaR for all confidence levels completely defines the distribution (superior to σ)
3. CVaR is a coherent risk measure
4. CVaR is continuous w.r.t. α
5. $CVaR_{\alpha}(w_1 X_1 + \ldots + w_n X_n)$ is a convex function w.r.t. $(w_1, \ldots, w_n)$
6. CVaR optimization can be reduced to *convex programming* and in some cases to *linear programming*

CVaR: Cons

1. CVaR is more sensitive than VaR to estimation errors
2. CVaR accuracy is heavily affected by accuracy of tail modeling

VaR or CVaR in financial applications?

- VaR is not restrictive as CVaR with the same confidence level

  $\longrightarrow$ a trader may prefer VaR
- A company owner may prefer CVaR; a board of director may prefer reporting VaR to shareholders
- VaR may be better for portfolio optimization when good models for the tails are not available
- CVaR should be used when good models for the tails are available
- CVaR has superior mathematical properties
- CVaR can be easily handled in optimization and statistics
- Avoid comparison of VaR and CVaR for the same level α

PSG: Portfolio Safeguard

- Adequate accounting for risks, classical and downside risk measures:

  - Value-at-Risk (VaR)
  - Conditional Value-at-Risk (CVaR)
  - Drawdown
  - Maximum Loss
  - Lower Partial Moment
  - Probability (e.g. default probability)
  - Variance
  - St.Dev.
  - and many others

- Various data inputs for risk functions: scenarios and covariances

  - Historical observations of returns/prices
  - Monte-Carlo based simulations, e.g. from RiskMetrics or S&P CDO evaluator
  - Covariance matrices, e.g. from Barra factor models

PSG: Portfolio Safeguard

- Powerful and robust optimization tools:
  - four environments:
    - Shell (Windows-dialog)
    - MATLAB
    - C++
    - Run-file
  - simultaneous constraints on many functions at various times (e.g., multiple constraints on standard deviations obtained by resampling in combination with drawdown constraints)

PSG: Risk Functions

matrix of scenarios

or

covariance matrix

Risk function

(e.g., St.Dev, VaR, CVaR, Mean,…)

matrix with one row

Linear function

PSG: Example

INPUT DATA: Matrix of Scenarios			
x	y	Scenario Benchmark	Scenario Probability
θ_{11}	θ_{12}	θ_{01}	p_1
θ_{21}	θ_{22}	θ_{02}	p_2

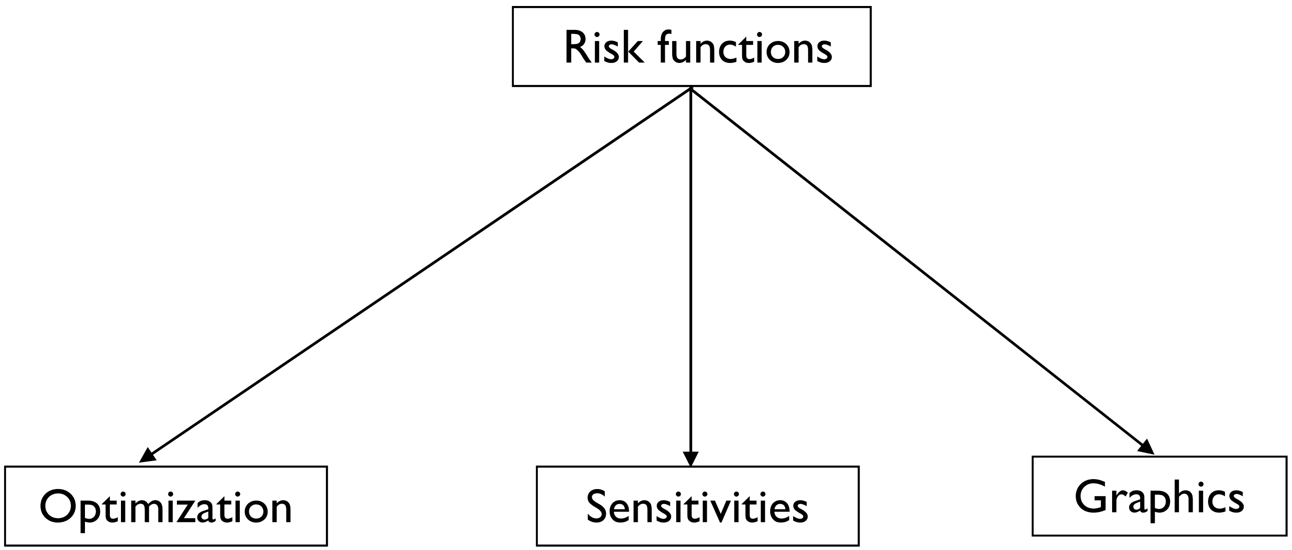
$\downarrow$

Loss Vector	
Loss Function	Scenario Probability
$\theta_{01} - (\theta_{11} * X + \theta_{21} * Y)$	p_1
$\theta_{02} - (\theta_{12} * X + \theta_{22} * Y)$	p_2

$\downarrow$

Function Evaluation	
	Function Value
Mean	$p_1 * [\theta_{01} - (\theta_{11} * X + \theta_{21} * Y)] + p_2 * [\theta_{02} - (\theta_{12} * X + \theta_{22} * Y)]$
Max Loss	$max[\theta_{01} - (\theta_{11} * X + \theta_{21} * Y)], [\theta_{02} - (\theta_{12} * X + \theta_{22} * Y)]$

PSG: Operations with Functions



Case Study: Risk Control using VaR

- Risk control using VaR may lead to paradoxical results for skewed distributions
- Undesirable feature of VaR optimization: VaR minimization may increase the extreme losses
- Case Study main result: minimization of 99%-VaR Deviation leads to 13% increase in 99%-CVaR compared to 99%-CVaR of optimal 99%-CVaR Deviation portfolio
- Consistency with theoretical results: CVaR is coherent, VaR is not coherent

Larsen, N., Mausser, H., Uryasev, S., "*Algorithms for Optimization of Value-at-Risk*", Financial Engineering, E-commerce, Supply Chain, P. Pardalos , T.K. Tsitsiringos Ed., Kluwer, 2000

Case Study: Risk Control using VaR

P.1

$$\min CVaR_{\alpha}^{\Delta}(x)$$
$$s.t. \sum_{i=1}^{n} r_i x_i \geq \bar{r}$$
$$\sum_{i=1}^{n} x_i = 1$$

Pb.2

$$\min VaR_{\alpha}^{\Delta}(x)$$
$$s.t. \sum_{i=1}^{n} r_i x_i \geq \bar{r}$$
$$\sum_{i=1}^{n} x_i = 1$$

	$\min \mathrm{CVaR}_{0.99}^{\Delta}$	$\min \mathrm{VaR}_{0.99}^{\Delta}$	Ratio
$\mathrm{CVaR}_{0.99}$	0.0073	0.0083	1.130
$\mathrm{CVaR}_{0.99}^{\Delta}$	0.0363	0.0373	1.026
$\mathrm{VaR}_{0.99}$	0.0023	0.0005	0.231
$\mathrm{VaR}_{0.99}^{\Delta}$	0.0313	0.0295	0.944
Max Loss $= \mathrm{CVaR}_1$	0.0133	0.0148	1.116
Max Loss Deviation $= \mathrm{CVaR}_1^{\Delta}$	0.0423	0.0438	1.036

Columns 2, 3 report value of risk functions at optimal point of Problem 1 and 2;
Column "Ratio" reports ratio of Column 3 to Column 2

Case Study: Linear Regression-Hedging

- Investigate performance of optimal hedging strategies based on different deviation measures
- Determining optimal hedging strategy is a *linear regression problem*

$$\hat{\theta} = x_1\theta_1 + \ldots + x_I\theta_I$$

- Benchmark portfolio value is the response variable, replicating financial instruments values are predictors, portfolio weights are coefficients of the predictors to be determined
- Coefficients $x_1, \ldots, x_I$ chosen to minimize a replication error function depending upon the residual $\theta_0 - \hat{\theta}$

Case Study: Linear Regression-Hedging

$$\text{Loss Function} = L(x,\theta) = L(x_1, \ldots, x_I, \theta_0, \ldots, \theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i$$

$$\text{Two Tail } \alpha\text{\%-VaR Deviation} = TwoTailVaR_{\alpha}^{\Delta}(L(x,\theta)) = \text{VaR}_{\alpha}(L(x,\theta)) + \text{VaR}_{\alpha}(-L(x,\theta))$$

(1) $\min_{x} \ \text{CVaR}_{0.9}^{\Delta}(L(x,\theta))$

(2) $\min_{x} \ MAD(L(X,\theta))$

(3) $\min_{x} \ \sigma(L(x,\theta))$

(4) $\min_{x} \ TwoTailVaR_{0.75}^{\Delta}(L(X,\theta))$

(5) $\min_{x} \ TwoTailVaR_{0.9}^{\Delta}(L(X,\theta))$

- Out-of-sample performance of hedging strategies significantly depends on the skewness of the distribution
- Two-Tailed 90%-VaR has the best out-of-sample performance
- Standard deviation has the worst out-of-sample performance

Case Study: Linear Regression-Hedging

Optimal Points	$CVaR^{\Delta}_{0.9}$	MAD	σ	$TwoTailVaR^{\Delta}_{0.75}$	$TwoTailVaR^{\Delta}_{0.9}$
$CVaR^{\Delta}_{0.9}$	0.690	0.815	1.961	0.275	1.122
MAD	1.137	0.714	1.641	0.379	1.880
σ	1.405	0.644	1.110	0.979	1.829
$TwoTailVaR^{\Delta}_{0.75}$	1.316	0.956	1.955	0.999	1.557
$TwoTailVaR^{\Delta}_{0.9}$	0.922	0.743	1.821	0.643	1.256

Out-of-sample performance of different deviation measures evaluated at optimal points of the 5 different hedging strategies (e.g., the first raw is for $CVaR^{\Delta}_{0.9}$ hedging strategy)

Optimal Points	Max Loss	$CVaR_{0.9}$	$VaR_{0.9}$
$CVaR^{\Delta}_{0.9}$	-18.01	-18.05	-18.08
MAD	-16.49	-17.44	-17.88
σ	-13.31	-15.29	-15.60
$TwoTailVaR^{\Delta}_{0.75}$	-15.31	-16.19	-16.71
$TwoTailVaR^{\Delta}_{0.9}$	-18.02	-18.51	-18.66

Each row reports value o different risk functions evaluated at optimal points of the 5 different hedging strategies (e.g., the first raw is for $CVaR^{\Delta}_{0.9}$ hedging strategy)

the best performance has *TwoTailVar* $^{\Delta}_{0.9}$

Example:
Chance and VaR constraints equivalence

- We illustrate numerically the equivalence:

$$\text{Prob}\{L(x,\theta) > \epsilon\} \leq 1-\alpha \quad \leftrightarrow \quad \text{VaR}_\alpha(L(x,\theta)) \leq \epsilon \ ,$$

Problem 1:

$$\begin{aligned} \max \ & E[-L(x,\theta)] \\ s.t. \ & \text{Prob}\{L(x,\theta) > \epsilon\} \leq 1-\alpha = 0.05 \\ & v_i \leq x_i \leq u_i \ , \quad i = 1,..,I \ , \\ & \textstyle\sum_{i=1}^I x_i = 1 \ . \end{aligned}$$

Problem 2:

$$\begin{aligned} \max \ & E[-L(x,\theta)] \\ s.t. \ & \text{VaR}_\alpha(L(x,\theta)) \leq \epsilon \ , \\ & v_i \leq x_i \leq u_i \ , \quad i = 1,..,I \\ & \textstyle\sum_{i=1}^I x_i = 1 \ . \end{aligned}$$

Optimal Weights	Prob ≤ 0.05	VaR $\leq \epsilon$
x_1	.051	.051
x_2	.055	.055
x_3	.071	.071
x_4	.053	.053
x_5	.079	.079
x_6	.289	.289
x_7	.020	.020
x_8	.300	.300
x_9	.063	.063
x_{10}	.020	.020

At optimality the two problems selected the same portfolio with the same objective function value

Case Study: Portfolio Rebalancing Strategies, Risks and Deviations

- We consider a portfolio rebalancing problem:

$$\min \; R(x,\theta) - k * E[-L(x,\theta)]$$
$$s.t.$$
$$\sum_{i=1}^{I} x_i = 1 \;,$$
$$v_i \leq (x_i) \leq u_i \;,\quad \text{i=1,..,I}$$

- We used as risk functions VaR, CVaR, VaR Deviation, CVaR Deviation, Standard Deviation
- We evaluated Sharpe ratio and mean value of each sequence of portfolios
- We found a good performance of VaR and VaR Deviation minimization
- Standard Deviation minimization leads to inferior results

Case Study: Portfolio Rebalancing Strategies, Risks and Deviations (Cont'd)

- Results depend on the scenario dataset and on *k*
- *In the presence of mean reversion* the tails of historical distribution are not good predictors of the tail in the future
- VaR disregards the unstable part of the distribution thus may lead to good out-of-sample performance

k	VaR	CVaR	VaR Deviation	CVaR Deviation	Standard Deviation
-1	1.2710	1.2609	1.2588	1.2693	1.2380
-3	1.2711	1.2667	1.2762	1.2652	1.2672
-5	1.2712	1.2666	1.2721	1.2743	1.2628

Sharpe ratio for the rebalancing strategy when different risk functions are used in the objective for different values of the parameter *k*

Conclusions: key observations

- CVaR has superior mathematical properties: CVaR is *coherent*, CVaR of a portfolio is a *continuous* and *convex* function with respect to optimization variables
- CVaR can be optimized and constrained with convex and linear programming methods; VaR is relatively difficult to optimize
- VaR does not control scenarios exceeding VaR
- VaR estimates are statistically more stable than CVaR estimates
- VaR may lead to bearing high uncontrollable risk
- CVaR is more sensitive than VaR to estimation errors
- CVaR accuracy is heavily affected by accuracy of tail modeling

Conclusions: key observations

- There is a one-to-one correspondence between Risk Measures and Deviation Measures
- CVaR Deviation is a strong competitor of Standard Deviation
- Mixed CVaR Deviation should be used when tails are not modeled correctly. Mixed CVaR Deviation gives different weight to different parts of the distribution
- Master Fund Theorem and CAPM can be generalized with the for different deviation measure.

Conclusions: Case Studies

- *Case Study 1*: risk control using VaR may lead to paradoxical results for skewed distribution. Minimization of VaR may lead to a stretch of the tail of the distribution exceeding VaR
- *Case Study 2*: determining optimal hedging strategy is a linear regression problem. Out-of-sample performance based on different Deviation Measures depends on the skewness of the distribution, we found standard deviation have the worst performance
- *Case Study 3*: chance constraints and percentiles of a distribution are closely related, VaR and Chance constraints are equivalent
- *Case Study 4*: the choice of the risk function to minimize in a portfolio rebalancing strategy depends on the scenario dataset. In the presence of mean reversion VaR neglecting tails may lead to good out-of-sample performance