Robust Measures of Time-Varying Skewness at Short and Long Horizons

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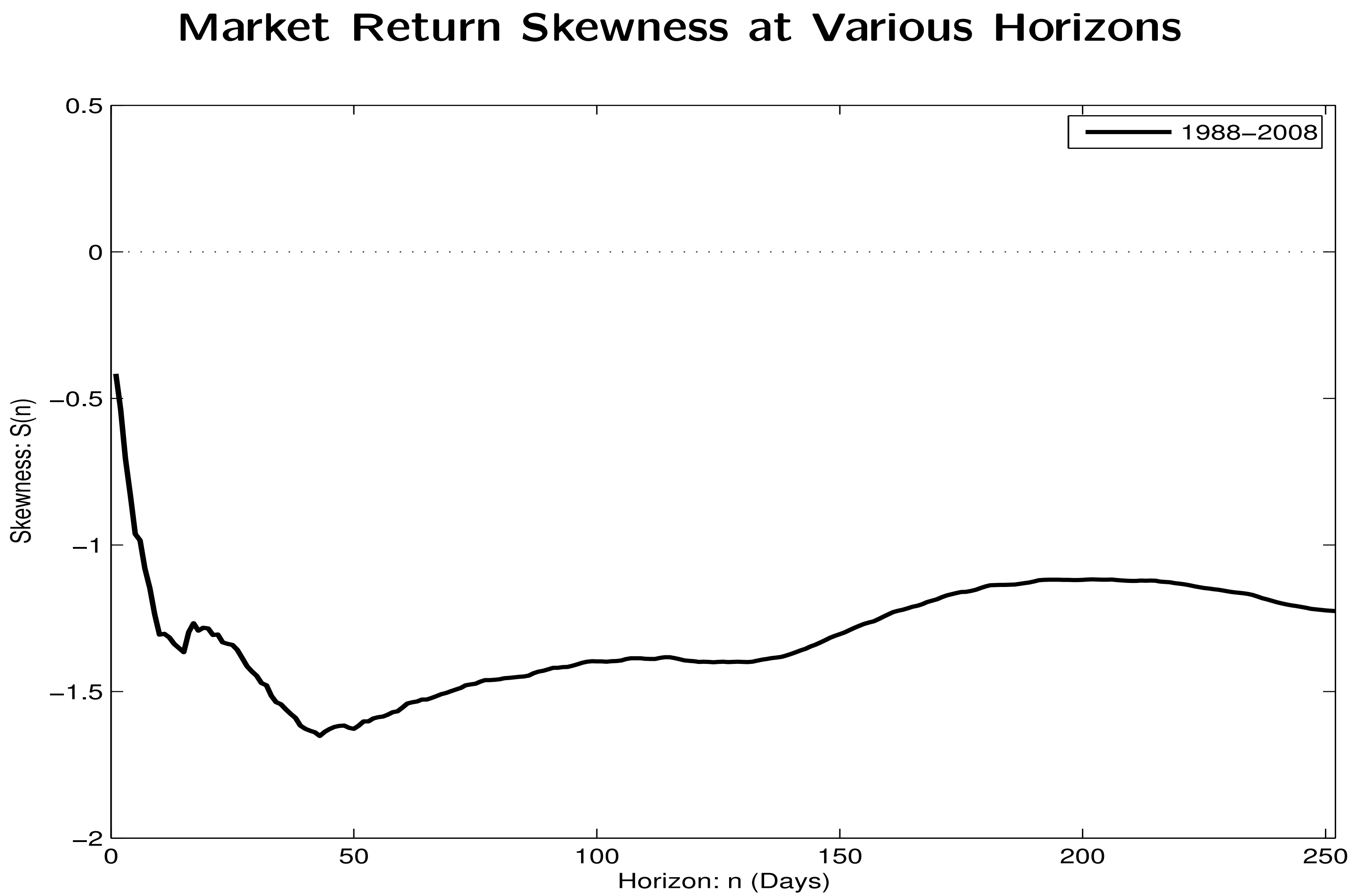
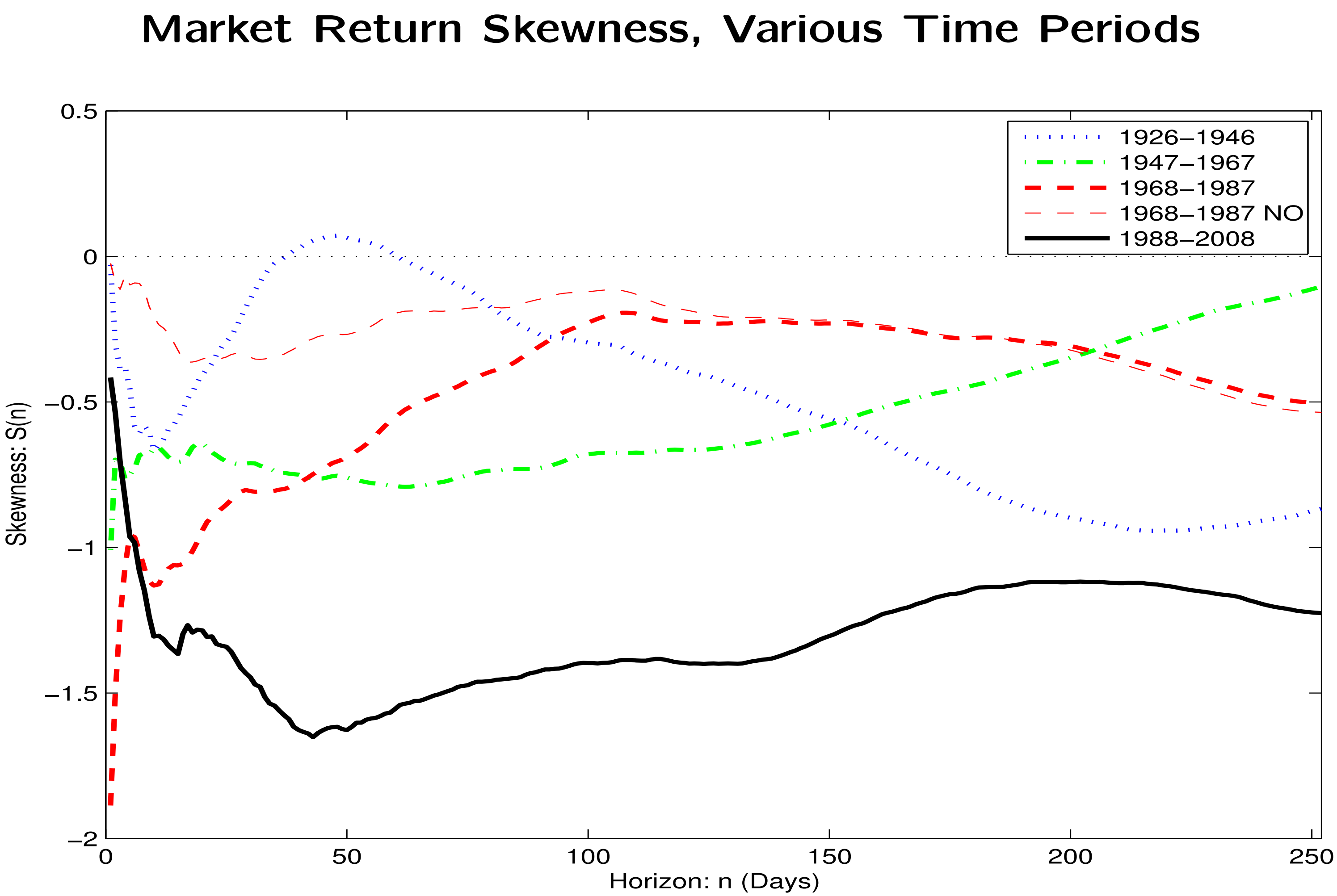
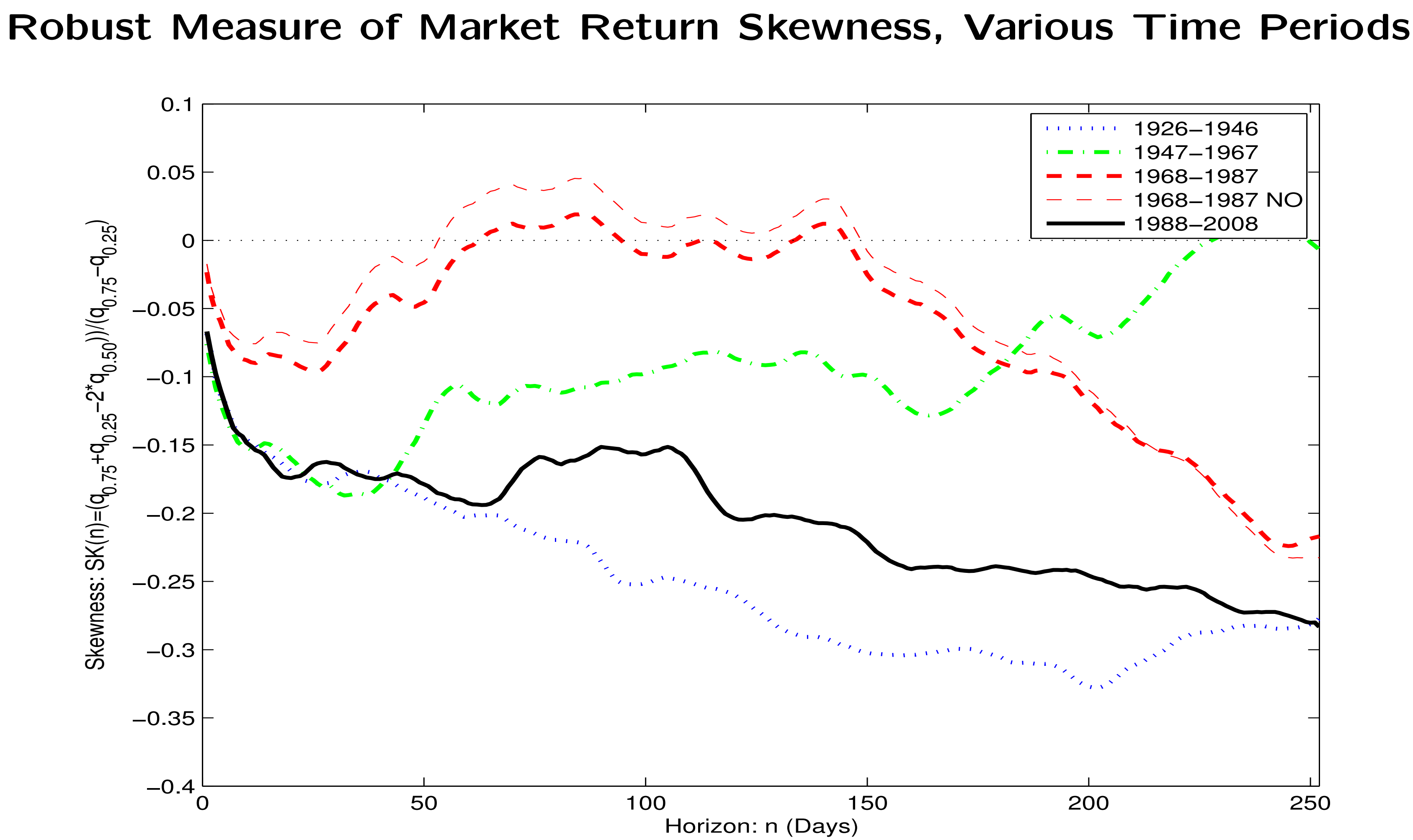
Fields Conference
Toronto, Canada, 2010



Market Return Skewness at Various Horizons
1988–2008
Skewness: S(n)
Horizon: n (Days)
0.5
0
−0.5
−1
−1.5
−2
0
50
100
150
200
250



Market Return Skewness, Various Time Periods
1926−1946
1947−1967
1968−1987
1968−1987 NO
1988−2008
0.5
0
−0.5
−1
−1.5
−2
Skewness: S(n)
0
50
100
150
200
250
Horizon: n (Days)


Robust Measure of Market Return Skewness, Various Time Periods
Skewness: $SK(n)=(q_{0.75}+q_{0.25}-2*q_{0.50})/(q_{0.75}-q_{0.25})$
Horizon: n (Days)
0.1
0.05
0
−0.05
−0.1
−0.15
−0.2
−0.25
−0.3
−0.35
−0.4
0
50
100
150
200
250
1926−1946
1947−1967
1968−1987
1968−1987 NO
1988−2008

Goal: Robust Measure of Skewness

- Estimating Skewness
  - Robust measure of asymmetry
  - From underlying returns
  - At various horizons–term structure
  - Conditional model

- Skewness Literature
  - Asymmetric Volatility/GARCH: French et al. (1987); Campbell and Hentschel (1992); Engle and Ng (1992); Glosten et al. (1992); Engle and Mistry (2009)
  - Leverage Effect/Stochastic Volatility/Jumps: Heston (1993); Bates (1996); Das and Sundaram (1999); Das and Uppal (2004); Eraker et al. (2003)
  - Skewness from Options: Dennis and Mayhew (2002); Bakshi et al. (2003); Mitton and Vorkink (2007); Duane and Wei (2009); Conrad et al. (2009)
  - Economics/Preferences/Pricing: Kraus and Litzenberger (1976); Harvey and Siddique (2000); Dittmar (2002); Hong et al. (2007)
  - Direct Estimates: Harvey and Siddique (1999), Leon et al. (2005), Kim and White (2004), White et al. (2008)

Overview of Idea

1. Construct robust measures of conditional skewness of the n-period log returns, r_{t+n}^n, using conditional quantiles:

$$SK_t^n = \frac{(q_t^n(\alpha_{0.75}) - q_t^n(\alpha_{0.50})) - (q_t^n(\alpha_{0.50}) - q_t^n(\alpha_{0.25}))}{q_t^n(\alpha_{0.75}) - q_t^n(\alpha_{0.25})}$$

- $q_t^n(\alpha_\theta)$: θ-conditional quantile, defined as the inverse of the conditional distribution of r_{t+n}^n, given information at time t
- For all symmetric distributions, $SK_t^n = 0$
- $-1 < SK_t^n < 1$
- Bowley (1920); Hinkley (1975); Groeneveld and Meeden (1984); Kim and White (2004), White et al. (2008)
- Alternative:

$$\overline{SK}_t^n = \frac{\int_0^{0.5} \left[\left(F^{-1}_{r_{t+n}^n|t}(1-\theta) - F^{-1}_{r_{t+n}^n|t}(0.5)\right) - \left(F^{-1}_{r_{t+n}^n|t}(0.5) - F^{-1}_{r_{t+n}^n|t}(\theta)\right)\right] d\theta}{\int_0^{0.5} \left[F^{-1}_{r_{t+n}^n|t}(1-\theta) - F^{-1}_{r_{t+n}^n|t}(\theta)\right] d\theta}$$

Overview of Idea

2. Model $q_t^n(\alpha_{\theta,n})$ of r_{t+n}^n as a function of lagged daily information, x_t.

$$q_t^n(\alpha_{\theta,n}) \equiv q_t^n(\alpha_{\theta,n}; \{x_t; x_{t-1}; ...\})$$

- $\alpha_{\theta,n}$: parameters to be estimated for each quantile θ and horizon n.
- Non-linear quantiles of r_{t+n}^n estimated with daily observations
  - Quantile MIDAS regressions–possible efficiency gains
- Quantiles at all horizons estimated with the same information $\{x_t; x_{t-1}; ...\}$
- Relation to CaViar models (Engle and Manganelli (2004))
- Results on efficiency gains from joint multi-quantile estimation not available.

Quantile MIDAS Specifications

- One parameterization: Absolute Values

$$
\begin{aligned}
q_t^n(\alpha_{\theta,n}) &= \mu_{\theta,n} + \gamma_{\theta,n} * \sum_{j=0}^{J} \beta_{\theta,n}(\kappa_{\theta,n} : j)|r_{t-j}| \\
\beta(\kappa : j) &= \frac{f((j/J), \kappa_1, \kappa_2)}{\sum_{j=1}^{J} f((j/J), \kappa_1, \kappa_2)} \\
f(x, a, b) &= \frac{x^{a-1}(1-x)^{b-1}\Gamma(a+b)}{\Gamma(a)\Gamma(b)}, \Gamma(a) = \int_0^\infty e^{-x}x^{a-1}dx
\end{aligned}
$$

- $\alpha_{\theta,n} = (\mu_{\theta,n}, \gamma_{\theta,n}, \kappa_{\theta,n})$
- Parsimonious parameterization of the lags
- Other specifications possible (asymmetric). Predictors other that $|r_{t-j}|$?

- Important: Need to insure that quantiles don't cross, or $q_t^n(\alpha_{0.25}) < q_t^n(\alpha_{0.50}) < q_t^n(\alpha_{0.75})$
- Estimation: Koenker and Bassett (1978, 1982); Koenker and Zhao (1996); Engle and Manganelli (2004)

Estimation: Nonlinear Quantile Regression

The estimator $\widehat{\alpha}_{\theta,n}$ is the solution to the following optimization problem:

$$\min_{\alpha_{\theta,n}\in A} T^{-1}\sum_{t=1}^{T}\rho_\theta\left(r_{t+n}^n - q_t^n\left(\alpha_{\theta,n}\right)\right)$$

where $\rho_\theta(z) = z*(\theta - 1_{[z\leq 0]})$ is the standard "check function."

Theorem 1. Suppose that Assumptions A1-A6 hold. Then $\widehat{\alpha}_{\theta,n} \xrightarrow{p} a_{\theta,n}$.

- Proof follows White (1994, 5.12), Engle and Manganelli (2004), and White et al. (2008)

Theorem 2. Suppose that Assumptions B1-B5 hold in addition to the assumptions in Theorem 1. Then

$$V_T^{-1/2} Q_T \sqrt{T}(\widehat{\alpha}_{\theta,n} - \alpha_{\theta,n}) \xrightarrow{d} N(0, I)$$

where $V_T \equiv E\left[T^{-1}\theta\left(1-\theta\right)\sum_{t=1}^{T}\nabla' q_t^n\left(\alpha_{\theta,n}\right)\nabla q_t^n\left(\alpha_{\theta,n}\right)\right]$,

$Q_T \equiv E\left[T^{-1}\sum_{t=1}^{T} f_{\theta,t}\left(0\right)\nabla' q_t^n\left(\alpha_{\theta,n}\right)\nabla q_t^n\left(\alpha_{\theta,n}\right)\right]$, and $\nabla q_t^n\left(\alpha_{\theta,n}\right)$ is the gradient vector of $q_t^n\left(\alpha_{\theta,n}\right)$ and $f_{\theta,t}\left(0\right)$ the value at zero of the density $f_{\theta,t}$ of $\varepsilon_{\theta,t+n} \equiv r_{t+n}^n - q_t^n\left(\alpha_{\theta,n}\right)$, conditional on information at time t.

- Proof follows Weiss (1991), Powell (1986, 1991), and Engle and Manganelli (2004)

Imposing Monotonicity: Triangular Specification of Quantiles

- For quantiles $\theta_1 < \theta_2 < ... < \theta_K$

$$q_t^n(\alpha_{\theta_k}) = q_t^n(\alpha_{\theta_{k-1}}) + h_t^n(\alpha_{\theta_k}), \quad k = 2, 3, ..., K$$

where $h_t^n(\alpha_{\theta_k}) > 0$.

- Specification 1: Absolute Value

$$\begin{aligned} q_t^n(\alpha_{\theta_1}) &= \mu_{\theta_1} + \gamma_{\theta_1} * \sum_{j=0}^{J} \beta_{\theta_1}(\kappa_{\theta_1} : j)|r_{t-j}| \\ q_t^n(\alpha_{\theta_k}) &= q_t^n(\alpha_{\theta_{k-1}}) + \sum_{j=0}^{J} \beta_{\theta_k}(\kappa_{\theta_k} : j)|r_{t-j}|, k = 2, 3, ..., K \end{aligned}$$

  - $\beta_{\theta_k}(\kappa_{\theta_k} : j)$ is non-normalized, for $k > 1$.

- Specification 2: Indirect ARCH

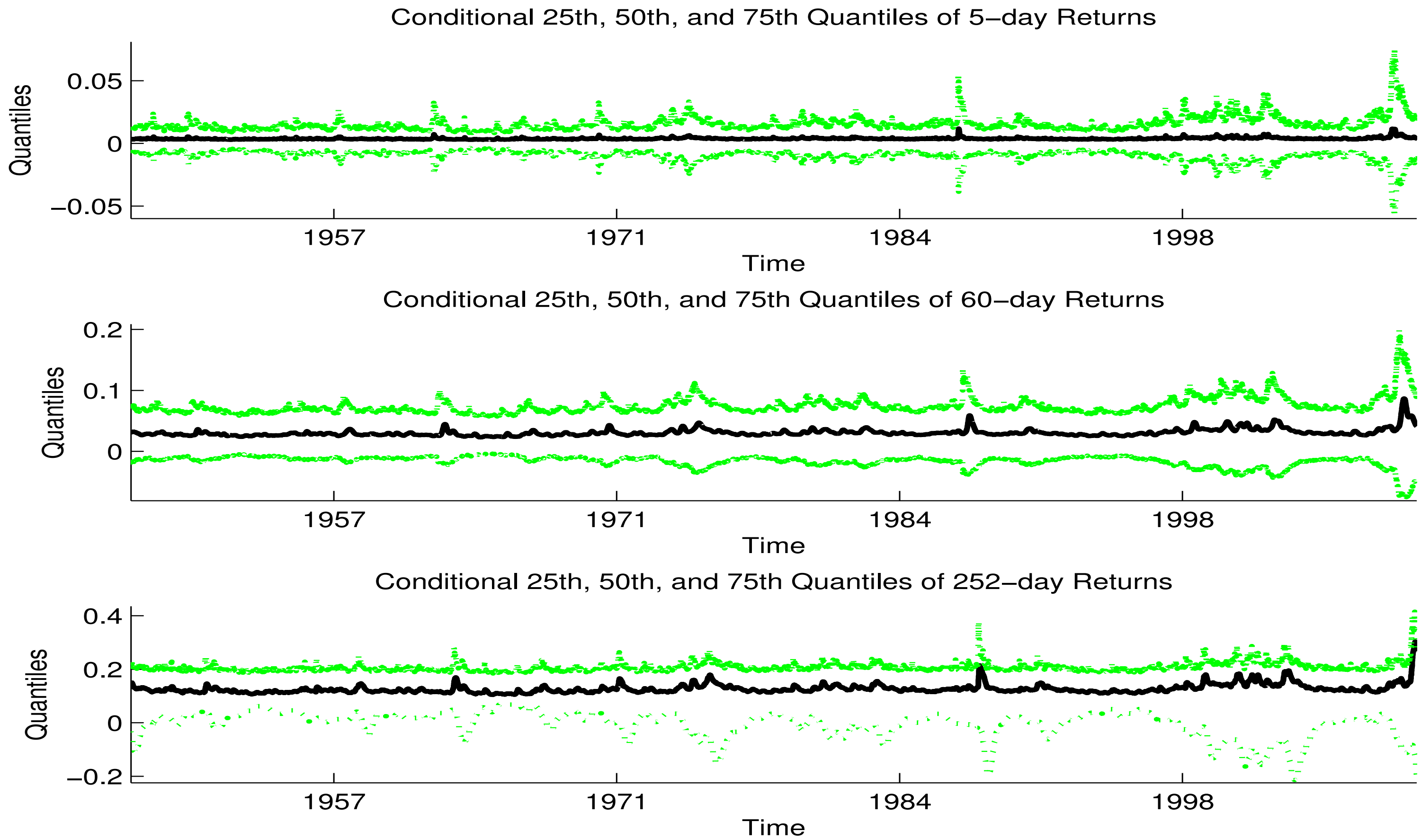
$$
\begin{aligned}
q_t^n(\alpha_{\theta_1}) &= \left(\mu_{\theta_1} + \gamma_{\theta_1} * \sum_{j=0}^{J} \beta_{\theta_1}(\kappa_{\theta_1} : j) r_{t-j}^2\right)^{1/2} \\
q_t^n(\alpha_{\theta_k}) &= q_t^n(\alpha_{\theta_{k-1}}) + \left(\mu_{\theta_k} + \gamma_{\theta_k} * \sum_{j=0}^{J} \beta_{\theta_k}(\kappa_{\theta_k} : j) r_{t-j}^2\right)^{1/2}, k = 2, 3, ..., K
\end{aligned}
$$

- Other specifications possible.

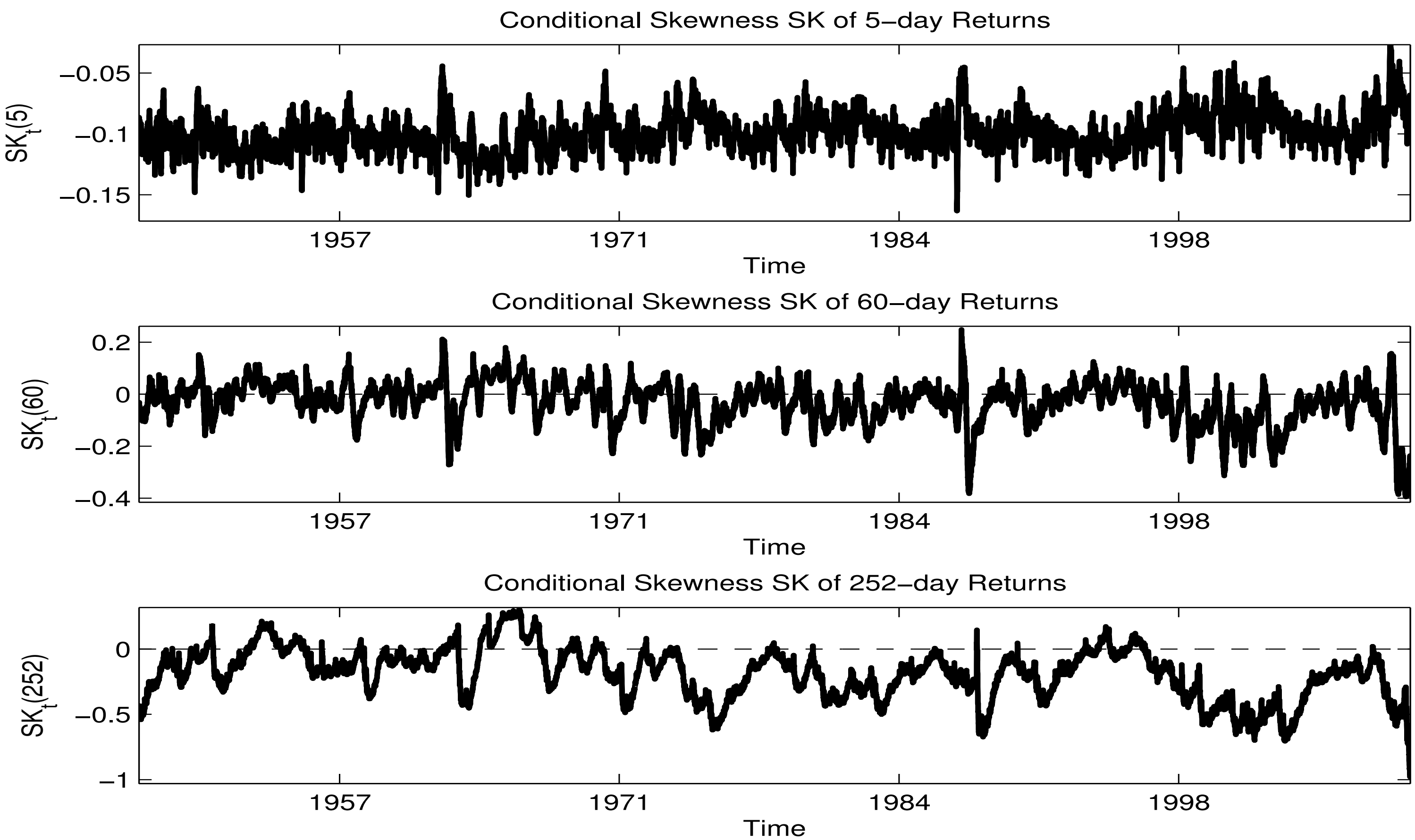
Results

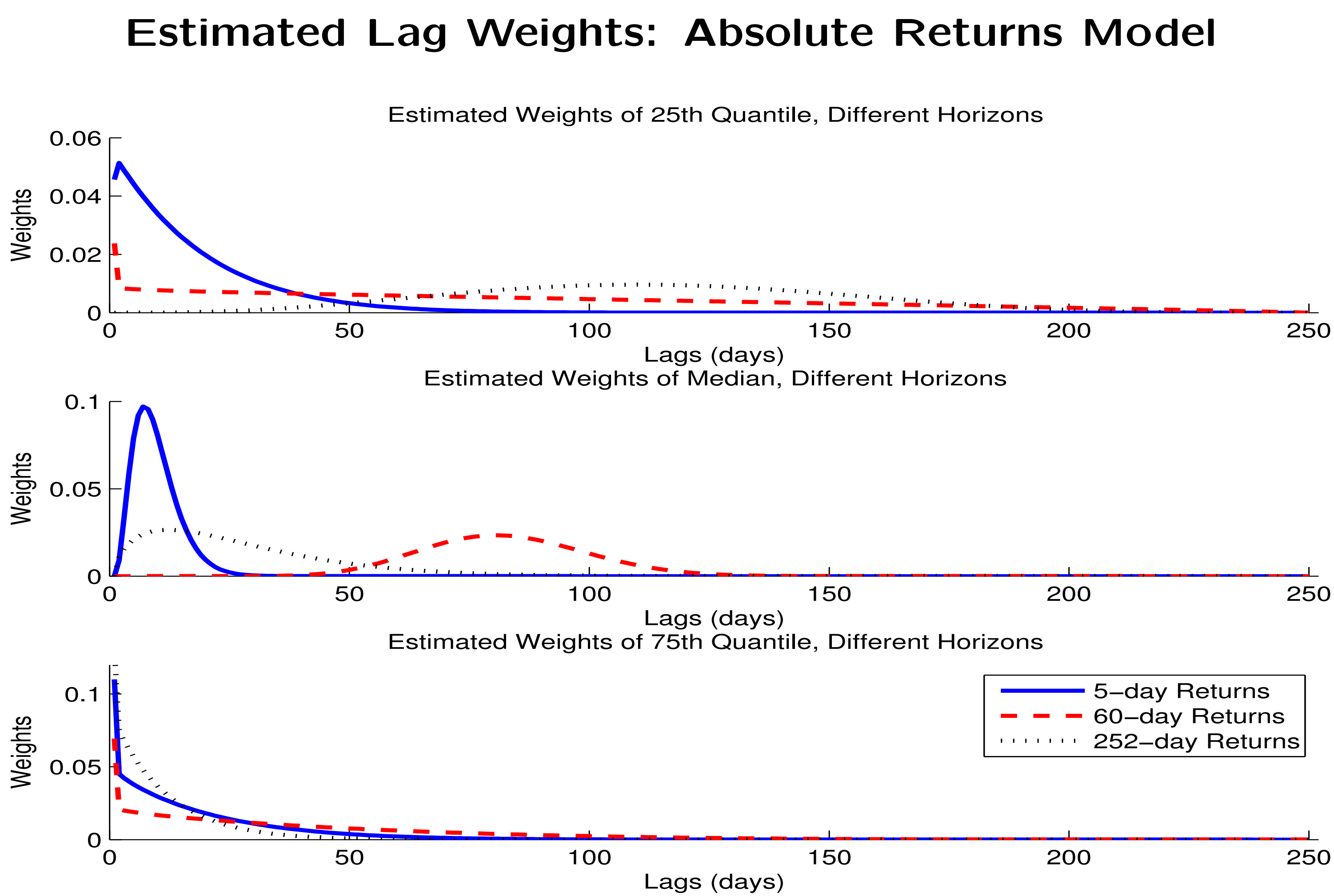
- Data: Market (value-weighted) returns
  - Daily frequency, 1946-2009.
- Specifications
  - Absolute returns (symmetric)
  - Horizons: 5, 60, 252
  - Overlapping and non-overlapping returns
  - Triangular specification

Estimated Conditional Quantiles: Absolute Returns Model



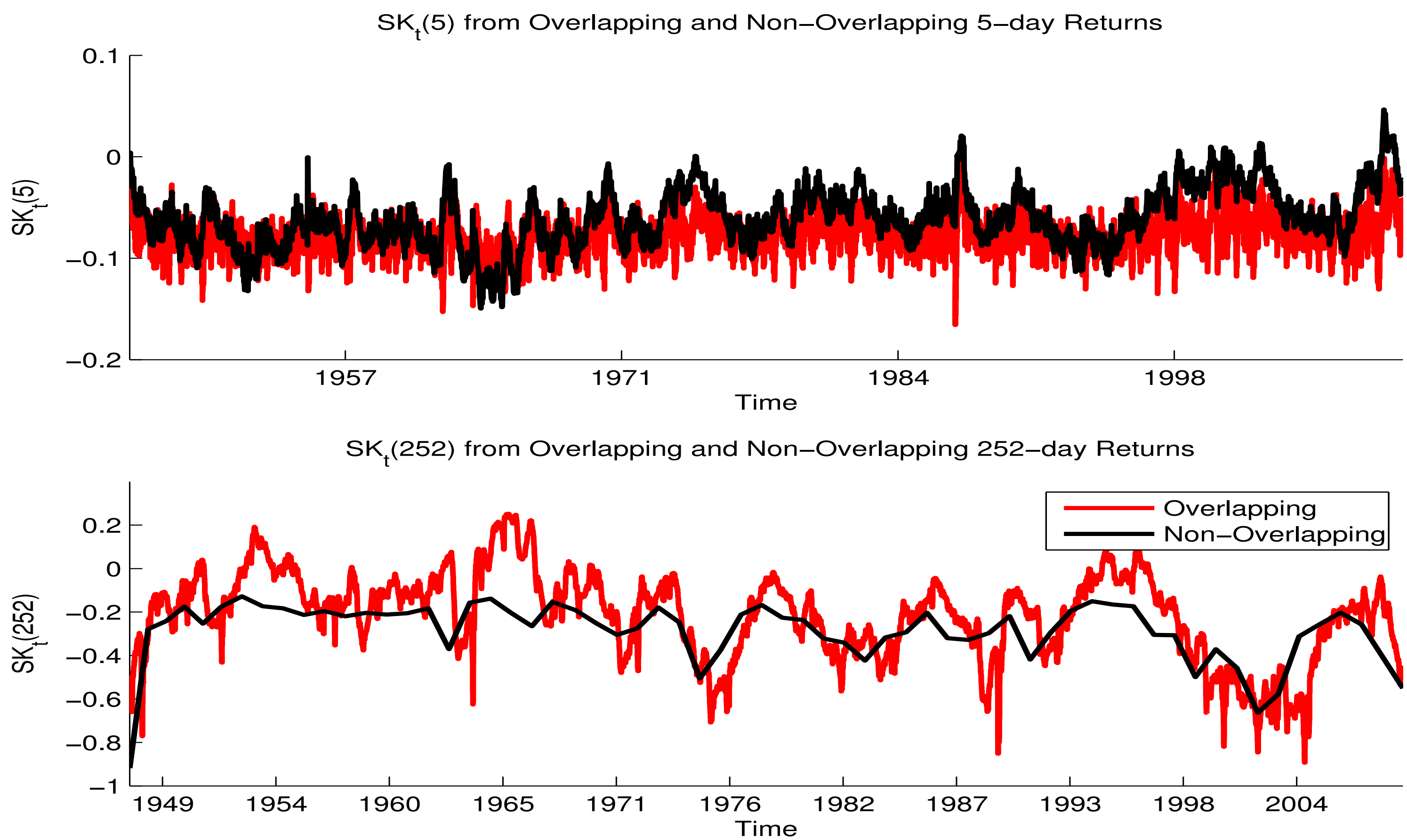
Estimated SK_t^n Measure: Absolute Returns Model



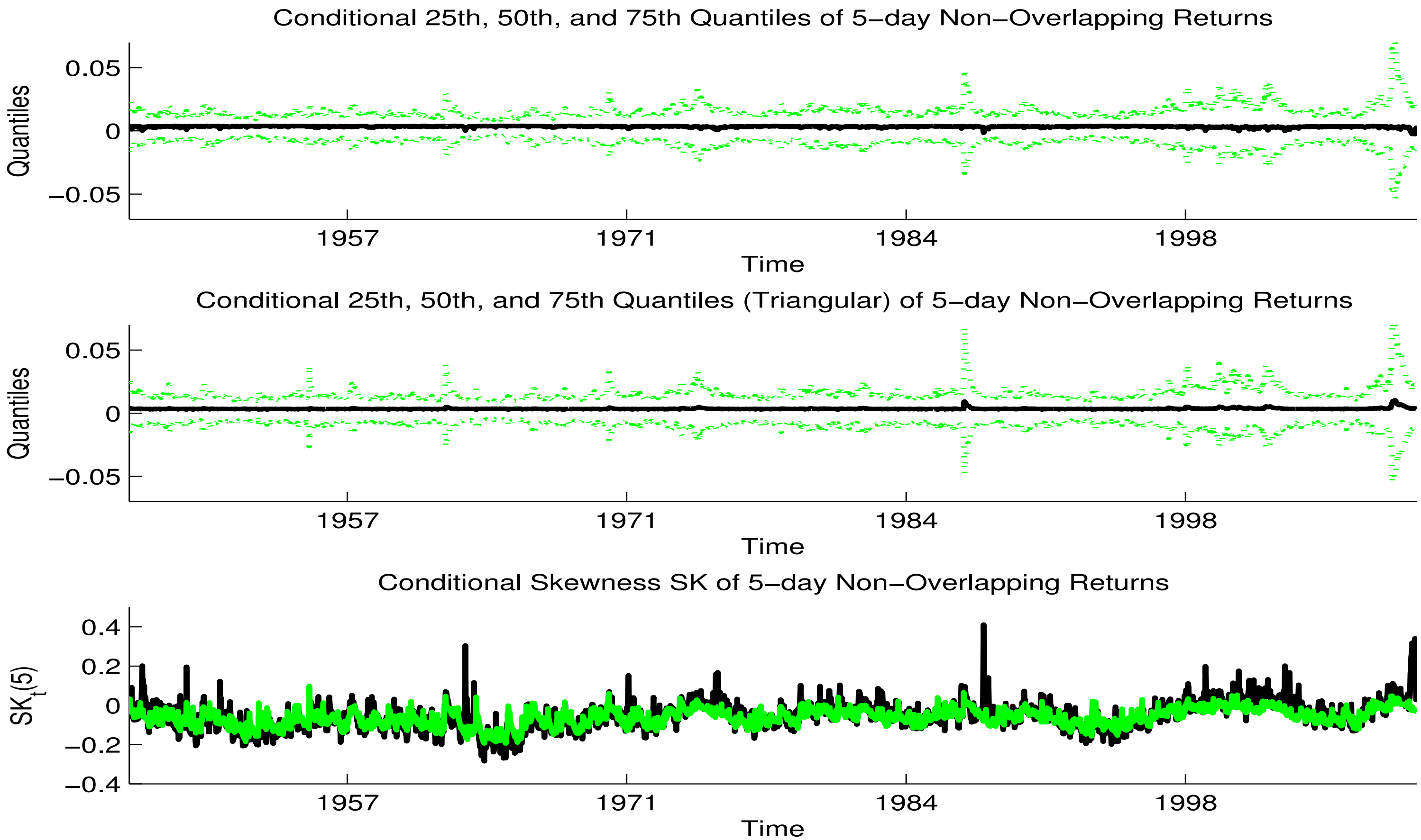


Estimated Lag Weights: Absolute Returns Model
Estimated Weights of 25th Quantile, Different Horizons
Weights
0.06
0.04
0.02
0
0
50
100
150
200
250
Lags (days)
Estimated Weights of Median, Different Horizons
Weights
0.1
0.05
0
0
50
100
150
200
250
Lags (days)
Estimated Weights of 75th Quantile, Different Horizons
Weights
0.1
0.05
0
0
50
100
150
200
250
Lags (days)
5−day Returns
60−day Returns
252−day Returns

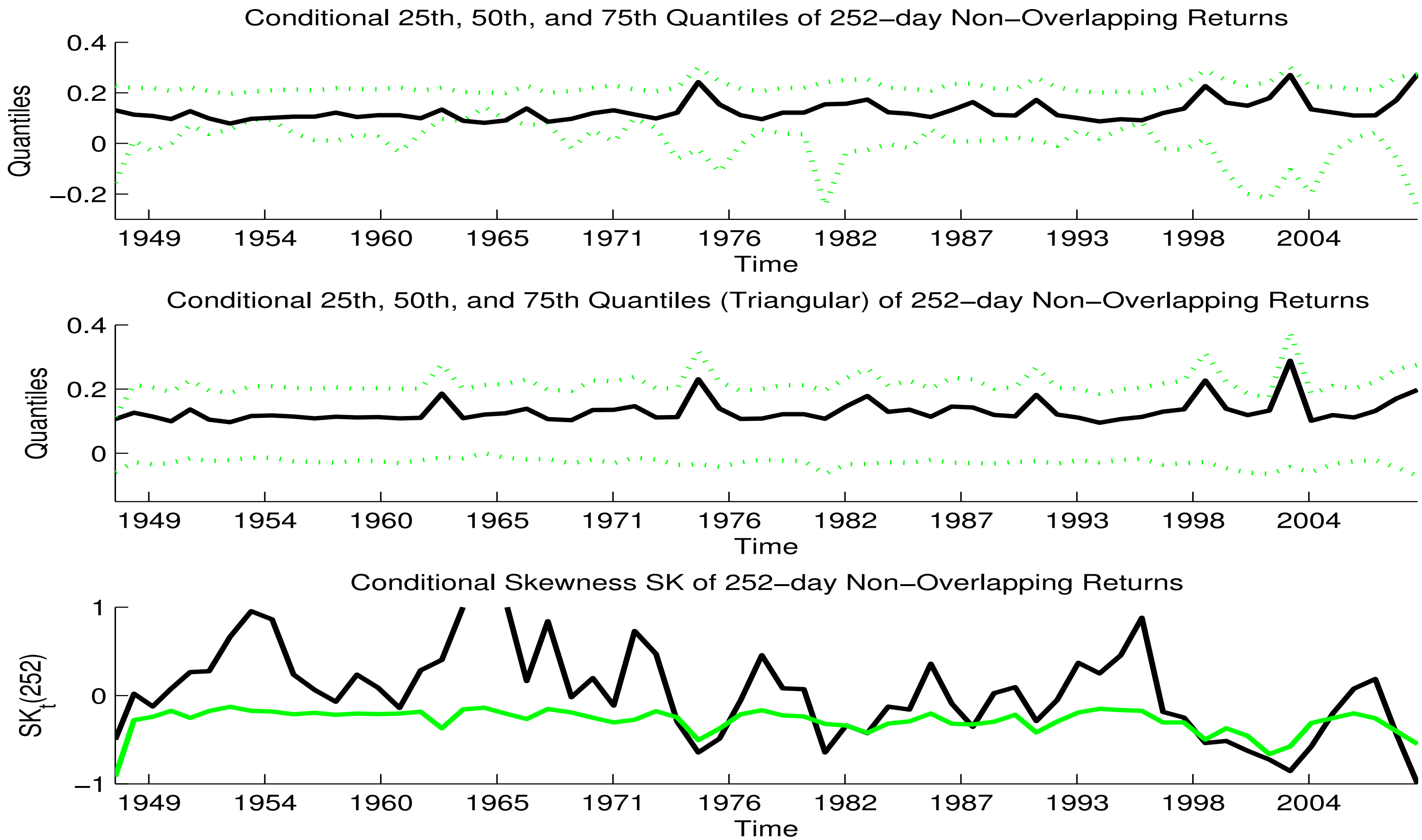
Overlapping vs Non-Overlapping Returns: Absolute Returns Model

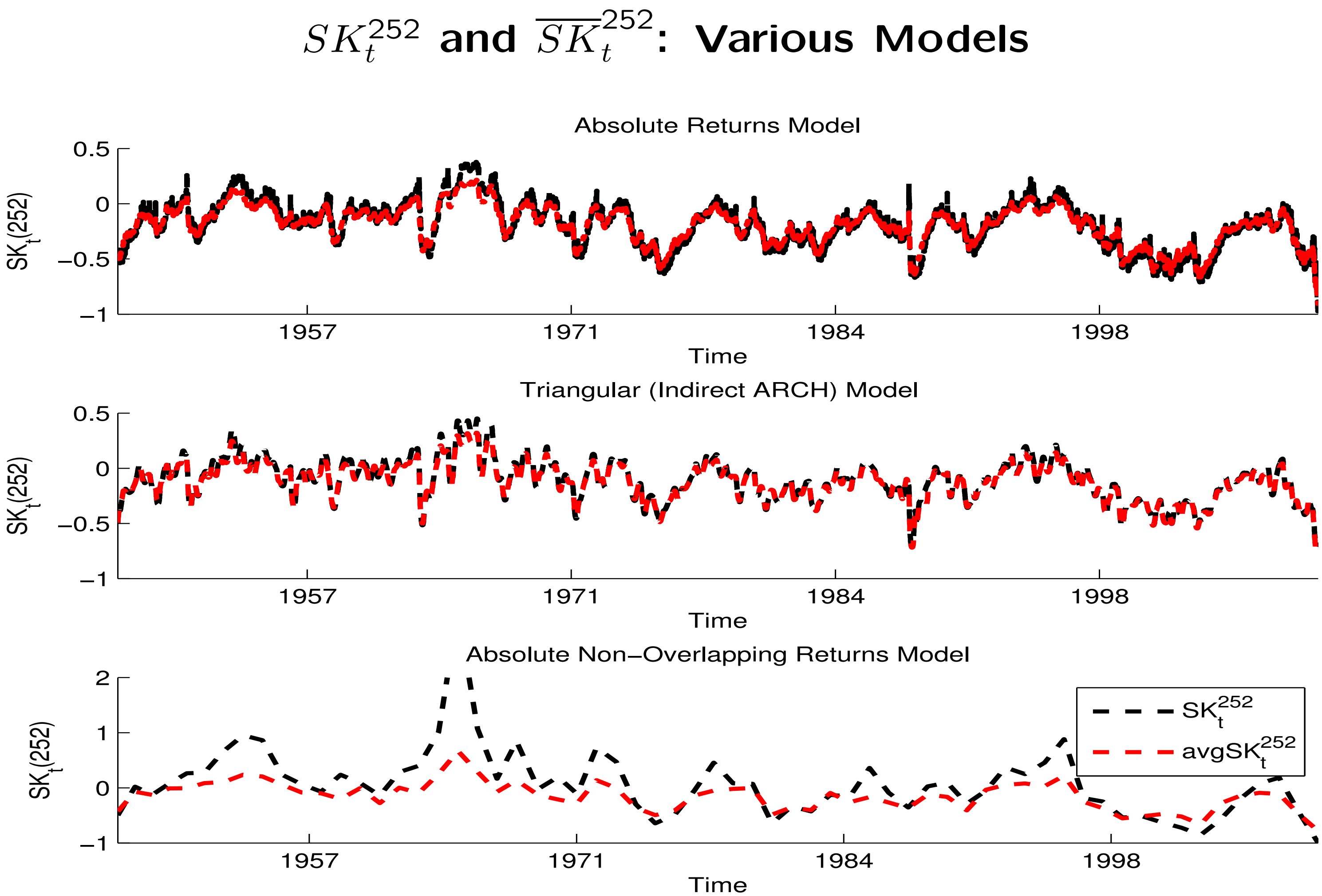


Non-Overlapping 5-Day Returns: Unrestricted and Triangular Models



Non-Overlapping Annual Returns: Unrestricted and Triangular Models





SK_t^{252} and $\overline{SK_t}^{252}$: Various Models
Absolute Returns Model
Triangular (Indirect ARCH) Model
Absolute Non−Overlapping Returns Model
$SK_t(252)$
Time
0.5
0
−0.5
−1
2
1
1957
1971
1984
1998
SK_t^{252}
$avgSK_t^{252}$

Conclusion

- Robust measures of conditional skewness of returns at different horizons
- Based on quantile MIDAS models
- Imposing the monotonicity of quantiles with a triangular specification

To Do:

- Efficiency gains from using daily returns to estimate quantiles
- Additional regressors, alternative specifications
- More asset classes
- Asset pricing and portfolio allocation implications
- Joint estimation of quantiles