A Stylized Markovian Copula Model of Counterparty Credit Risk: CVA computation under netting and collateralization

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Outline

1. General Counterparty Risk
2. Counterparty Credit Risk
3. A Benchmark Problems of Counterparty Credit Risk
4. Portfolio Case
5. Conclusion

Basic Concept

Risk that some value is lost by a party in OTC derivatives contracts due to the default of the other party [Canabarro and Duffie 03, Brigo et al. *]

- Promised payments not paid
- Early termination of a contract with positive value at time of default of the other party

The primary form of (credit) risk – vulnerability
Very significant during the crisis
An important dynamic modeling issue/challenge, particularly in connection with credit derivatives

General Set-Up

$(\Omega, \mathbb{F}, \mathbb{P})$, $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ risk-neutral pricing model (with $r = 0$ for notational simplicity, except in the numerical part)

$\mathbb{E}_t$ Conditional expectation under $\mathbb{P}$ given $\mathcal{F}_t$

τ_{-1} and τ_0 Default times of the two parties, referred to henceforth as the investor, labeled -1, and its counterparty, labeled 0

- $[0, +\infty]$-valued $\mathbb{F}$-stopping times
- Bilateral counterparty risk $\leftrightarrow$ counterparty risk on both sides is considered $\leftrightarrow \tau_{-1} < +\infty,\ \tau_0 < +\infty$
- Unilateral counterparty risk$\leftrightarrow \tau_{-1} = +\infty$

R_{-1} and R_0 Recovery rates, given as $\mathcal{F}_{\tau_{-1}}$- and $\mathcal{F}_{\tau_0}$- measurable $[0, 1]$-valued random variables

τ $\tau_{-1} \wedge \tau_0$, with related default and non-default indicator processes denoted by H and J, so $H_t = \mathbb{1}_{\tau\le t}$ and $J = 1 - H$.

- No actual cash flow after τ

All cash flows and prices considered from the perspective of the investor

Cash Flows

General case reduced to that of a

Fully netted and collateralized portfolio

Δ Counterparty risky cumulative cash flows

D Counterparty clean cumulative cash flows

$\Longrightarrow$

$$\Delta = JD + HD_{\tau-} + H\Gamma_{\tau}$$
$$+ (R_0\chi^{+} - \chi^{-})[H, H^0] - (R_{-1}\chi^{-} - \chi^{+})[H, H^{-1}] - \chi[[H, H^0], H^{-1}]$$

Γ_{τ} Value of the collateral (or margin account) at time τ

$\chi = P_{(\tau)} + (D_{\tau} - D_{\tau-}) - \Gamma_{\tau}$ Algebraic 'debt' of the counterparty to the investor at time τ

$P_{(\tau)}$ 'Fair (ex-dividend) value' of the portfolio at τ

$D_{\tau} - D_{\tau-}$ Promised cash flow at τ

Collateral Formation Qualification

Remark

We need to stress that in order to simplify our presentation we give a highly stylized model for the collateral process. In particular we do not explicitly account for such aspects of the collateral formation as

- haircut provisions,
- margin period of risk,
- minimum transfer amounts,
- collateral thresholds.

We shall incorporate these important considerations into our model in a future paper.

Representation Formulas: I

$\Pi_t := \mathbb{E}_t\left[\Delta_T - \Delta_t\right]$ Counterparty Risky Value of the portfolio

$P_t := \mathbb{E}_t\left[D_T - D_t\right]$ Counterparty Clean Value of the portfolio

Market 'Legal Value' standard $P_{(\tau)} = P_\tau$ assumed for simplicity

CVA (Credit Valuation Adjustment)

$$\mathrm{CVA}_t := J_t(P_t - \Pi_t)$$

can be represented as

$$\mathrm{CVA}_t = J_t \mathbb{E}_t\left[\xi\right] ,$$

where the $\mathcal{F}_\tau$-measurable Potential Future Exposure at Default (PFED) ξ is given by

$$\xi = (1 - R_0)\mathbb{1}_{\tau=\tau_0}\chi^+ - (1 - R_{-1})\mathbb{1}_{\tau=\tau_{-1}}\chi^- = \xi^+ - \xi^-$$

Representation Formulas: II

Expected Exposures (EEs) and CVA

$$\begin{aligned}\mathrm{CVA}_0 &= \int_0^T \beta_s \mathrm{EE}_+(s)\mathbb{P}(\tau_0 \in ds,\, \tau_{-1} \geq s) \\ &- \int_0^T \beta_s \mathrm{EE}_-(s)\mathbb{P}(\tau_{-1} \in ds,\, \tau_0 \geq s)\end{aligned}$$

where the Expected (Positive) Exposures $\mathrm{EE}_\pm$, also known as the Asset Charge and the Liability Benefit, respectively, are the functions of time defined by, for $t \in [0, T]$,

$$\mathrm{EE}_+(t) = \mathbb{E}\left[(1 - R_0)\chi^+ | \tau_0 = t \leq \tau_{-1}\right],$$

$$\mathrm{EE}_-(t) = \mathbb{E}\left[(1 - R_{-1})\chi^- | \tau_{-1} = t \leq \tau_0\right] .$$

Representation Formulas: III

Expected Exposures ($\mathcal{E}\mathcal{E}$s) and CVA

$$\begin{aligned} \mathrm{CVA}_0 = & \int_0^T \int_s^T \beta_s \mathcal{E}\mathcal{E}_+(s,u)\mathbb{P}(\tau_0 \in ds,\, \tau_{-1} \in du) \\ & - \int_0^T \int_s^T \beta_s \mathcal{E}\mathcal{E}_-(u,s)\mathbb{P}(\tau_{-1} \in ds,\, \tau_0 \in du) \end{aligned}$$

with

$$\begin{aligned} \mathcal{E}\mathcal{E}_+(t,r) &= \mathbb{E}\left[(1-R_0)\chi^+ | \tau_0 = t, \tau_{-1} = r\right], \\ \mathcal{E}\mathcal{E}_-(r,t) &= \mathbb{E}\left[(1-R_{-1})\chi^- | \tau_0 = r, \tau_{-1} = t\right] \ . \end{aligned}$$

Remark

Need of a dynamic, tractable model for P_t, Γ_t

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Counterparty Risk is a crucial issue in connection with valuation and risk management of credit derivatives in the crisis

- Wrong Way Risk [Redon 06]
  - Cycle and contagion effects $\rightarrow$ Time of default of a counterparty selling credit protection typically given as a moment of high value of credit protection
- 'Joint Defaults Component' of the PFED hardly collateralizable

$\rightarrow$ Need of an adequate, dynamic and tractable model of dependence between default times

More Set-Up

$\mathbb{N}_n$	$\{-1, 0, \ldots, n\}$
τ_i's	Default times (stopping times) of the investor, its counterparty and n credit names underlying a portfolio of credit derivatives
H^i's	Default indicator processes, so $H^i_t = \mathbb{1}_{\tau_i \leq t}$
R_i's	Recovery rates, assumed to be constant for simplicity

Common Shocks Model [Elouerkhaoui 07, Brigo et al. 07]

Let $\mathcal{I} = \{I_1, \ldots, I_m\}$ denote (few) pre-specified subsets of $\mathbb{N}_n$

Sets of obligors susceptible to default simultaneously

Set $Y = \mathbb{N}_n \cup \mathcal{I}$
Define, for $\imath \in Y$, an intensity function $\lambda_\imath(t)$, and

$$\widehat{\tau}_\imath = \inf\{t > 0; \int_0^t \lambda_\imath(s)ds \geq E_\imath\} ,$$

for IID exponential random variables $E_\imath$s
One then sets, for every $i \in \mathbb{N}_n$

$$\tau_i = \widehat{\tau}_i \wedge \bigwedge_{I \in \mathcal{I};\, I \ni i} \widehat{\tau}_I$$

Immediate extension to stochastic intensities $\lambda_i(t, X_t^i)$, for $i \in \mathbb{N}_n$, for a factor Markov process $\mathrm{X} = (X^i)_{i \in \mathbb{N}_n}$ independent of the $E_\imath$s

Dynamic Perspective

- Let $\mathrm{H} = (H^i)_{i \in \mathbb{N}_n}$
- Model filtration $\mathbb{F} = \mathbb{F}^{\mathrm{X}} \vee \mathbb{F}^{\mathrm{H}}$
- We propose a Markovian bottom-up model of multivariate default times, which will have the following key features:
  (i) The pair (X, H) is Markov in its natural filtration $\mathbb{F}$,
  (ii) Each pair (X^i, H^i) is a Markov process,
  (iii) At every instant, each alive obligor can default individually, or all the surviving names whose indices are in the set I can default simultaneously, for every $I \in \mathcal{I}$
  (iv) No direct contagion effects
  (v) Defaults dependence and Wrong way risk via Joint Defaults

Dynamic Perspective

- We thus define a certain number of groups $I_l \subseteq \mathbb{N}_n$, of obligors who are likely to default simultaneously. Let $\mathcal{I} = (I_l)_l$.
- We define the generator of process $(\mathrm{X}, \mathrm{H}) = (X^i, H^i)_{i \in \mathbb{N}_n}$ as, for $u = u(t, \chi, \varepsilon)$ with $\chi = (x_{-1}, \ldots, x_n) \in \mathbb{R}^{n+2}, \varepsilon = (e_{-1}, \ldots, e_n) \in \{0,1\}^{n+2}$:

$$
\begin{aligned}
\mathcal{A}_t u(t, \varepsilon, \chi) = & \sum_{i \in \mathbb{N}_n} \left(b_i(t, x_i) \partial_{x_i} u(t, \chi, \varepsilon) + \frac{1}{2} \sigma_i^2(t, x_i) \partial^2_{x_i^2} u(t, \chi, \varepsilon) \right) \\
& + \sum_{i,j \in \mathbb{N}_n;\, i<j} \varrho_{i,j}(t) \sigma_i(t, x_i) \sigma_j(t, x_j) \partial^2_{x_i, x_j} u(t, \chi, \varepsilon) \\
& + \sum_{i \in \mathbb{N}_n} \left(\eta_i(t, x_i) - \sum_{I \in \mathcal{I};\, I \ni i} \lambda_I(t, \chi) \right) \left(u(t, \chi, \varepsilon^i) - u(t, \chi, \varepsilon) \right) \\
& + \sum_{I \in \mathcal{I}} \lambda_I(t, \chi) \left(u(t, \chi, \varepsilon^{\mathcal{I}}) - u(t, \chi, \varepsilon) \right) ,
\end{aligned}
$$

Dynamic Perspective

where, for $i, j \in \mathbb{N}_n$ and $I \in \mathcal{I}$:
• b_i, σ_i^2, $\varrho_{i,j}(t)$ and η_i denote suitable drift, variance, correlation and pre-default intensity function-coefficients,
• ε^i, resp. ε^I, denotes the vectors obtained from ε by replacing the component e_i, resp. the components e_j for $j \in I$, by number one,
• the non-negative bounded functions $\lambda_I(t)$ are chosen so that the following holds, for every t, i:

$$\sum_{I \in \mathcal{I};\, I \ni i} \lambda_I(t, \chi) \leq \eta_i(t, x_i) \, .$$

Dynamic Perspective

Markovian Copulae Properties (cf. [Bielecki et al. 08])

The pair (X, H) is a Markov process.
For $i \in \mathbb{N}_n$, the pair (X^i, H^i) is a jointly Markov process admitting the following generator, for $u = u(t, x_i, e_i)$ with $(x_i, e_i) \in \mathbb{R} \times \{0, 1\}$:

$$\begin{aligned} \mathcal{A}^i_t u(t, x_i, e_i) &= b_i(t, x_i)\partial_{x_i} u(t, x_i, e_i) + \frac{1}{2}\sigma_i^2(t, x_i)\partial^2_{x_i^2} u(t, x_i, e_i) \\ &+ \eta_i(t, x_i)\big(u(t, x_i, 1) - u(t, x_i, e_i)\big) \ . \end{aligned}$$

A Tractable Model of Counterparty Credit Risk

- Semi-explicit pricing formulas for single-name credit derivatives like individual CDSs (assuming, say, affine processes X^is)
- Fast recursive convolution pricing schemes for static basket credit derivatives like CDO tranches
- Independent calibration of the model marginals and dependence structure
- Model simulation very fast

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Bilateral CCR on a Payer CDS

'AIG selling protection on LEH to Tom Bielecki'

Investor	Buyer of default protection on a firm ('Tom Bielecki')
Counterparty	Seller of default protection on the firm ('AIG')
Firm	Reference credit underlying the CDS ('LEH')

$\tau_{-1} \vee \tau_0 < +\infty,\ n = 1$

Bilateral CCR on a Payer CDS

Remark

We consider the pre-Big-Bang covenants regarding the cash flows of the CDS contract. That is, we do not include the up-front payment in the cash flows. The developments below can however be easily adapted to the post-Big-Bang universe of CDS contracts.

We consider a counterparty risky payer CDS on name 1 (CDS protection on name 1 bought by the investor, or credit name -1, from its counterparty, represented by the credit name 0). Denoting by T the maturity, κ the contractual spread and $R_1 \in [0,1]$ the recovery, we write

$$C_t = -\kappa(t \wedge T)\,, \;\; \delta_t = (1 - R_1)\mathbb{1}_{t<T}\ .$$

Bilateral CCR on a Payer CDS

Proposition

For a counterparty risky payer CDS, one has,

$$\begin{aligned}\xi &= (1-R_0)\mathbb{1}_{\tau=\tau_0}\Big(P_\tau + \mathbb{1}_{\tau_1=\tau<T}(1-R_1) - \Gamma_\tau\Big)^+ \\ &\quad -(1-R_{-1})\mathbb{1}_{\tau=\tau_{-1}}\Big(P_\tau + \mathbb{1}_{\tau_1=\tau<T}(1-R_1) - \Gamma_\tau\Big)^- \ .\end{aligned}$$

So, in case of no collateralization ($\Gamma = 0$),

$$\xi = (1-R_0)\mathbb{1}_{\tau=\tau_0}\Big(P_\tau^+ + \mathbb{1}_{\tau_1=\tau<T}(1-R_1)\Big) - (1-R_{-1})\mathbb{1}_{\tau=\tau_{-1}}P_\tau^- \ ,$$

and in the case of extreme collateralization ($\Gamma_\tau = P_{\tau-}$),

$$\begin{aligned}\xi &= (1-R_0)\mathbb{1}_{\tau=\tau_0=\tau_1<T}(1-R_1-P_{\tau-})^+ \\ &\quad -(1-R_{-1})\mathbb{1}_{\tau=\tau_{-1}=\tau_1\le T}(1-R_1-P_{\tau-})^- \ .\end{aligned}$$

Bilateral CCR on a Payer CDS: Numerics I

The intensities of default $\eta_i(t, X^i), i = -1, 0, 1$ are assumed to be of the form

$$\eta_i(t, X^i) = a_i + X^i,\ i = -1, 0, 1.$$

where a_i, $i = -1, 0, 1$ are constants and X^i, $i = -1, 0, 1$ are homogenous Cox-Ingersoll-Ross (CIR) processes with stochastic differential equation (SDE) given by

$$dX_t^i = \zeta_i(\mu_i - X_t^i)\, dt + \sigma_i \sqrt{X^i}\, dW_i, i = -1, 0, 1.$$

Bilateral CCR on a Payer CDS: Numerics II

Each collection of the parameters $(a_i, \zeta_i, \mu_i, \sigma_i)$, $i = -1, 0, 1$, may take values corresponding to the "low", "medium" and "high" regime. The values are shown below:

Credit Risk Level	a	ζ	μ	σ	X_0	Market CDS Spread
low	α^l	0.9	0.001	σ^l	0.001	10
middle	α^m	0.80	0.02	σ^m	0.02	120
high	α^h	0.50	0.05	σ^h	0.05	300

Bilateral CCR on a Payer CDS: Numerics III

Scenario: an investor with a very low risk profile, a counterparty which has middle credit risk profile and a reference name with high risk profile.

$(\alpha_{I_1}, \alpha_{I_2})$	$\sigma_1^h = 0.01$	$\sigma_1^h = 0.20$
(0,0)	1.4(0.1)	4.5(0.1)
(0.1,0)	28(0.7)	29(0.6)
(0.2,0)	55(0.9)	55(0.9))
(0.3,0)	82(1.1)	82(1.1)
(0.4,0)	110(1.3)	109(1.3)
(0.5,0)	138(1.5)	136(1.4))
(0.6,0)	166(1.6)	164(1.6)
(0.7,0)	195(1.7)	191(1.7)
(0.8,0)	224(1.8)	220(1.8)
(0.9,0)	253(1.9)	248(1.8)
(1.0,0)	281(2.0)	276(2.0)

Unilateral CCR on a Payer CDS

'AIG selling protection on LEH to BofA'

Investor	Buyer of default protection on a firm ('BofA')
Counterparty	Seller of default protection on the firm ('AIG')
Firm	Reference credit underlying the CDS ('LEH')

$\tau_{-1} = +\infty$, $\tau = \tau_0$, $n = 1$

[Huge and Lando 99, Hull and White 01, Jarrow and Yu 01, Leung and Kwok 05, Brigo and Chourdakis 08, Brigo and Capponi 08, Blanchet-Scalliet and Patras 08, Lipton and Sepp 09]

PFED (no margining)

$$\xi = (1 - R_0)\Big(\mathbb{1}_{\tau<\tau_1\wedge T}P_\tau^+ + \mathbb{1}_{\tau=\tau_1<T}(1 - R_1)\Big)$$

- P Clean Price of the CDS
- T Maturity of the CDS
- R_1 Recovery rate on the underlying firm

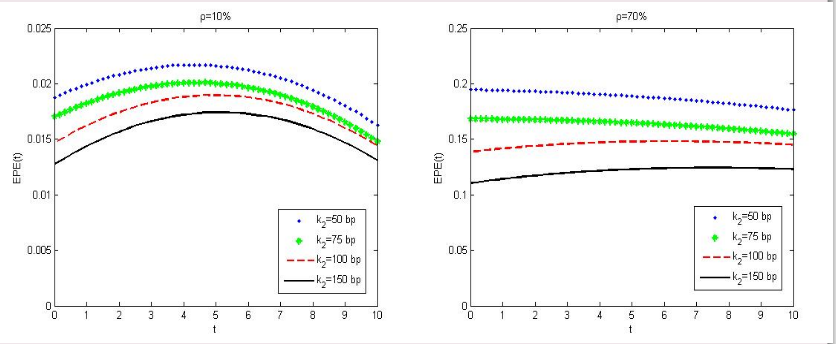
Assessing the impact on the counterparty risk of the investor of

- the (clean) CDS spread $\kappa_0 (= \kappa_2)$ of the counterparty
- the asset correlation ρ between the underlying firm and the counterparty

Limited impact of the factor process
Deterministic intensities below (affine in time)

EE(t)

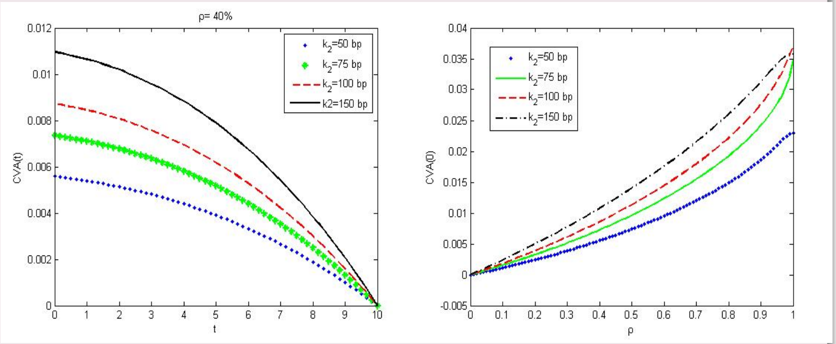
Left: $\rho = 10\%$, Right: $\rho = 70\%$



EE(t): Left column: affine intensities, Right column: constant intensities.

CVA

Left: CVA(t) (ρ=40%), Right: CVA(0) as a function of ρ



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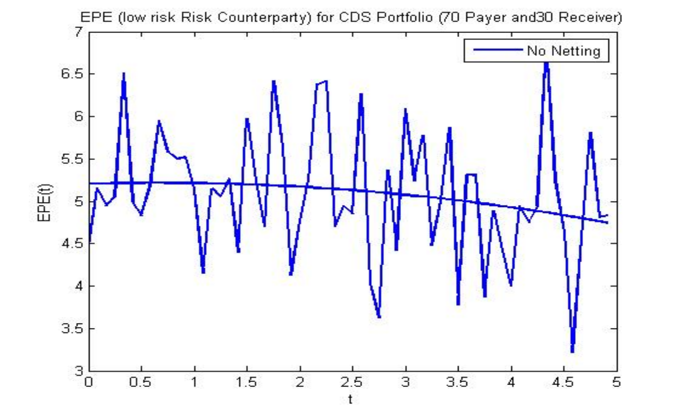
CDS Portfolio Unilateral CCR

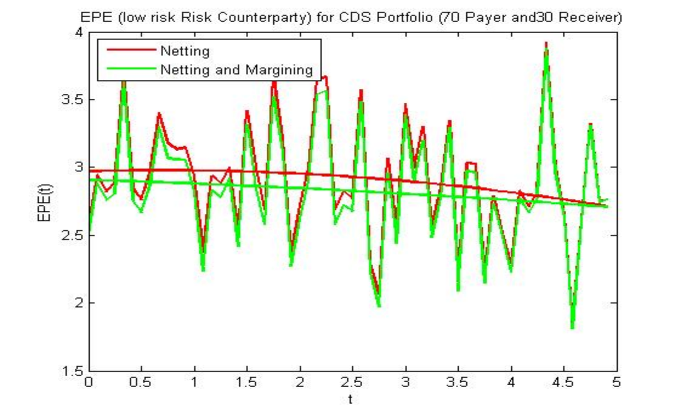
- Portfolio of 70 payer and 30 receiver CDSs
- Individual intensities of the form $a_i + X^i$ where a_i is a constant and X^i is a CIR process.
- Three homogenous groups of obligors
- Three nested groups of joint defaults

type of cpty	CDS Spread of cpty	$(\alpha_{I_1}, \alpha_{I_2}, \alpha_{I_3})$	CVA no Nettg	CVA Nettg	CVA Nettg Margining
low	10	(0.3, 0.3, 0.3)	369(9)	211(5)	205(5)
middle	120	(0.3, 0.3, 0.3)	5746(33)	4761(27)	4676(27)
high	300	(0.3, 0.3, 0.3)	5809(32)	4731(27)	4562(27)

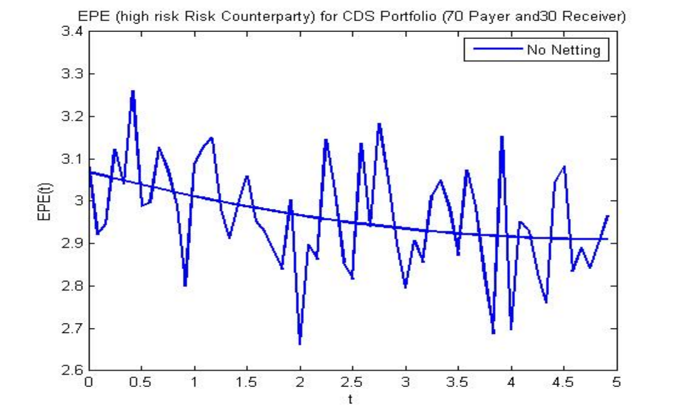
EE(t) for portfolio: I



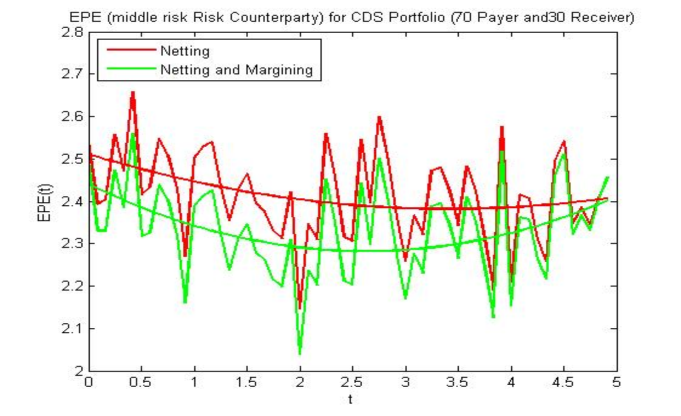
EE(t) for portfolio: II



EE(t) for portfolio: III



EE(t) for portfolio: IV



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To Sum-up

A Markovian Copula Model of Counterparty Credit Risk
Simplicity and Consistency of a ‘dynamized copula’ set-up

- Fast single-name and static basket credit derivatives pricing schemes
- Decoupled Calibration Methodology
    - Automatically calibrated marginals
    - Model dependence parameters calibrated independently
- Model simulation very fast

Adequation of the model’s CVA and EE with stylized features

Perspectives

Assessing systematically the impact of

- Netting
- Collateralization
  - Dealing with the issue of optimal collateralization as a control problem
- Factors

Facing the simulation computational challenge of CCR on real-life porfolios with tens of thousands of contracts

- More intensive than (Credit-)VaR or other risk measure computations
  - Value the portfolio at every time point of every simulated trajectory
- Devise appropriate variance reduction techniques
  - Importance Sampling exploiting the Markovian structure of the model
  - Particle methods (Sequential Monte Carlo)
- Devise appropriate approximate or simulation/regression procedures for non-analytic (dynamic basket) instruments