

Real World Pricing of Long Term Contracts

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Introduction

- **variable annuities**
 - fund-linked
 - tax-deferred
 - guarantees
- **guaranteed minimum death benefits (GMDBs)**
 - roll-ups:
 - original investment accrued at a pre-defined interest rate

- GMDB put, floating put and/or look-back put option

long maturities of contracts

standard option pricing theory ?

no obvious best choice for price:

IFRS Phase II

CFO-Forum (2008)

CRO-Forum (2008)

Solvency II

- pricing and hedging of GMDBs

benchmark approach in

Pl. (2002, 2004), Pl. & Heath (2006)

Bühlmann & Pl. (2003)

best performing portfolio as benchmark

does **not** require

existence of an equivalent risk neutral probability measure

exploits real trends

may provide significantly lower prices

Financial Model

- **underlying risky security (unit)**

$$dS_t = (\mu_t - \gamma)S_t dt + \sigma_t S_t dW_t$$

$\gamma \geq 0$ management fee rates

W_t - Brownian motion

- **savings account**

$$dB_t = r_t B_t dt$$

- market price of risk

$$\theta_t = \frac{\mu_t - \gamma - r_t}{\sigma_t}$$

- risky asset

$$\frac{dS_t}{S_t} = r_t dt + \sigma_t \theta_t dt + \sigma_t dW_t$$

- instantaneous portfolio return

$$\begin{aligned}\frac{dV_t}{V_t} &= \pi_t^0 \frac{dB_t}{B_t} + \pi_t^1 \frac{dS_t}{S_t} \\ &= r_t dt + \pi_t^1 \sigma_t (\theta_t dt + dW_t)\end{aligned}$$

- fractions

$$\pi_t^0 = \delta_t^0 \frac{B_t}{V_t}, \quad \pi_t^1 = \delta_t^1 \frac{S_t}{V_t}$$

$$\pi_t^0 + \pi_t^1 = 1$$

- numeraire portfolio (NP) as benchmark

Long (1990)

V^* is best performing in several ways

- is growth optimal portfolio

Kelly (1956)

$$V^* = \max E(\log V_T)$$

$$\pi_t^{1*} = \frac{\mu_t - \gamma - r_t}{\sigma_t^2} = \frac{\theta_t}{\sigma_t}$$

$$\frac{dV_t^*}{V_t^*} = r_t dt + \theta_t (\theta_t dt + dW_t)$$

Merton (1992)

- NP best performing portfolio

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log \left(\frac{V_T^*}{V_0^*} \right) \geq \limsup_{T \rightarrow \infty} \frac{1}{T} \log \left(\frac{V_T}{V_0} \right)$$

global well diversified index \approx NP

Pl. (2005)

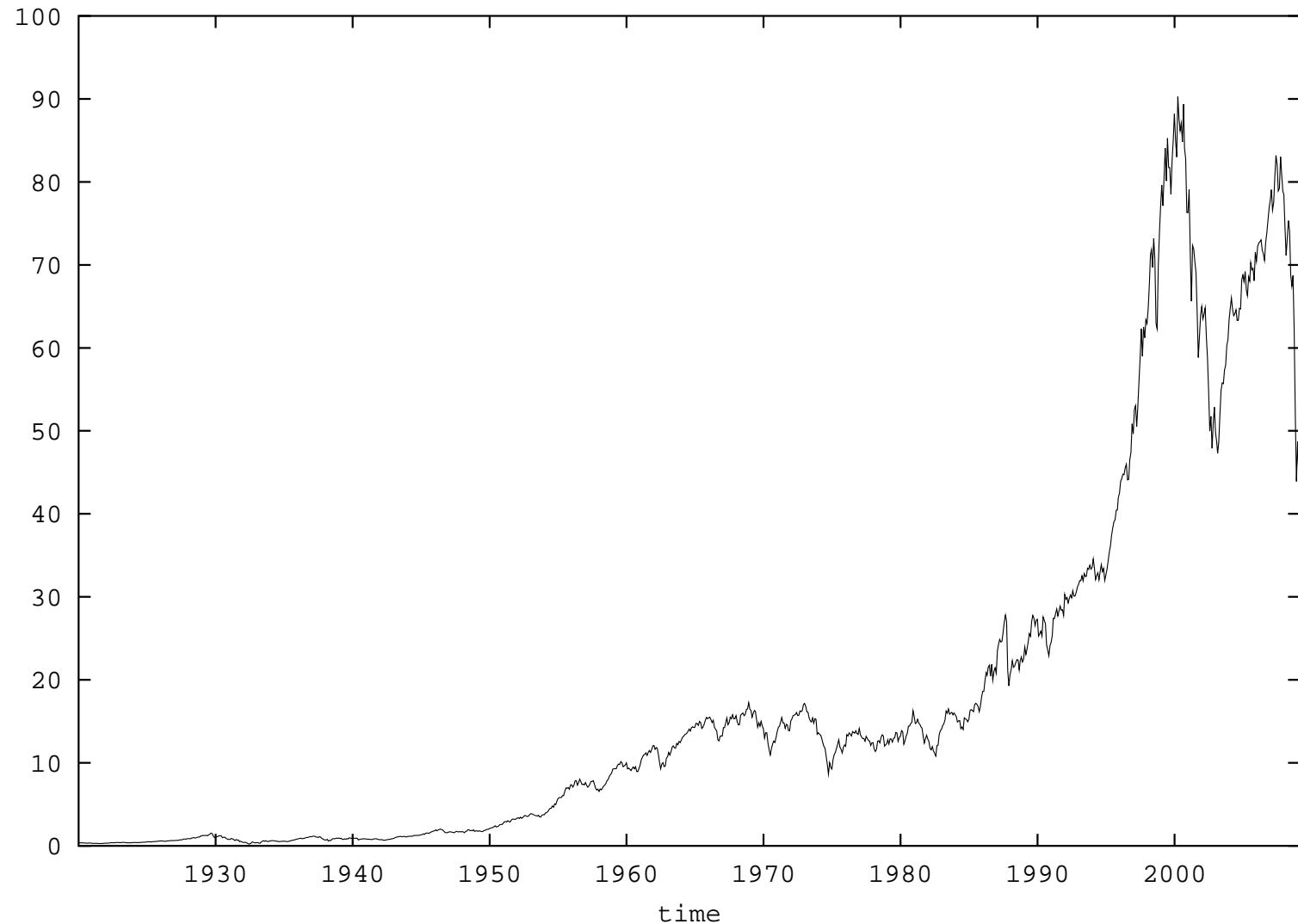


Figure 1: Discounted S&P500 total return index.

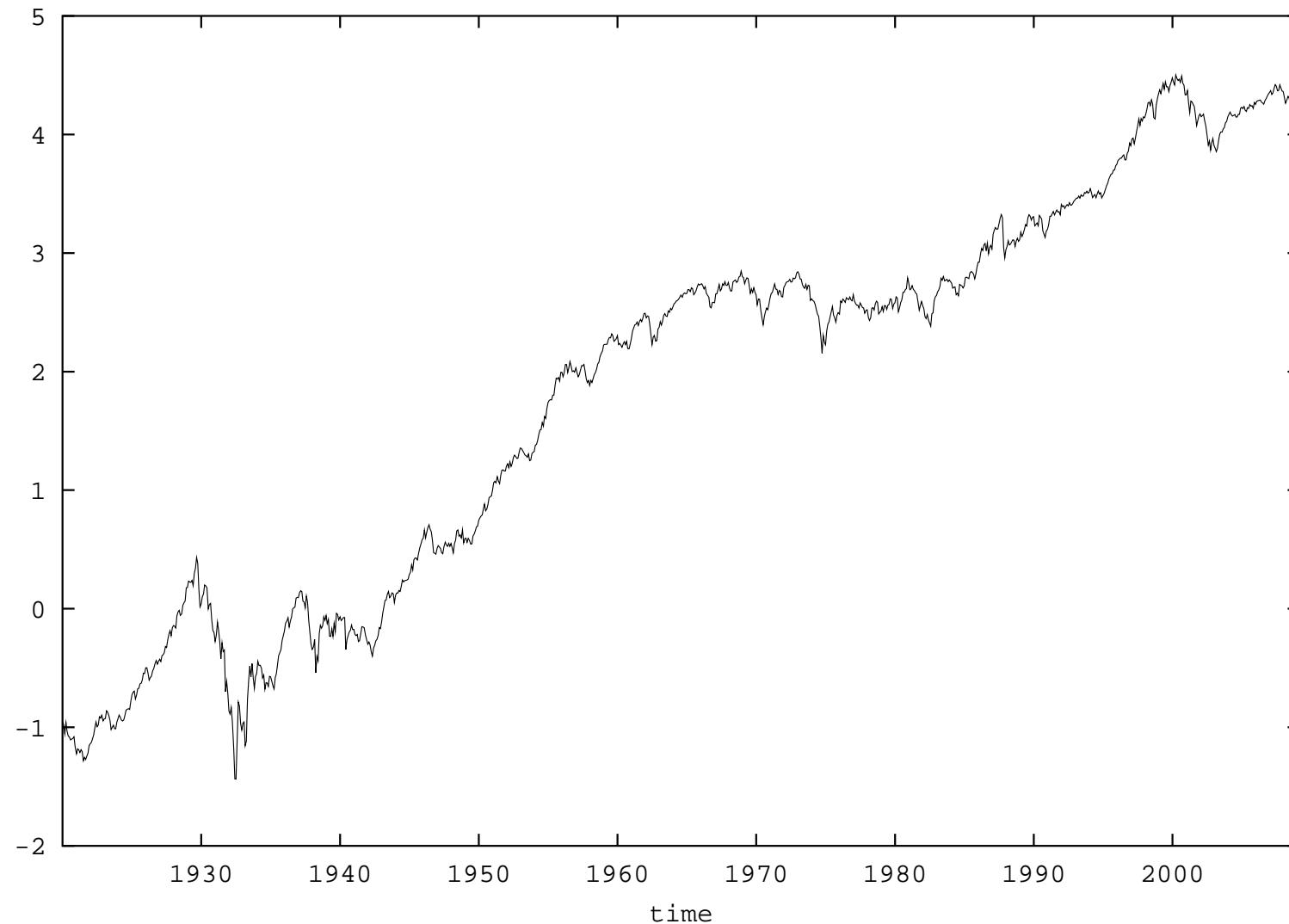


Figure 2: Logarithm of discounted S&P500.

- **benchmarked price**

$$\hat{U}_t = \frac{U_t}{V_t^*}$$

\hat{U} nonnegative \implies

$$\hat{U}_t \geq E_t (\hat{U}_s)$$

$$t \leq s$$

supermartingales (no upward trend)

Pl. (2002)

no strong arbitrage

- benchmarked securities that form martingales are called **fair**.

$$\hat{U}_t = E_t (\hat{U}_s)$$

$$t \leq s$$

- **Law of the Minimal Price**

Pl. (2008)

Fair prices are minimal prices

V nonnegative fair portfolio

V' nonnegative portfolio

$$V_T = V'_T$$

supermartingale property

\implies

$$V_t \leq V'_t$$

$t \in [0, T]$,

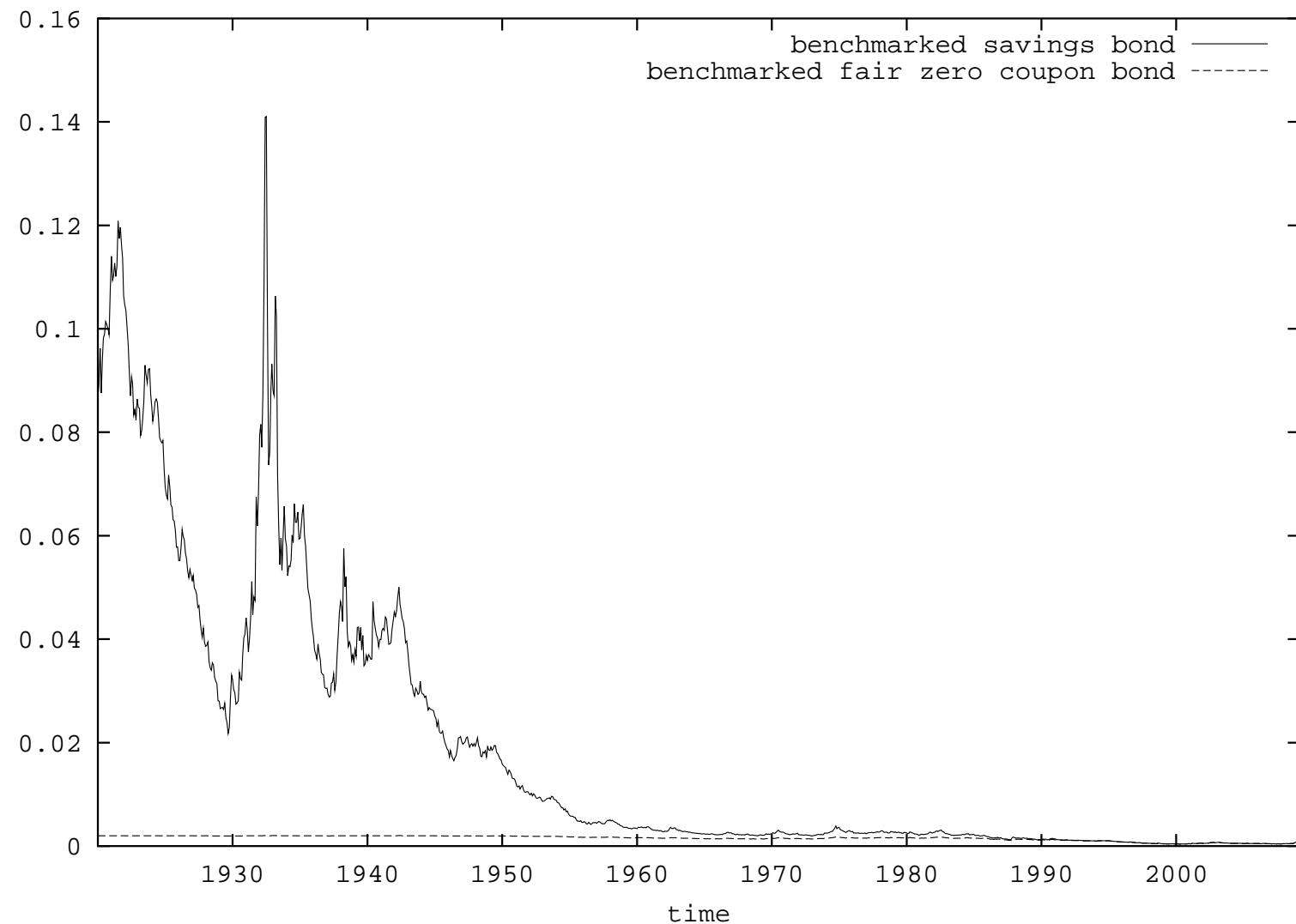


Figure 3: Benchmarked savings bond and benchmarked fair zero coupon bond.

- real world pricing formula

for

$$E_t \left(\frac{H_T}{V_T^*} \right) < \infty$$

$$U_H(t) = V_t^* E_t \left(\frac{H_T}{V_T^*} \right)$$

$$t \in [0, T]$$

no risk neutral probability needs to exist

- actuarial pricing formula

when H_T is independent of V_T^*

\implies

$$U_H(t) = P(t, T) E_t(H_T)$$

zero coupon bond

$$P(t, T) = V_t^* E_t \left(\frac{1}{V_T^*} \right)$$

- standard risk neutral pricing

Ross (1976), Harrison & Pliska (1983)

complete market

candidate Radon-Nikodym derivative

$$\Lambda_t = \frac{dQ}{dP} \Big|_{\mathcal{A}_t} = \frac{B_t V_0^*}{B_0 V_t^*}$$

supermartingale \implies

$$1 = \Lambda_0 \geq E_0(\Lambda_T)$$

\implies

$$\begin{aligned} U_H(0) &= E \left(\frac{V_0^*}{V_T^*} H_T \right) \\ &= E \left(\Lambda_T \frac{B_0}{B_T} H_T \right) \leq \frac{E \left(\Lambda_T \frac{B_0}{B_T} H_T \right)}{E(\Lambda_T)} \end{aligned}$$

Only in the special case when Λ_T martingale

\implies risk neutral pricing formula:

$$U_H(0) = E_Q \left(\frac{B_0}{B_T} H_T \right)$$

Ignores any real trend !

- Trends ignored also when using:

stochastic discount factor, Cochrane (2001);

deflator, Duffie (2001);

pricing kernel, Constantinides (1992);

state price density, Ingersoll (1987);

NP as Long (1990)

- Example zero coupon bond
benchmarked fair zero coupon

$$\hat{P}(t, T) = \frac{P(t, T)}{V_t^*} = E_t \left(\frac{1}{V_T^*} \right)$$

martingale (no trend)

for r_t - deterministic

$$P(t, T) = V_t^* E_t \left(\frac{1}{V_T^*} \right) = \exp \left\{ - \int_t^T r_s ds \right\} E_t \left(\frac{\bar{V}_t^*}{\bar{V}_T^*} \right),$$

$\bar{V}_t^* = \frac{V_t^*}{B_t}$ - discounted NP

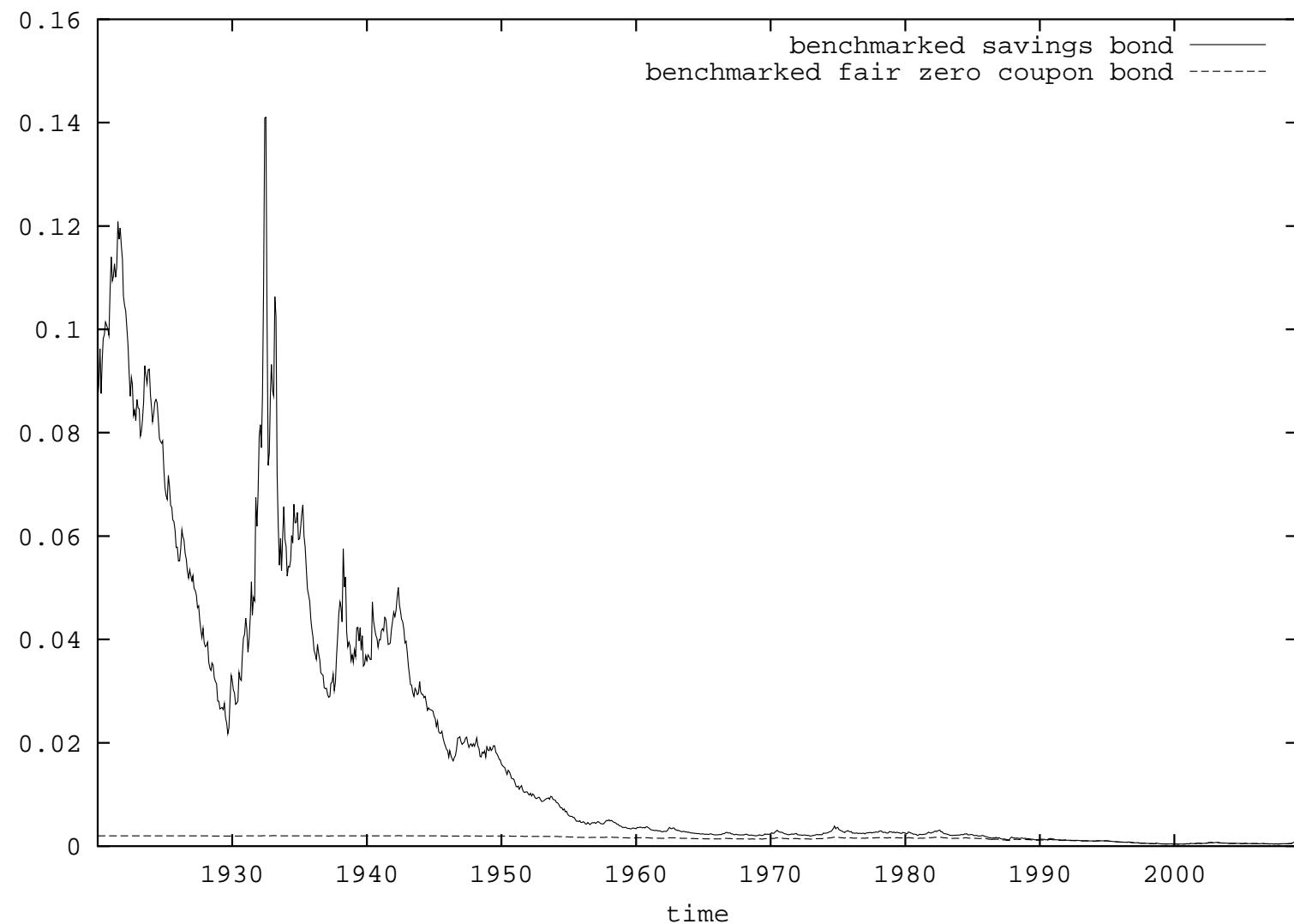


Figure 4: Benchmarked savings bond and benchmarked fair zero coupon bond.

\implies downward trend reflects equity premium

\implies Λ strict supermartingale (downward trend), then

$$P(t, T) < \exp \left\{ - \int_t^T r_s \, ds \right\} = \frac{B_t}{B_T}$$

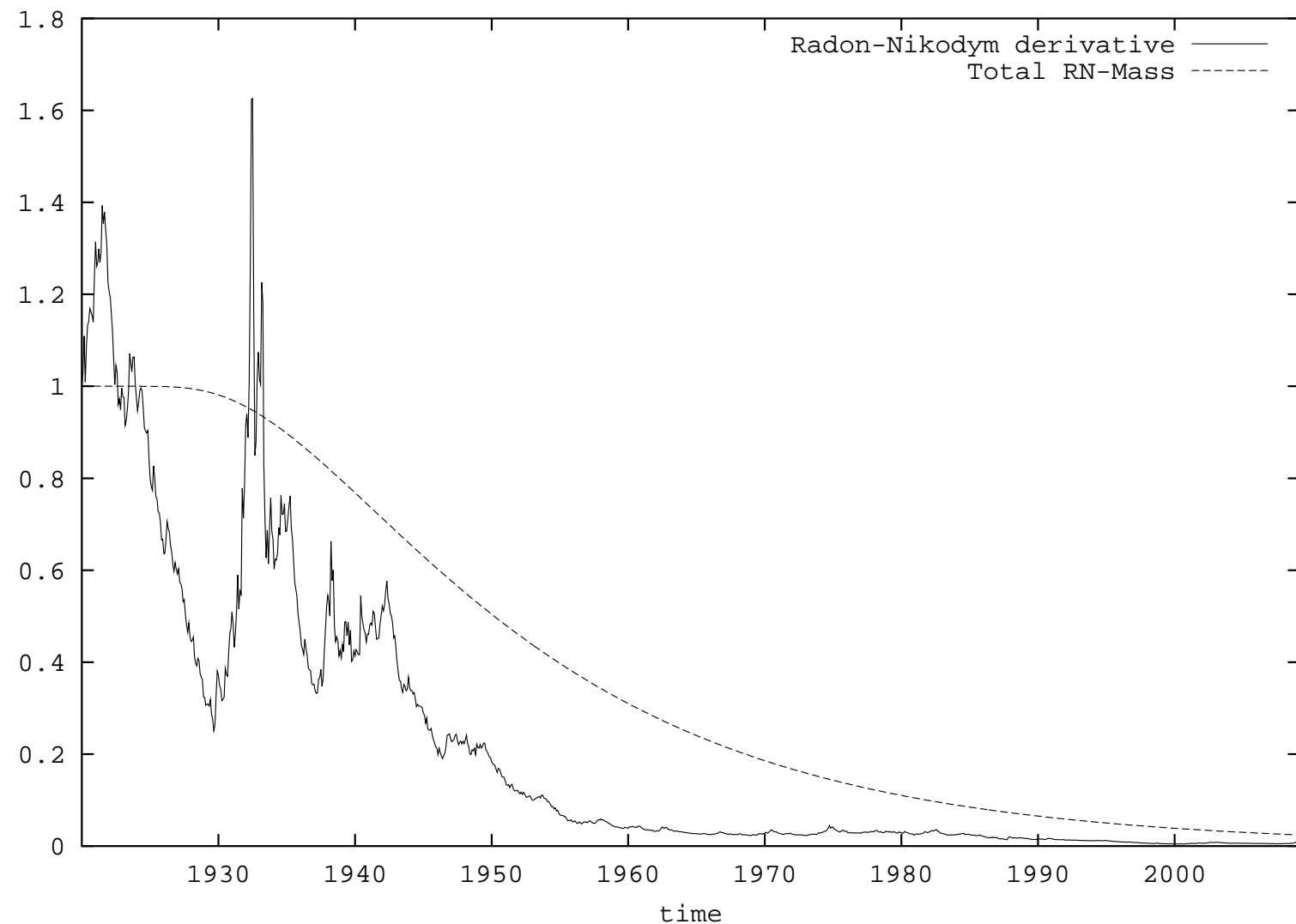


Figure 5: Radon-Nikodym derivative and total mass of putative risk neutral measure.

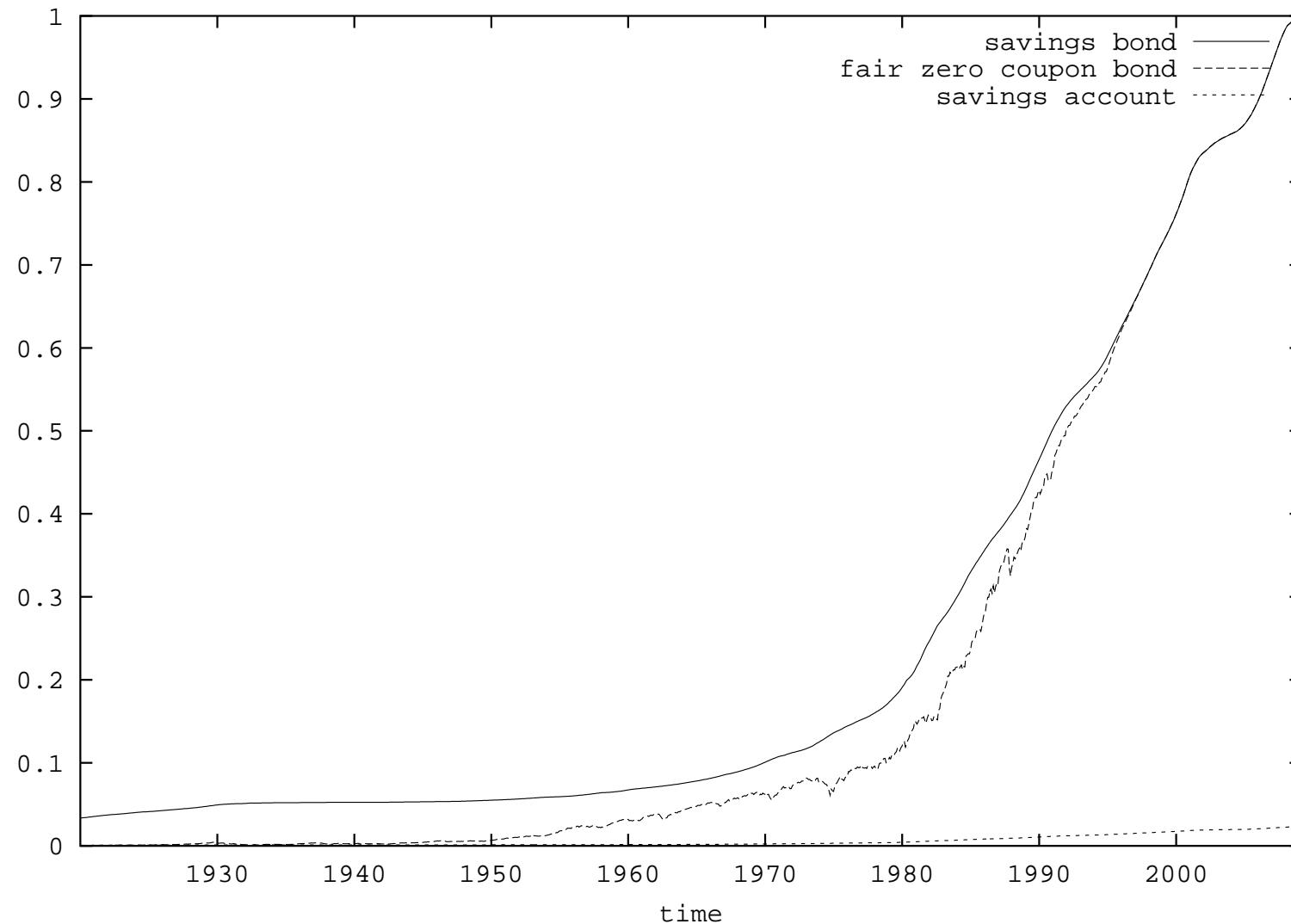


Figure 6: Savings bond, fair zero coupon bond and savings account.

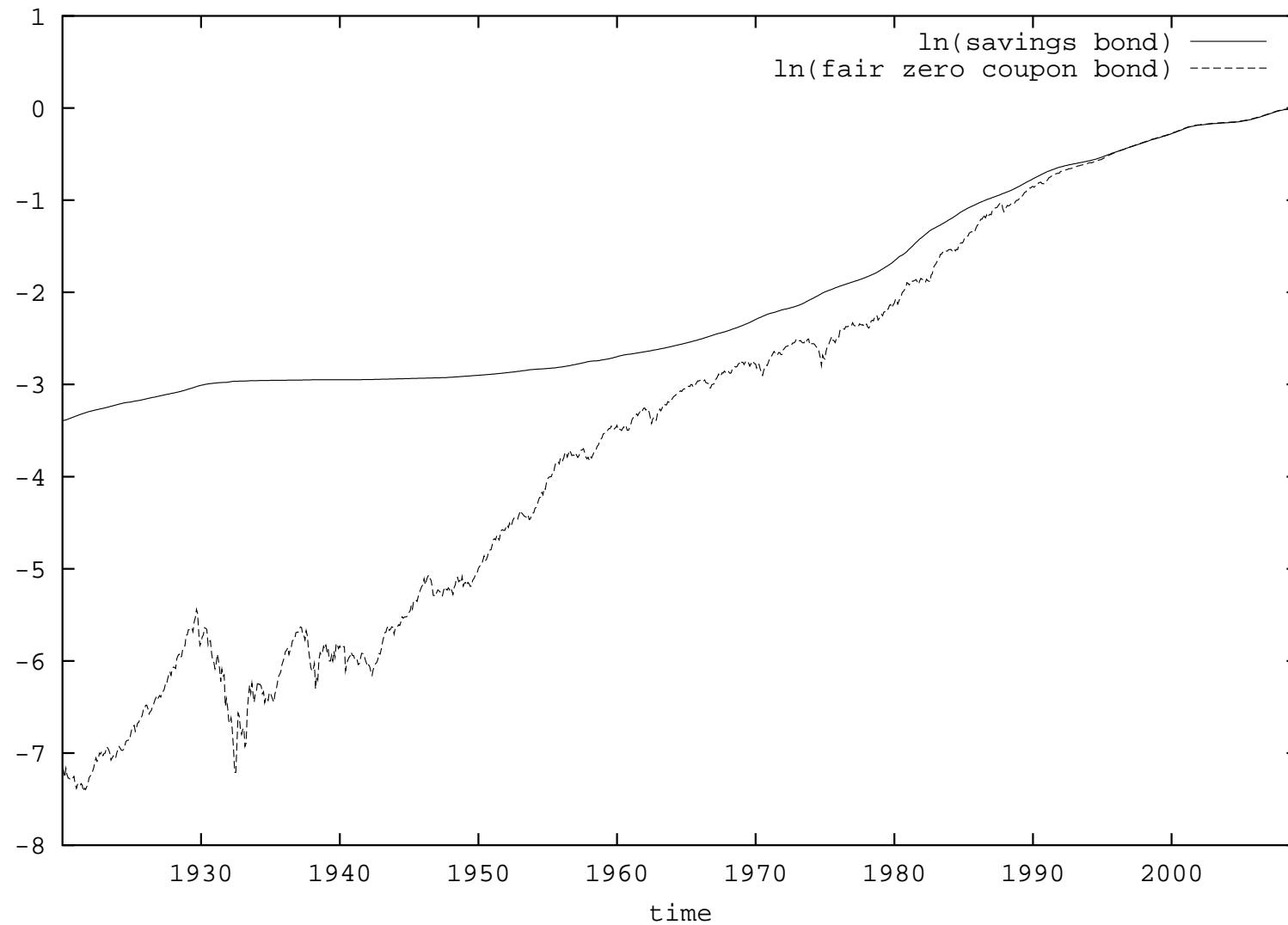


Figure 7: Logarithms of savings bond and fair zero coupon bond.

- discounted NP

$$\bar{V}_t^* = \frac{V_t^*}{B_t}$$

$$\begin{aligned} d\bar{V}_t^* &= \bar{V}_t^* \theta_t (\theta_t dt + dW_t) \\ &= \alpha_t dt + \sqrt{\alpha_t \bar{V}_t^*} dW_t \end{aligned}$$

- discounted NP drift

$$\alpha_t := \bar{V}_t^* \theta_t^2$$

\implies volatility

$$\theta_t = \sqrt{\frac{\alpha_t}{\bar{V}_t^*}}$$

reflects leverage effect

- **minimal market model**

Pl. (2001, 2002)

MMM

discounted NP drift

assume

$$\alpha_t = \alpha_0 \exp\{\eta t\}$$

$$\alpha_0 > 0$$

$$\text{net growth rate } \eta > 0$$

\implies MMM

- normalized NP

$$Y_t = \frac{\bar{V}_t^*}{\alpha_t}$$

$$dY_t = (1 - \eta Y_t) dt + \sqrt{Y_t} dW_t$$

square root process of dimension four

Only one parameter !

- **volatility of NP**

$$\theta_t = \frac{1}{\sqrt{Y_t}} = \sqrt{\frac{\alpha_t}{\bar{V}_t^*}}$$

leverage effect

- Zero coupon bond under the **MMM**

r_t - deterministic

$$P(t, T) = \exp \left\{ - \int_t^T r_s \, ds \right\} \left(1 - \exp \left\{ - \frac{\bar{V}_t^*}{2(\varphi(T) - \varphi(t))} \right\} \right)$$

Explicit formula !

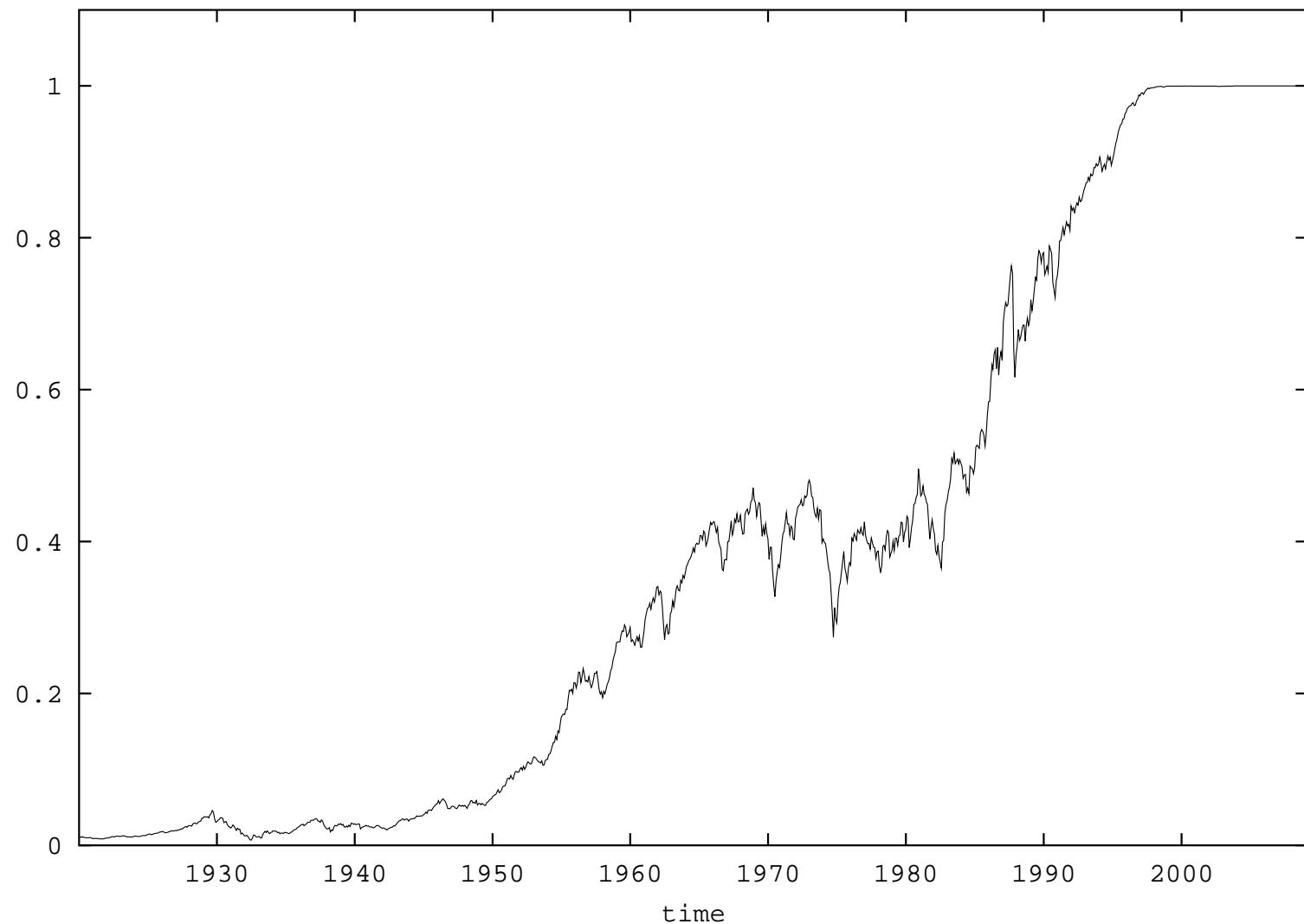


Figure 8: Fraction invested in the savings account.

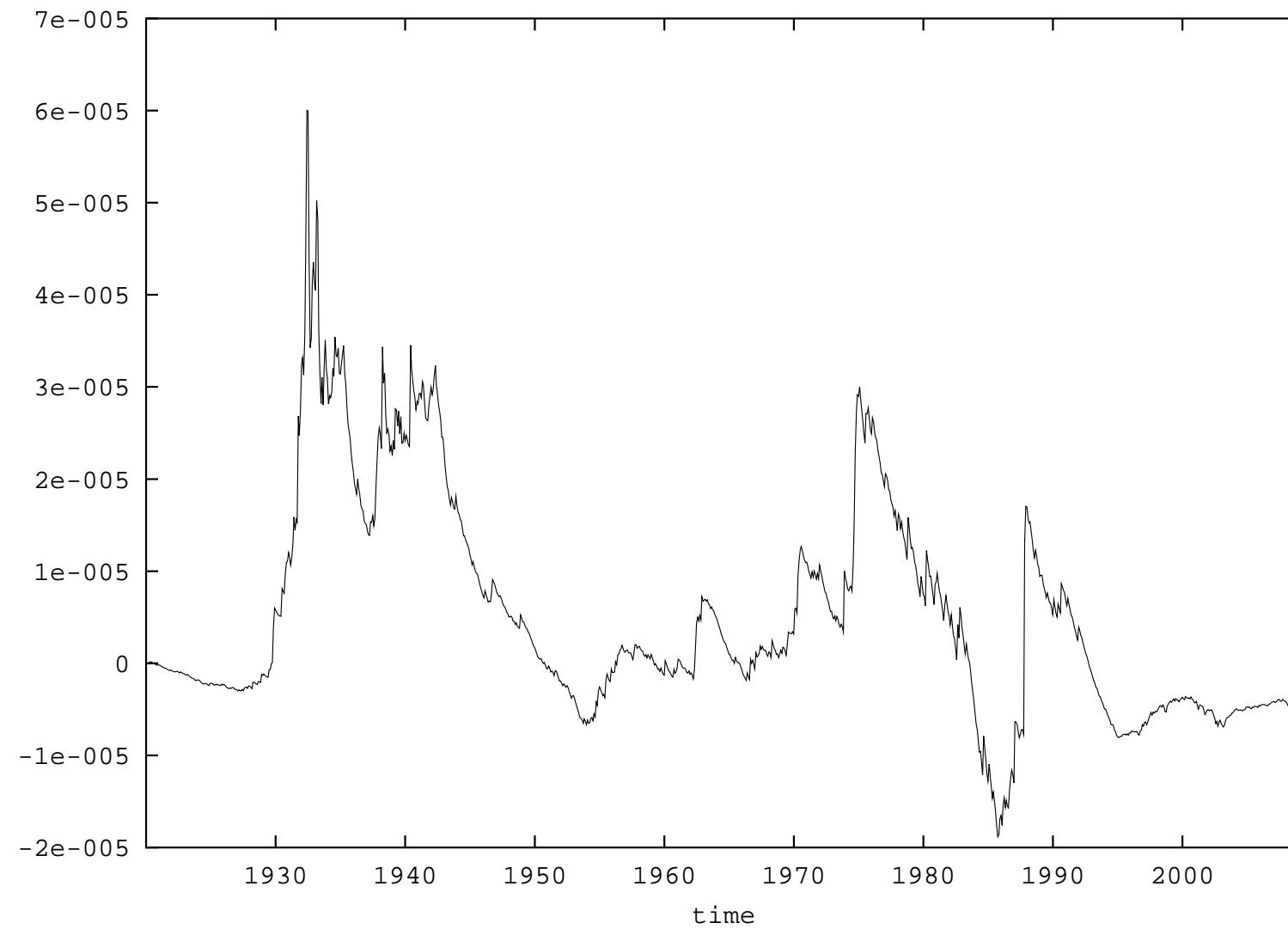


Figure 9: Benchmarked profit and loss.

- European put option under the MMM

Hulley, Miller & Pl. (2005)

$$p(t, V_t^*, T, K, r) = V_t^* E_t \left(\frac{(K - V_\tau^*)^+}{V_\tau^*} \right)$$

$$\begin{aligned} p(t, V_t^*, T, K, r) &= -V_t^* \chi^2(d_1; 4, l_2) \\ &\quad + K e^{-r(T-t)} (\chi^2(d_1; 0, l_2) - \exp \{-l_2/2\}) \end{aligned}$$

with

$$d_1 = \frac{4\eta K \exp\{-r(T-t)\}}{B_t \alpha_t (\exp\{\eta(T-t)\} - 1)}$$

and

$$l_2 = \frac{4\eta V_t^*}{B_t \alpha_t (\exp\{\eta(T-t)\} - 1)}$$

$\chi^2(x; n, l)$ non-central chi-square distribution function

$n \geq 0$ degrees of freedom

non-centrality parameter $l > 0$

$$\chi^2(x; n, l) = \sum_{k=0}^{\infty} \frac{\exp\left\{-\frac{l}{2}\right\} \left(\frac{l}{2}\right)^k}{k!} \left(1 - \frac{\Gamma\left(\frac{x}{2}; \frac{n+2k}{2}\right)}{\Gamma\left(\frac{n+2k}{2}\right)}\right)$$

- putative risk neutral price

$$\tilde{p}(t, V_t^*, T, K, r) = p(t, V_t^*, T, K, r) + K e^{-r(T-t)} \exp\left\{-\frac{\bar{V}_t^*}{2(\varphi(T) - \varphi(t))}\right\}$$

risk neutral is overpricing

$$\text{for } \bar{V}_t^* \rightarrow 0 \implies \tilde{p}(t, V_t^*, T, K, r) \rightarrow 0$$

and

$$\tilde{p}(t, V_t^*, T, K, r) \rightarrow K e^{-r(T-t)} > 0$$

risk neutral ignores trends

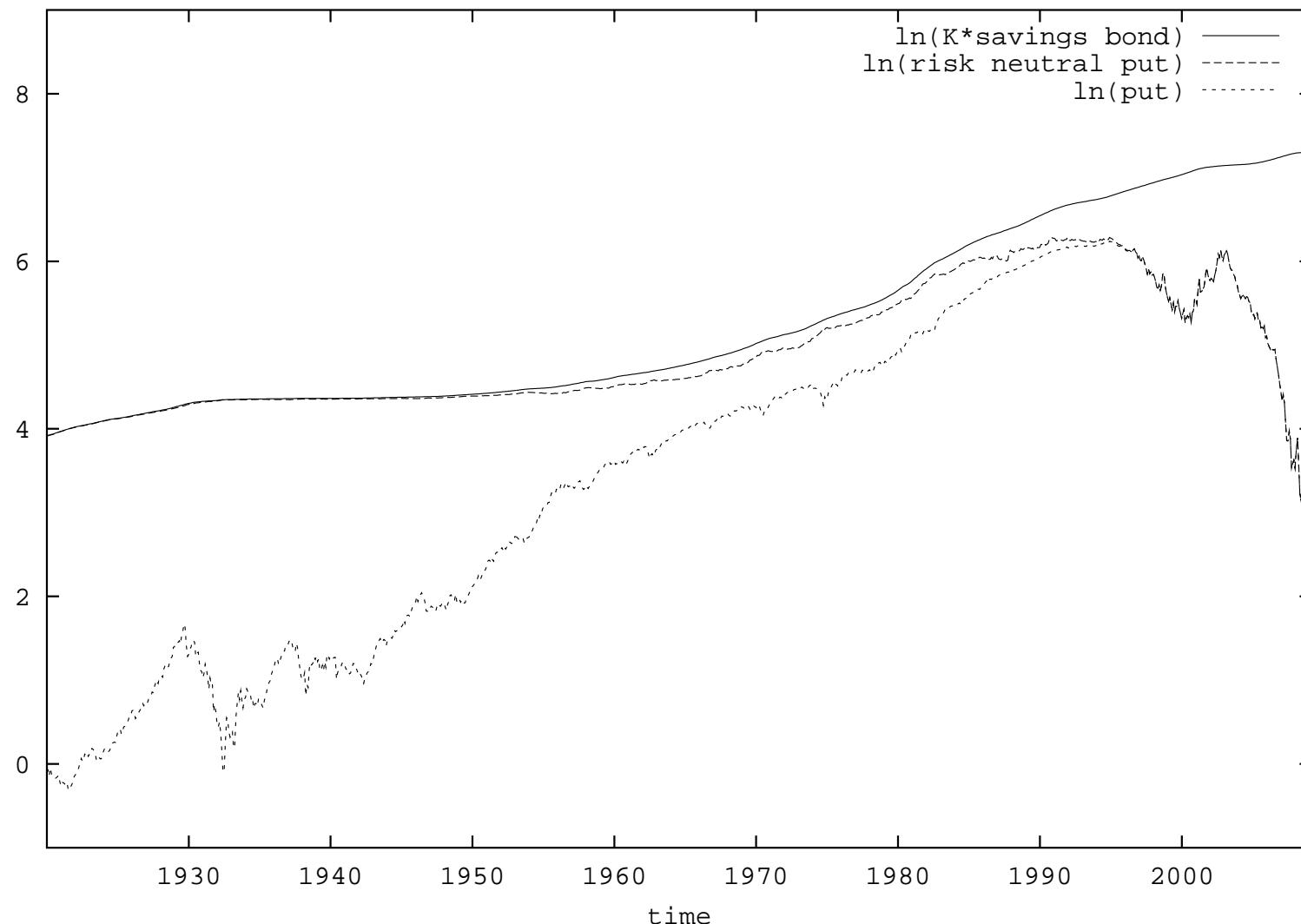


Figure 10: Logarithms of savings bond times K , risk neutral put and fair put.

Guaranteed Minimum Death Benefit (GMDB)

- payout to the policyholder

$$\max(e^{g\tau} V_0, V_\tau)$$

time of death τ

$g \geq 0$ is the guaranteed instantaneous growth rate

V_0 is the initial account value

V_τ is the unit value of the policyholder's account at time of death τ

- insurance charges $\xi \geq 0$

\implies policyholder's unit value

$$V_t = e^{-\xi t} V_t^*$$

\implies

$$\begin{aligned} \max(e^{g\tau} V_0, V_\tau) &= \max(e^{g\tau} V_0, e^{-\xi\tau} V_\tau^*) \\ &= e^{-\xi\tau} \max(e^{(g+\xi)\tau} V_0, V_\tau^*) \end{aligned}$$

insurance company invests the entire fund value V in the NP V^*

\implies payoff

$$H_T = GMDB_T = e^{-\xi T} \left[(e^{(g+\xi)T} V_0^* - V_T^*)^+ + V_T^* \right]$$

fair value $GMDB_0$ of the total claim

real world pricing formula

$$\begin{aligned} GMDB_0 &= V_0^* E \left(\frac{GMDB_T}{V_T^*} \right) \\ &= e^{-\xi T} \left(p(0, V_0^*, T, e^{(g+\xi)T} V_0^*, r) + V_0^* \right) \end{aligned}$$

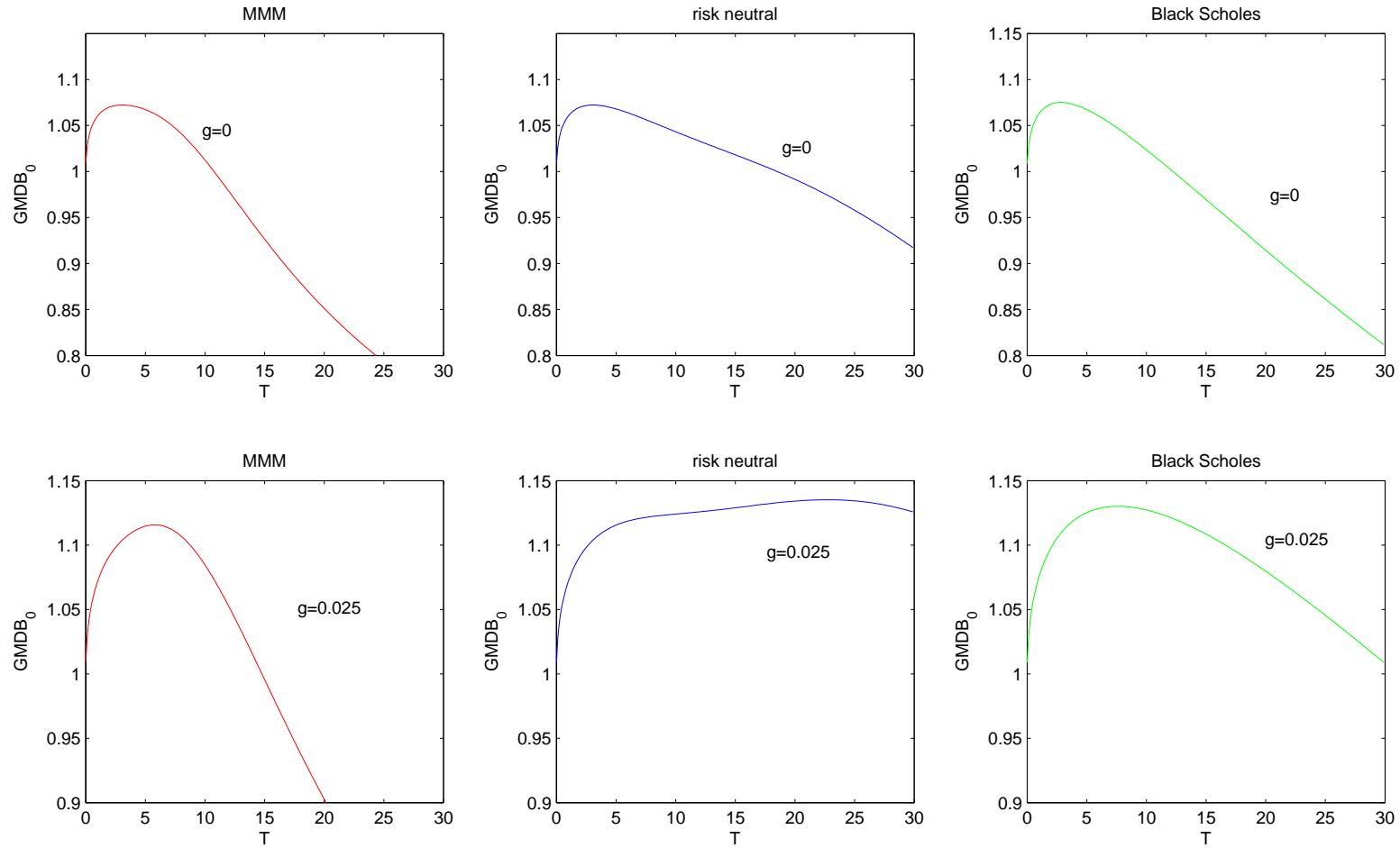


Figure 11: Present value of the GMDB under the real world pricing formula (left), the risk neutral pricing formula (middle) and the Black Scholes formula (right) for $\eta = 0.05$, $\alpha_0 = 0.05$, $r = 0.05$, $\xi = 0.01$ and $Y_0 = 20$.

- lifetime τ is stochastic

$$GMDB_0 = V_0^* E \left(E \left(\frac{GMDB_\tau}{V_\tau^*} \mid \mathcal{F}_\tau \right) \right)$$

$$GMDB_0 = \int_0^T \left(p(0, V_0^*, t, e^{(g+\xi)t} V_0^*, r) + V_0^* \right) e^{-\xi t} f_\tau(t) dt$$

$f_\tau(\cdot)$ - mortality density

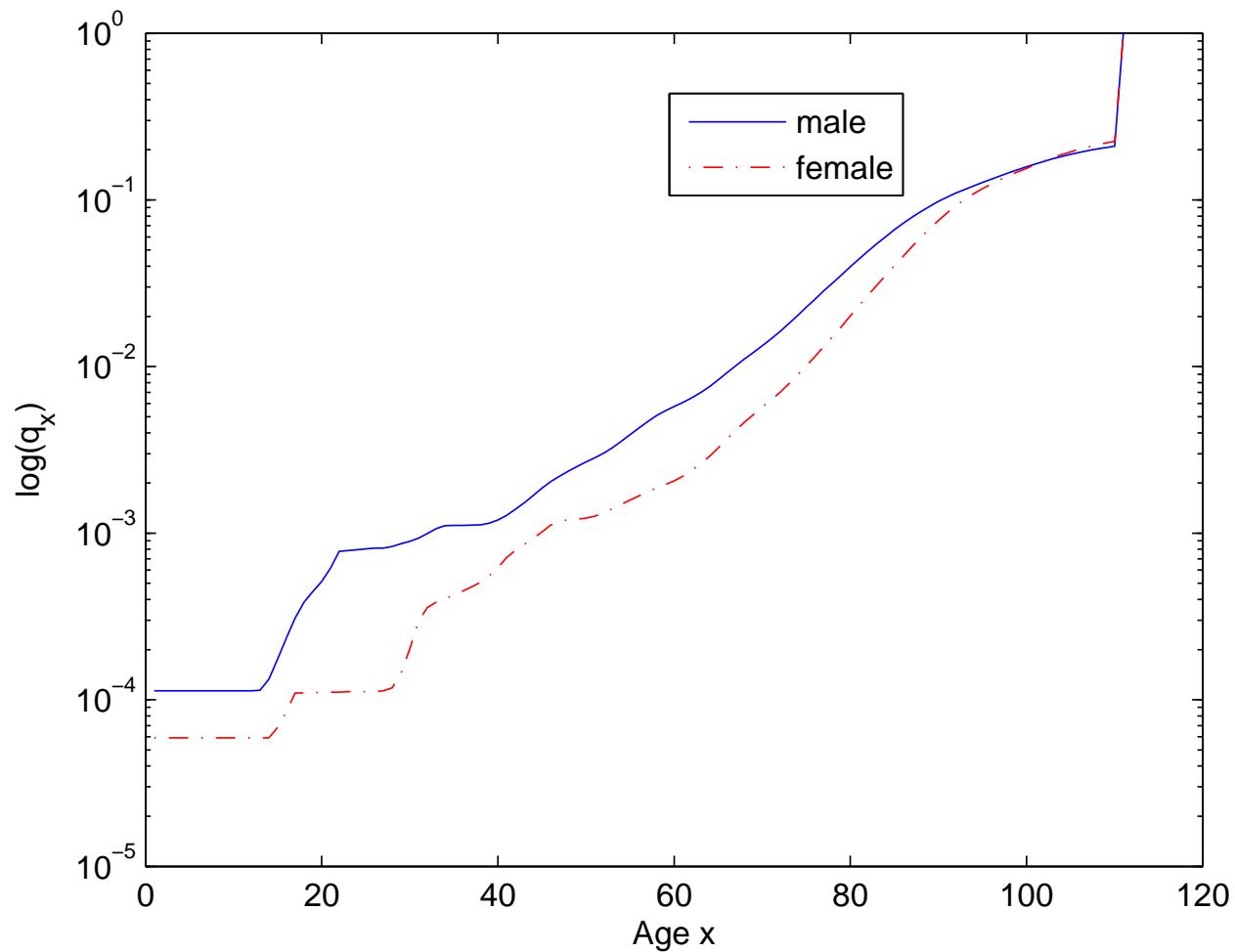


Figure 12: German mortality data

rational investors will lapse when embedded put-option out of the money
GMDBs have stochastic maturity
Titanic options, Milevsky & Posner (2001)

- **roll-up GMDB payoff**

$$H_t = \begin{cases} (1 - \beta_t) V_t^*, & \text{if lapsed at time } t, \\ \max(e^{g\tau} V_0, V_\tau^*), & \text{if death occurs at time } t = \tau \end{cases}$$

surrender charge β_t

$$\beta_t = \begin{cases} (8 - \lceil t \rceil)\%, & t \leq 7, \\ 0, & t > 7 \end{cases}$$

no credit risk

no accumulation phase

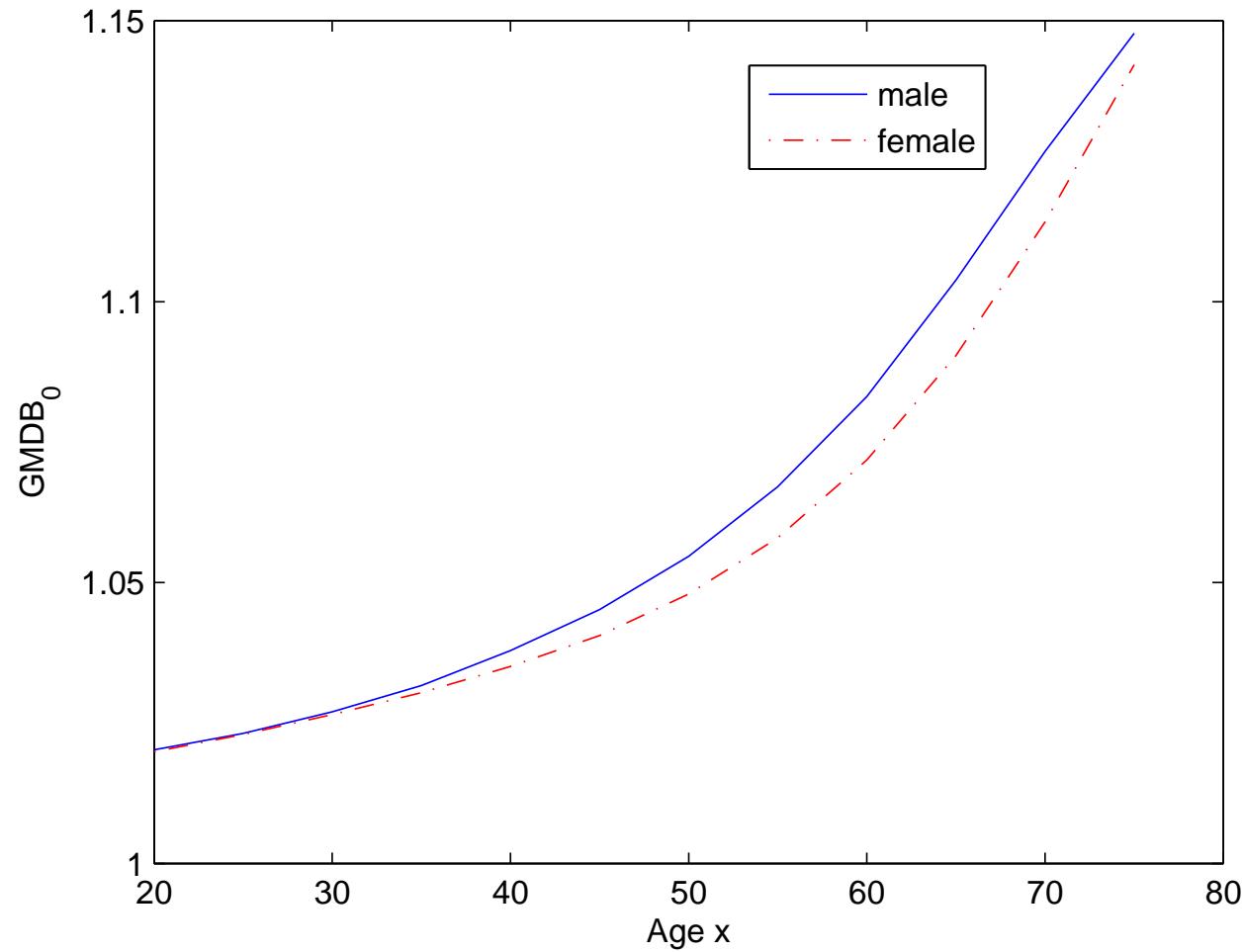


Figure 13: Value of the GMDB under the MMM for male and female policyholders aged x , assuming an irrational lapsation of $l = 1\%$.

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