Credit Risk via First Passage for Time Changed Brownian Motions

Tom Hurd

Mathematics and Statistics, McMaster University

Quantitative Finance Seminar, Fields Institute 2009
Joint work with Zhuowei Zhou

Capital Structure of a Firm

A stylized balance sheet:

1. Assets equal Liabilities:

$$V_t(\text{assets}) = D_t + E_t \quad (\text{debt} + \text{equity});$$

2. Solvency condition: log-leverage $\log(V_t/D_t) > 0$;
3. Time of Default $t^* = \inf\{t | \log(V_t/D_t) \leq 0\}$.

Desirable Features of Credit Models

We want models of V_t, D_t, E_t, t^* that are

1. Realistic and flexible;
2. APT consistent (no arbitrage): physical and risk neutral measures;
3. Time consistent (no ad hoc time assumptions);
4. computationally efficient;
5. realistic default dependence (in multifirm setting).

Credit Model Ratings

	Realism Flexibility	APT	Time Consistent	Efficient	Dependence
M	*	***	-	***	***
BC	*	***	***	**	*
RFM	*	***	***	***	*
CWL	*	***	***	**	-
TCBM	**	***	***	**	*

Table:
Desirable features of Credit Models: $-$ (bad) to $***$ (excellent). M=Merton 74; BC=Black-Cox 76; RFM=Reduced Form Model 95; CWL=Carr-Wu-Linetsky 07; TCBM=Time-Changed Brownian Motion 08

TCBM Modeling

1. $(\Omega, \mathcal{F}, \mathbb{F}, P)$, a filtered probability space;
2. $x + \sigma W_t + \beta\sigma^2 t$, Brownian motion starting at x having constant drift $\beta\sigma^2$ and volatility $\sigma > 0$;
3. Time-change, an independent strictly increasing cádlág process G_t with $G_0 = 0$;
4. *Laplace exponent process* for G:

$$\psi_s(u,t) = -\log E[e^{-uG_t} | \mathcal{F}_s], \quad s < t.$$

TCBM Credit Modeling Assumptions

Assumption

1. *Firm log-leverage process* $X_t = \log(V_t/D_t)$ *is the (stopped) time-changed Brownian motion (TCBM)* $X_t = x + \sigma W_{G_t} + \beta\sigma^2 G_t$ *generated by* W *and the normalized* G*:*
$$\lim_{T\to\infty} T^{-1} E[G_T] = 1.$$
2. *The firm's default time* t^* *is the first passage time of the second kind for* X_t *to hit zero, i.e. the stopping time*
$$t^* = \inf\{t | G_t \geq \tilde{t}\}, \quad \tilde{t} = \inf\{t | x + \sigma W_t + \beta\sigma^2 t \leq 0\}.$$

Examples of Time Changes

1. Exponential jump model: G_t is the Lévy subordinator with

$$\psi^{exp}(u,t) = t[(1-a)u + au/(u+b)], a \in [0,1], b > 0$$

Time increases uniformly at the rate $1-a$ in between jumps that occur at rate a/b and are $exp(b)$ distributed.

2. Variance Gamma model: G_t is the drifting gamma process with

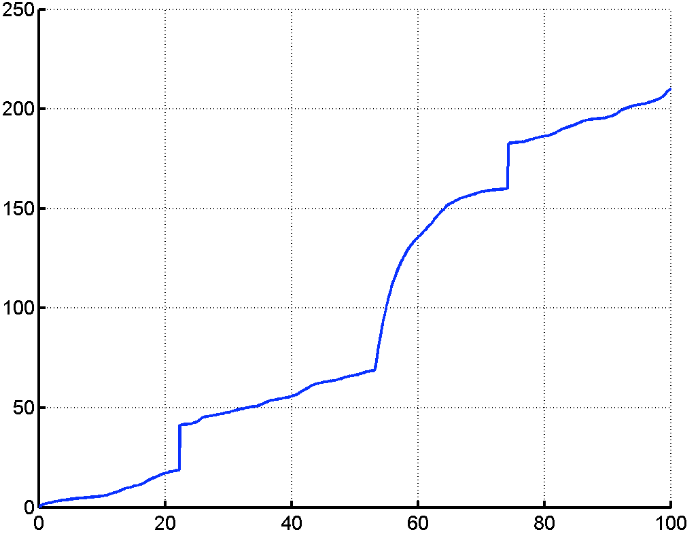
$$\psi^{VG}(u,t) = t[(1-a)u + ab\log(1+u/b)], a \in [0,1], b > 0$$

Here, small jumps occur with infinite activity.

3. Heston stochastic volatility model (subclass): $G_t = \int_0^t \lambda_s ds$ where λ is the CIR process:

$$d\lambda_t = b(1-\lambda)dt + \sqrt{2a\lambda_t}dW_t, a, b > 0$$

Sample Time Change Path



Formulas for Brownian Motion

Joint distribution function

$$
\begin{aligned}
P^{BM}(x,y,t) &= P_x[\tilde{t} > t, x + \sigma W_t + \beta\sigma^2 t \geq y] \\
&= N\left(\frac{x - y + \beta\sigma^2 t}{\sigma\sqrt{t}}\right) - e^{-2\beta x} N\left(\frac{-x - y + \beta\sigma^2 t}{\sigma\sqrt{t}}\right)
\end{aligned}
$$

$$
= \frac{e^{\beta(y-x)}}{\pi}\int_{-\infty}^{\infty} \frac{[z\cos(zy) - \beta\sin(zy)]\sin(zx)}{z^2+\beta^2} e^{-\sigma^2(z^2+\beta^2)t/2} dz,
$$

for $x, y \geq 0$. While the first of these is well known, the second is more useful for TCBMs.

Formula for TCBMs

Joint distribution function

$$P^{TCBM}(x,y,t) = E_x[\mathbf{1}_{\{\tilde{t}>G_t\}}\mathbf{1}_{\{x+\sigma W_{G_t}+\beta\sigma^2 G_t\geq y\}}]$$

$$= E[P^{BM}(x,y,G_t)]$$

$$= \frac{e^{\beta(y-x)}}{\pi}\int_{-\infty}^{\infty}\frac{[z\cos(zy)-\beta\sin(zy)]\sin(zx)}{z^2+\beta^2}e^{-\psi_0(\sigma^2(z^2+\beta^2)/2,t)}dz$$

Bond and CDS prices follow immediately.

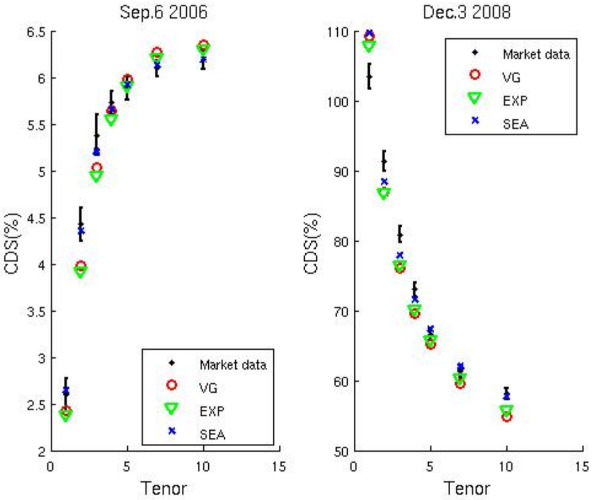
Ford 06-08: VG Model Results

		Jan 06-June 07	July 07-Dec 08
	number of weeks	78	78
	$\hat{\sigma}$	0.222(0.018)	0.226(0.027)
	$\hat{b}$	0.25	0.25
	$\hat{c}$	0.822(0.021)	0.574(0.020)
VG Model	$\hat{\beta}$	-1.125(0.013)	-1.003(0.020)
	$\hat{x}_{av}$	0.544	0.370
	$\hat{x}_{std}$	0.137	0.183
	RMSE (bid/ask units)	1.496	0.939

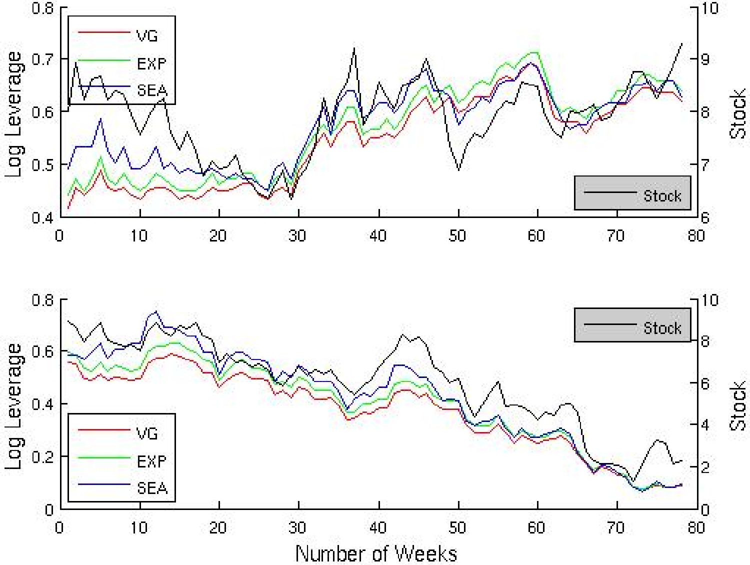
VG Default Model Results: Ford

Ford CDS curves: VG model vs Market

Ford CDS curves: 06/09/2006 (left) and 3/12/08(right)



Ford's log-leverage ratio versus stock price 2006-08



The TCBM Program

Implement and extend this framework.

1. *Bond and CDS pricing;*
2. *Portfolio credit and CDO pricing (work in progress);*
3. *Systemic risk (work in progress);*
4. *Joint Credit-Equity modeling...*

Market-based View of Capital Structure

The capital structure of a major firm is “measured” not just by balance sheets but also by prices of a large number of liquid financial securities written on it or by it.

- Prices of bonds and Credit Default Swaps of different maturities reflect the value of the firm’s debt, PD (probability of default), and LGD (loss given default);

Market-based View of Capital Structure

The capital structure of a major firm is "measured" not just by balance sheets but also by prices of a large number of liquid financial securities written on it or by it.

- Prices of bonds and Credit Default Swaps of different maturities reflect the value of the firm's debt, PD (probability of default), and LGD (loss given default);
- Stock price S_t plus dozens of stock options with different strikes and maturities measure market value of the firm's equity and (indirectly) firm value;

Market-based View of Capital Structure

The capital structure of a major firm is "measured" not just by balance sheets but also by prices of a large number of liquid financial securities written on it or by it.

- Prices of bonds and Credit Default Swaps of different maturities reflect the value of the firm's debt, PD (probability of default), and LGD (loss given default);
- Stock price S_t plus dozens of stock options with different strikes and maturities measure market value of the firm's equity and (indirectly) firm value;
- Market data is available instantaneously, while balance sheets are updated only quarterly.

Market-based View of Capital Structure

The capital structure of a major firm is "measured" not just by balance sheets but also by prices of a large number of liquid financial securities written on it or by it.

- Prices of bonds and Credit Default Swaps of different maturities reflect the value of the firm's debt, PD (probability of default), and LGD (loss given default);
- Stock price S_t plus dozens of stock options with different strikes and maturities measure market value of the firm's equity and (indirectly) firm value;
- Market data is available instantaneously, while balance sheets are updated only quarterly.
- How well does market data reveal capital structure?

Modeling Literature Review

1. Capital Structure Models: Merton 74, Black-Cox 76, Geske 77, Leland-Toft 96, Hull-Nelkin-White 03;
2. Hybrid Models: Madan-Unal 00, Albanese-Chen 04, Carr-Wu 05, Carr-Linetsky 06, Bayraktar-Yang 09;
3. Convertible Bonds: Grimwood-Hodges 02, Andersen-Buffum 03.

Aim of this Model

A market-based "structural" approach to joint valuation of debt and equity that respects the tenets of corporate finance.

It should

1. Be internally consistent, that is arbitrage-free;

Aim of this Model

A market-based "structural" approach to joint valuation of debt and equity that respects the tenets of corporate finance.
It should

1. Be internally consistent, that is arbitrage-free;
2. Reflect the capitalization structure of a typical firm;

Aim of this Model

A market-based "structural" approach to joint valuation of debt and equity that respects the tenets of corporate finance.
It should

1. Be internally consistent, that is arbitrage-free;
2. Reflect the capitalization structure of a typical firm;
3. Be simple enough, yet flexible;

Aim of this Model

A market-based "structural" approach to joint valuation of debt and equity that respects the tenets of corporate finance.

It should

1. Be internally consistent, that is arbitrage-free;
2. Reflect the capitalization structure of a typical firm;
3. Be simple enough, yet flexible;
4. Give a good fit to reality for a variety of types of firms.

Promising Areas of Application

1. Capital Structure Arbitrage: an arbitrage-free framework for credit and equity can spot mispricing in the market, and suggest improved "hedge fund" strategies;

Promising Areas of Application

1. Capital Structure Arbitrage: an arbitrage-free framework for credit and equity can spot mispricing in the market, and suggest improved "hedge fund" strategies;
2. Hedging credit securities with equity options and vice versa;

Promising Areas of Application

1. Capital Structure Arbitrage: an arbitrage-free framework for credit and equity can spot mispricing in the market, and suggest improved "hedge fund" strategies;
2. Hedging credit securities with equity options and vice versa;
3. Pricing convertible bonds, equity default swaps and other "hybrid" products;

Promising Areas of Application

1. Capital Structure Arbitrage: an arbitrage-free framework for credit and equity can spot mispricing in the market, and suggest improved "hedge fund" strategies;
2. Hedging credit securities with equity options and vice versa;
3. Pricing convertible bonds, equity default swaps and other "hybrid" products;
4. Market and credit risk management.

Model Assumptions

1. Risk neutral probability space $(\Omega, \mathcal{F}, \tilde{P})$;

Model Assumptions

1. Risk neutral probability space $(\Omega, \mathcal{F}, \tilde{P})$;
2. Constant interest rate r, wlog $r = 0$;

Model Assumptions

1. Risk neutral probability space $(\Omega, \mathcal{F}, \tilde{P})$;
2. Constant interest rate r, wlog $r = 0$;
3. all non-dividend paying traded assets are $\tilde{P}$-martingales;

Model Assumptions

1. Risk neutral probability space $(\Omega, \mathcal{F}, \tilde{P})$;
2. Constant interest rate r, wlog $r = 0$;
3. all non-dividend paying traded assets are $\tilde{P}$-martingales;
4. Stock price process $S_t \geq 0$ (observed directly);

Model Assumptions

1. Risk neutral probability space $(\Omega, \mathcal{F}, \tilde{P})$;
2. Constant interest rate r, wlog $r = 0$;
3. all non-dividend paying traded assets are $\tilde{P}$-martingales;
4. Stock price process $S_t \geq 0$ (observed directly);
5. log-leverage process $X_t \geq 0$ (not observed directly);

Model Assumptions

1. Risk neutral probability space $(\Omega, \mathcal{F}, \tilde{P})$;
2. Constant interest rate r, wlog $r = 0$;
3. all non-dividend paying traded assets are $\tilde{P}$-martingales;
4. Stock price process $S_t \geq 0$ (observed directly);
5. log-leverage process $X_t \geq 0$ (not observed directly);
6. Time of default

$$t^* = \inf\{t | S_t = 0\} = \inf\{t | X_t = 0\};$$

Model Assumptions

1. Risk neutral probability space $(\Omega, \mathcal{F}, \tilde{P})$;
2. Constant interest rate r, wlog $r = 0$;
3. all non-dividend paying traded assets are $\tilde{P}$-martingales;
4. Stock price process $S_t \geq 0$ (observed directly);
5. log-leverage process $X_t \geq 0$ (not observed directly);
6. Time of default
$$t^* = \inf\{t | S_t = 0\} = \inf\{t | X_t = 0\};$$
7. For $t > t^*$, $X_t = S_t = 0$.

Related Processes

We can define V_t, D_t by

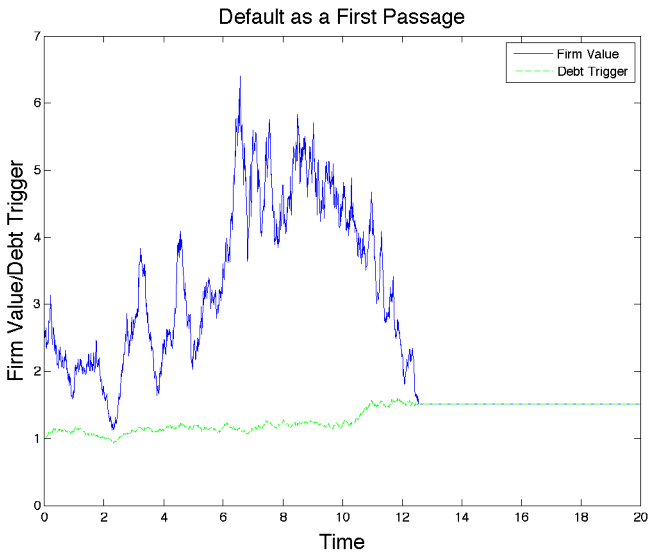
$$V_t := e^{v_t} = \frac{S_t}{1 - e^{-X_t}} \;\geq\; D_t := e^{d_t} = \frac{e^{-X_t} S_t}{1 - e^{-X_t}} > 0.$$

Then

$$X_t = \log\left(V_t / D_t\right), \quad S_t = V_t - D_t.$$

1. V_t is reminiscent of "market value of the firm", per share;
2. D_t is reminiscent of "market value of the debt", per share;
3. D_t plays the role of a stochastic default trigger in structural models.

Default Simulation



Modeling Considerations

1. For simplicity: only two independent factors.

Modeling Considerations

1. For simplicity: only two independent factors.
2. $S, X, V = e^v, D = e^d$ should be positively, but not perfectly correlated;

Modeling Considerations

1. For simplicity: only two independent factors.
2. $S, X, V = e^v, D = e^d$ should be positively, but not perfectly correlated;
3. Healthy firms have $X \gg \mathsf{0}$ and $V \gg D$, and hence $S \sim V$. Suggests that V be modeled analogously to traditional stock models, eg as geometric BM or TCBM.

Modeling Considerations

1. For simplicity: only two independent factors.
2. $S, X, V = e^v, D = e^d$ should be positively, but not perfectly correlated;
3. Healthy firms have $X \gg 0$ and $V \gg D$, and hence $S \sim V$. Suggests that V be modeled analogously to traditional stock models, eg as geometric BM or TCBM.
4. Computability: Models for X_t must lead to solvable first passage problem.

Further Model Assumptions

Motivated by these considerations we assume:

1. V_t, D_t, S_t are $\tilde{P}$-martingales;
2. $v = \mathsf{log}\, V, d = \mathsf{log}\, D, v - d$ are stopped TCBMs with a common time change G.

Securities Valuation

1. Zero coupon bond price (with stochastic recovery RD_{t^*}/D_0):

$$\bar{P}_0(T) = \tilde{E}[\mathbf{1}_{\{t^*>T\}} + \mathbf{1}_{\{t^*\le T\}} RD_{t^*}/D_0]$$

2. Credit derivatives such as CDS, coupon bonds etc, follow from $\bar{P}_t(T)$: these depend strongly only on factor X.
3. Equity derivatives depend strongly on both factors:
4. Call option price:

$$\mathsf{Call}^{KT}(v_0, d_0) = \tilde{E}_{v_0,d_0}[(e^{v_T} - e^{d_T} - K)^+ \mathbf{1}_{\{t^*>T\}}]$$

A barrier spread option.

New Result for Spread Options

We can efficiently compute spread options using a new result:

Proposition (Hurd-Z. Zhou, 2009)

The payoff function has the Fourier representation

$$(e^{x_1} - e^{x_2} - 1)^+ = (2\pi)^{-2} \iint_{\mathbb{R}^2 + i\epsilon} e^{i(u_1 x_1 + u_2 x_2)} \hat{P}(u_1, u_2) du_1 du_2$$

for any $\epsilon = (\epsilon_1, \epsilon_2)$ *with* $\epsilon_2 > 0$ *and* $\epsilon_1 + \epsilon_2 < 0$. *Here*

$$\hat{P}(u_1, u_2) = \frac{\Gamma(iu_1 + iu_2 - 1)\Gamma(-iu_2)}{\Gamma(iu_1 + 1)},$$

where $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ *is the complex gamma function defined for* $\Re z > 0$.

Call Option Formula

Corollary

In the case of a single time change G,

$$\mathsf{Call}_0^{1T}(v_0, d_0) = \iint_{\mathbb{R}^2 + i(0,\tilde{\epsilon}_2)} e^{i(z - i\beta_1)X_0 + iu\tilde{X}_0^{\perp}} F(z,u;T)Q(z,u)\,dz\,du.$$

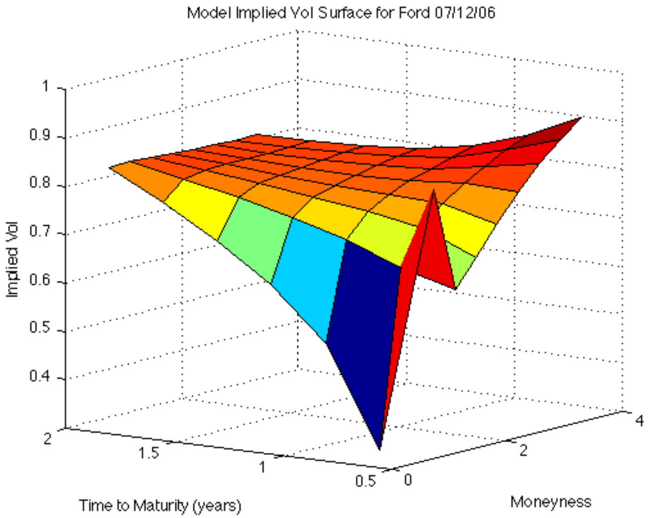
where

$$\begin{aligned} F(z,u;T) &= \exp\left[-\psi_G\left(r + \tilde{\sigma}_1^2(z^2 + \beta_1^2)/2 + \tilde{\sigma}_2^2(u^2/2 - i\beta_2 u), T\right)\right] \\ Q(z,u) &= \frac{iz(\det M)}{4\pi^3} \int_{\mathbb{R} + i\tilde{\epsilon}_1} \frac{1}{(v - i\beta_1)^2 - z^2} \hat{P}(M'[v;u])\,dv \end{aligned}$$

and $M = [1,\ -1; 1\ m]$.

This gives call option price as a function of the current values S_0, X_0 in an explicit two-dimensional FFT.

Ford Model Implied Vol Surface: 12/07/06



Conclusions

1. New results in first passage expand the range of feasible structural credit models;

Conclusions

1. New results in first passage expand the range of feasible structural credit models;
2. Two factor models for (S_t, X_t, V_t, D_t) are a simple, natural generalization of traditional "structural credit models" whose detailed features have not yet been studied;

Conclusions

1. New results in first passage expand the range of feasible structural credit models;
2. Two factor models for (S_t, X_t, V_t, D_t) are a simple, natural generalization of traditional "structural credit models" whose detailed features have not yet been studied;
3. FFT methods lead to fast computations.

Conclusions

1. New results in first passage expand the range of feasible structural credit models;
2. Two factor models for (S_t, X_t, V_t, D_t) are a simple, natural generalization of traditional “structural credit models” whose detailed features have not yet been studied;
3. FFT methods lead to fast computations.
4. TCBMs extend efficiently to multiple firms, enabling CDO pricing and portfolio credit VaR.

Conclusions

1. New results in first passage expand the range of feasible structural credit models;
2. Two factor models for (S_t, X_t, V_t, D_t) are a simple, natural generalization of traditional "structural credit models" whose detailed features have not yet been studied;
3. FFT methods lead to fast computations.
4. TCBMs extend efficiently to multiple firms, enabling CDO pricing and portfolio credit VaR.
5. Much work to be done!

Some Papers

Available (or will be soon) at www.math.mcmaster.ca/tom

1. T. R. Hurd, "Credit Risk Modelling using time-changed Brownian motion"
2. T. R. Hurd, A. Kuznetsov, "On the first passage time for Brownian motion subordinated by a Levy process"
3. T. R. Hurd and Z. Zhou. "Modeling credit derivatives via time changed Brownian motions"
4. T. R. Hurd and Z. Zhou. "A Fourier transform method for spread option pricing"