What is the Shape of the Risk-Return Relation?

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Introduction

Risk-return trade-off is central to modern finance

Little empirical consensus on the relationship btw the ex-ante equity premium and risk as measured by conditional stock market volatility:

- Negative:
  - Campbell (1987), Breen, Glosten, Jagannathan (1989), Glosten, Jagannathan, Runkle (1993), Whitelaw (1994), Brandt and Kang (2004).
- Positive:
  - French, Schwert and Stambaugh (1995), Bollerslev, Engle and Wooldridge (1988), Harvey (1989), Ghysels et al (2005), Ludvigsson and Ng (2007).

Risk-return trade-off: Theoretical insights

Should we be surprised by the inconclusive empirical evidence?

- Backus and Gregory (1993, page 177): "... the relation between the risk premium and the conditional variance of the excess return can have **virtually any shape**: It can be increasing, decreasing, flat, or even nonmonotonic. The shape depends on both the preferences of the representative agent and the probability structures across states."

Expected Return-Covariance Risk Trade-off: Theory

Merton's ICAPM establishes a trade-off between the conditional variance $\sigma_t^2(r_{t+1})$ and the expected market excess return:

$$E_t[r_{t+1}] = a_W \sigma_t^2(r_{t+1}) + b_W cov_t(r_{t+1}, x_{t+1}),$$

x_{t+1}: state vector
$cov_t(r_{t+1}, x_{t+1})$: hedging component

- stock market volatility
- covariance with 'investment opportunities'

First source of bias: conditional mean and volatility

- Expected return versus expected risk trade-off: both are unobserved - need model-based proxies

Econometric estimates may be biased:

- too few variables in the conditioning set
- use of linear forecasting models due to difficulties with large sets of predictor variables

Second source of bias: shape of risk-return relation

Analysis based on restrictive assumptions (such as linearity) on the risk-return relation could produce biased results.

$\Rightarrow$ Allow for flexible functional form in risk-return mapping.

- Boosted Regression Trees
  - piece-wise constant approximation to unknown functional form
  - can handle large sets of predictor variables
  - no restrictions such as monotonicity
  - control overfitting through *ensemble learning*: model combination, sub-sampling and shrinkage

Third source of bias: How is risk measured?

- Conditional volatility of stock returns?
- Conditional covariance between economic activity (consumption growth) and stock returns?
- Merton's ICAPM: both should matter

What do we find?

- Strongly non-linear relationship between variations in the conditional mean and volatility:
  - At low-to-medium levels of volatility: positive risk-return trade-off
  - At high levels of volatility: the relationship is inverted
  - This finding also holds for the Great Depression, the recent financial crisis and in other subsamples
- Strongly monotonic (increasing) and significant relation between a new conditional covariance measure and expected returns

Models for Conditional Mean and Volatility

Empirical analysis of the risk-return relation typically relies on model-based proxies of the form

$$\begin{aligned} \hat{\mu}_{t+1|t} &= f_{\mu}(x_t|\hat{\theta}_{\mu}) \\ \hat{\sigma}_{t+1|t} &= f_{\sigma}(x_t|\hat{\theta}_{\sigma}) \end{aligned}$$

Conventional to use an RV proxy and assume a linear model

$$\begin{aligned} r_{t+1} &= \beta_{\mu}' x_t + \varepsilon_{rt+1} \\ \hat{\sigma}_{t+1} &= \beta_{\sigma}' x_t + \varepsilon_{\sigma t+1} \end{aligned}$$

No theoretical justification for linearity assumption - model misspecification errors and biases.

Regression Trees

Determine

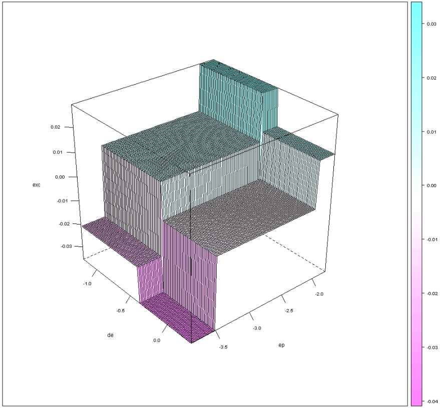
1. splitting variable
2. split point

A regression tree, $\mathcal{T}_J$, with J regions (states) and parameters $\Theta_J = \{S_j, c_j\}_{j=1}^{J}$ can be written

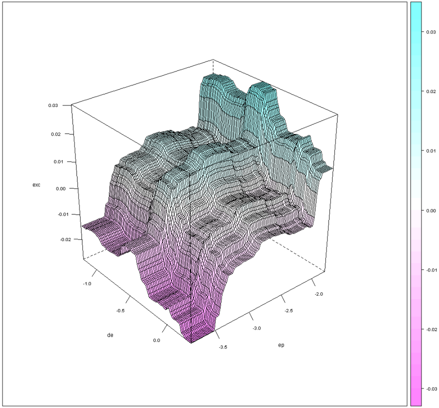
$$\mathcal{T}(x, \Theta_J) = \sum_{j=1}^{J} c_j \, I\,(x \in S_j).$$

- $S_1, S_2, ..., S_J$: J disjoint states
- $x = (x_1, x_2, ..., x_P)$: P predictor ("state") variables
- The dependent variable is constant, c_j, within each state, S_j

Returns as a function of log d/e and log e/p



(a) 3 Iterations



(b) 5000 Iterations

Implementation - Boosting and Ensemble Learning

- Stochastic gradient boosting
- 10,000 boosting iterations
- Number of states, $J = 2$
- Shrinkage parameter, $\lambda = 0.001$, determines how much each tree contributes to the overall fit:

$$f_B(x_t) = f_{B-1}(x_t) + \lambda \sum_{j=1}^{J} c_{j,B} I\{x_t \in S_{j,B}\}.$$

- Subsampling, using half the data for fitting
- Objective function is MAE: $T^{-1} \sum_{t=1}^{T} |y_{t+1} - f(x_t)|$

Data

Monthly excess returns on S&P 500 index, 1927-2008.

- 12 predictor variables
  - log dividend-price ratio (symbol: dp);
  - log earnings-price ratio (ep);
  - three-month T-bill rate (rfree);
  - de-trended T-bill rate (rrel);
  - yield on long term government bonds (lty);
  - term spread (tms);
  - default yield spread (defspr);
  - lagged excess return (exc);
  - long term return (ltr);
  - stock variance (vol);
  - log dividend-earnings ratio (de)
  - inflation rate (infl).
- Realized variance

$$\hat{\sigma}_t^2 = \sum_{i=1}^{N_t} r_{i,t}^2.$$

Relative influence of predictor variables: Excess Returns

1927-2008	infl	ep	rrel	lty	vol	de	Top 3	Top 5
relative influence	17.48%	17.31%	10.10%	9.47%	8.30%	7.50%	44.89%	62.66%
p-value	0.1%	0.4%	11.5%	11.0%	29.5%	38.8%	0.0%	0.0%

1927-1967	infl	ep	de	dp	exc	ltr	Top 3	Top 5
relative influence	12.41%	11.01%	10.35%	9.30%	9.27%	8.99%	33.77%	52.34%
p-value	11.0%	24.5%	37.8%	51.0%	60.4%	57.5%	28.7%	52.50%

1968-2008	infl	ep	dp	rrel	exc	vol	Top 3	Top 5
relative influence	16.49%	12.54 %	12.52%	10.97%	8.45%	7.31%	41.55%	60.97%
p-value	0.6%	6.3%	7.3%	12.6%	47.1%	67.7%	0.3%	0.0%

Relative Influence of predictor variables: Realized Volatility

1927-2008	vol	defspr	exc	de	infl	ep	Top 3	Top 5
relative influence	66.10%	8.87%	7.39%	6.25%	2.89%	2.42%	82.36%	91.50%
p-value	0.0%	0.0%	0.0%	0.0%	1.3%	2.8%	0.0%	0.0%

1927-1967	vol	de	defspr	dp	lty	exc	Top 3	Top 5
relative influence	41.99%	13.92%	13.61%	6.12%	6.09%	5.94%	69.52%	81.73%
p-value	0.0%	0.0%	0.0%	0.4%	0.1%	0.8%	0.0%	0.0%

1968-2008	vol	exc	ep	dp	infl	rfree	Top 3	Top 5
relative influence	60.16%	11.65%	6.25%	5.40%	4.35%	3.99%	78.06%	87.81%
p-value	0.0%	0.0%	0.0%	1.0%	4.4%	5.3%	0.0%	0.0%

Unknown shape of Risk-Return Trade-off

Theoretical analysis suggests the need to avoid imposing strong functional form assumptions on the risk-return relationship

Explore a general (non-linear) risk-return relationship:

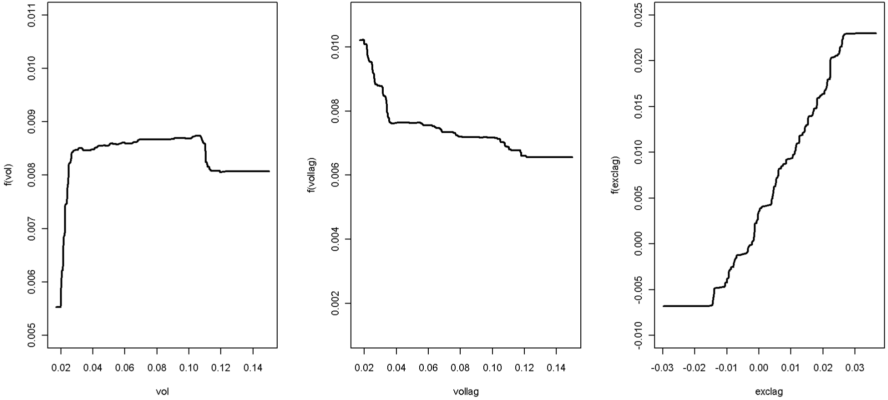
$$\hat{\mu}_{t+1|t} = f(\hat{\sigma}_{t+1|t}, \hat{\sigma}_{t|t-1}, \hat{\mu}_{t|t-1}) + \varepsilon_{t+1}$$

Relative influence of variables in the flexible risk-return trade-off: Volatility Model

Model: $\mu_{t+1|t} = f(\sigma_{t+1|t}, \sigma_{t|t-1}, \mu_{t|t-1})$

Sub-Samples	$\sigma_{t+1\|t}$	$\sigma_{t\|t-1}$	$\mu_{t\|t-1}$
1927-2008	7.6% (1.8%)	9.4% (0.0%)	83.0% (0.0%)
1927-1967	21.1% (0.0%)	19.5% (0.0%)	59.4% (0.0%)
1968-2008	9.9% (70.6%)	9.9% (53.9%)	80.2% (0.0%)
High-Volatility Periods			
1927-1939	24.5% (8.8%)	35.4% (0.0%)	40.1% (0.0%)
2001-2008	23.0% (55.3%)	20.6% (88.8%)	56.4% (0.0%)

Risk-Return Trade-off



Consumption vs Activity Based Asset Pricing

Under concave utility and a positive relation between consumption growth and stock returns:

$$\frac{\partial E_t[r_{t+1}]}{\partial cov_t(\Delta c_{t+1}, r_{t+1})} > 0$$

Under a monotonically increasing relationship btw consumption growth and changes in economic activity, ΔEA_{t+1}:

$$\frac{\partial E_t[r_{t+1}]}{\partial cov_t(\Delta EA_{t+1}, r_{t+1})} > 0$$

"Realized" Covariance

ADS index (Aruoba-Diebold-Scotti, 2009) tracks high frequency (daily) business conditions.

- ADS index blends high- and low-frequency information using the Kalman filter in a dynamic factor framework.

We compute monthly "realized" covariances btw stock returns and changes in the ADS index from daily observations

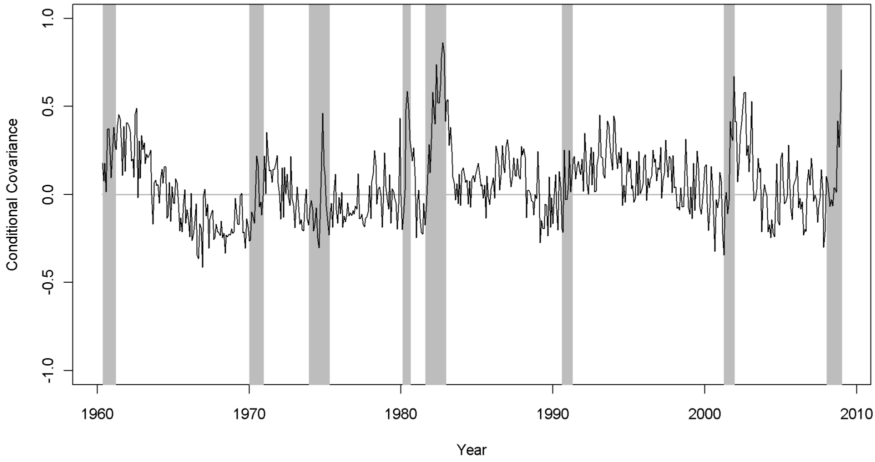
$$\widehat{cov}_t = \sum_{i=1}^{N_t} \Delta ADS_{i,t} \times r_{i,t}$$

$\Delta ADS_{i,t}$: change in the ADS index on day i during month t
$r_{i,t}$: daily stock market return

Correlation between the ADS index and aggregate consumption

	Monthly	Quarterly	Semiannual	Annual
Consumption	19.80%	34.06%	49.07%	54.67%
Durable Consumption	16.12%	20.11%	39.01%	45.69%
Non-Durable Consumption	15.81%	35.11%	45.17%	54.72%

Conditional Covariance



Expected Return-Covariance Risk Trade-off

We estimate a linear model relating expected returns to the conditional covariance as well as lags of these variables:

$$\hat{\mu}_{t+1|t} = \alpha + \beta_1 \widehat{cov}_{t+1|t} + \beta_2 \widehat{cov}_{t|t-1} + \beta_3 \hat{\mu}_{t|t-1} + \varepsilon_{t+1}.$$

We also consider a model that is not restricted to be linear

$$\hat{\mu}_{t+1|t} = g(\widehat{cov}_{t+1|t}, \widehat{cov}_{t|t-1}, \hat{\mu}_{t|t-1}) + \varepsilon_{t+1}.$$

Risk-return relation for the Covariance Model

A. Linear Model

Model: $\mu_{t+1|t} = \alpha + \beta_1 \ cov_{t+1|t} + \beta_2 \ cov_{t|t-1} + \beta_3 \ \mu_{t|t-1} + \epsilon_{t+1}$

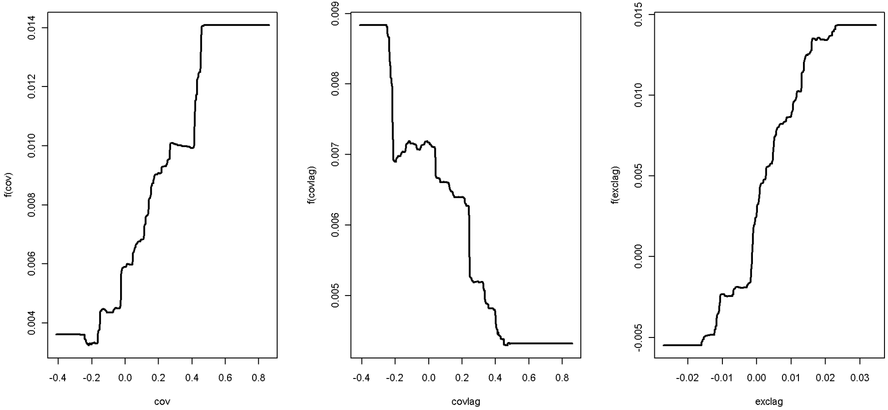
Sample	$cov_{t+1\|t}$ (t-stat)	$cov_{t\|t-1}$ (t-stat)	$\mu_{t\|t-1}$ (t-stat)	R^2
1960-2008	0.018 (8.71)	-0.007 (-3.37)	0.580 (17.23)	45.34%

B. Flexible Risk-Return Model

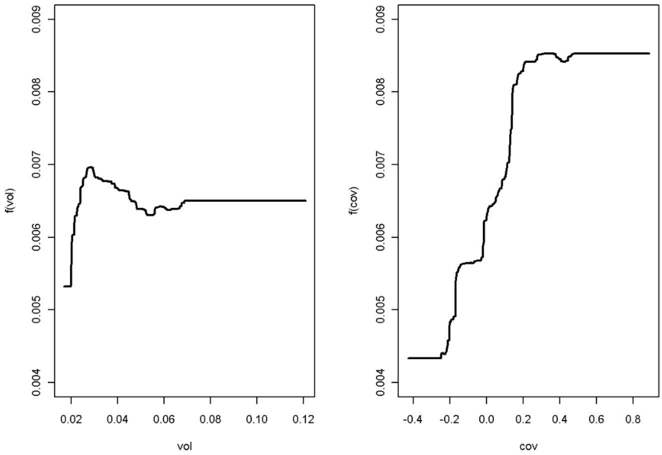
Model: $\mu_{t+1|t} = f(cov_{t+1|t}, cov_{t|t-1}, \mu_{t|t-1})$

Sample	$cov_{t+1\|t}$	$cov_{t\|t-1}$	$\mu_{t\|t-1}$
1960-2008	26.3 % (0.0%)	9.6% (43.3%)	64.1% (0.0%)

Risk-Return Trade-off for the Covariance Model



Risk-return: Joint Specification



$$\hat{\mu}_{t+1|t} = g\Big(\underset{(6.6\%^{*})}{\hat{\sigma}_{t+1|t}}, \quad \underset{(13.1\%^{**})}{\widehat{cov}_{t+1|t}}, \quad \underset{(67.4\%^{**})}{\hat{\mu}_{t|t-1}}, \quad \underset{(6.4\%)}{\hat{\sigma}_{t|t-1}}, \quad \underset{(6.5\%)}{\widehat{cov}_{t|t-1}} \Big) + \varepsilon_{t+1}$$

Monotonicity test for the risk-return relation

Horizon (months)	Number of Observations per Portfolio		
	Small	Medium	Large
A. Volatility Estimates			
1	0.000	0.018	0.010
2	0.000	0.000	0.017
3	0.000	0.000	0.039
B. Covariance Estimates			
1	0.420	0.984	0.994
2	0.320	0.898	0.949
3	0.000	0.960	0.899
C. VIX-based Estimates			
1	0.027	0.041	0.091

Conclusion: Is there a risk-return trade-off?

- Explored a new method for estimating the conditional equity premium and the conditional volatility
  - flexible, avoids imposing strong functional form assumptions
  - can handle large-dimensional sets of predictor variables
  - controls overfitting
- Evidence of a non-monotonic risk-return relationship if risk is measured by volatility
  - at low-to-medium conditional volatility levels: positive risk-return trade-off
  - at high conditional volatility levels: inverted risk-return trade-off
- The risk-return relationship is monotonically increasing if risk is measured through the "realized" (conditional) covariance btw changes in economic activity and stock returns