The Realized Laplace Transform of Volatility

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Motivation

- Strong parametric and nonparametric evidence for presence of persistent stochastic volatility and jumps on the stock markets.
- Volatility typically associated with a slowly-moving investment opportunities set that adds intertemporal component in an asset position (e.g. Merton's intertemporal hedging demand).
- However
  - there is recent nonparametric evidence that volatility has relatively short-lived spikes, i.e., jumps that are triggered by a jump on the stock market (Todorov and Tauchen (2009), Jacod and Todorov (2009)) ...
  - ... and such moves in volatility are of real concerns to investors: evidence from options markets in Bollerslev and Todorov (2009) suggests those risks bear non-trivial premia.

Motivation

- Therefore, we need a much better understanding of the statistical features of the stochastic volatility...
- ... and high-frequency data is the natural source for providing a robust (i.e. model-free) way of doing this.

Main Results

In this study:

- We propose an alternative way to aggregate high-frequency data into a realized measure that we name **Realized Laplace Transform of Volatility** (RLT).
- RLT is a measure for the stochastic volatility and in the jump-diffusion case is robust to jumps.
- RLT is an estimate for the empirical Laplace transform of the **spot volatility** and thus provides direct information for its **marginal distribution**.
- We show that recovering the statistical features of stochastic volatility crucially depends on the **small scale behavior** of the asset price.
- The concept of RLT can be extended to the estimation of the **conditional** Laplace transform of the volatility.

Connection with previous work

Estimating spot volatility from high-frequency data is very difficult:

- it goes back at least as early as the work of Foster and Nelson (1996) in which they analyze the spot variance estimator in a diffusion setting...
- ... however, such estimator has very slow rate of convergence.

Alternatives that possess much better statistical properties (e.g. a rate of convergence of $\sqrt{n}$) are

- to aggregate in time, i.e., use fill-in asymptotics to estimate integrated over time quantities - the leading example being Realized Volatility
- to aggregate over sample paths in the probability space, i.e., use long-span asymptotics to estimate functionals over the invariant distribution of the spot volatility

Connection with previous work

- The jump-robust extensions of the Realized Volatility estimate $\int_0^t \sigma_s^2 ds$ and moments of the latter **do not have**, in general, a one-to-one mapping with the moments of the spot volatility. Hence the time-aggregation embodied in those measures distorts the connection with the statistical properties of the **spot volatility**.

- Our **RLT** estimator, instead, aggregates the high-frequency data in a different way (by cosine transformation of the returns) and estimates $\int_0^t e^{-u\sigma_s^2} ds$. The latter when averaged over time is a direct estimate of the Laplace transform of the spot volatility and hence **does not** suffer from the time-distortions above.

Construction of the Realized Laplace Transform

Recall for a generic non-negative random variable X and scalar $u \geq 0$ we denote by

$$\mathcal{L}_X(u) = \mathbb{E}\left(e^{-uX}\right)$$

the real Laplace transform of X. The family of functions $\{e^{ux}\}_{u\geq 0}$ is separating within the class of distribution functions supported on $[0,\infty)$, so the mapping from $F(x)$ to $\mathcal{L}_X(u)$ is one-to-one.

Our Goal: To estimate the Laplace transform of the spot volatility (or some power of it).

Construction of the Realized Laplace Transform

In the case when σ_t^2 is observed directly things are easy

$\Longrightarrow$ use the associated empirical process:

$$\frac{1}{T}\sum_{t=1}^{T} e^{-u\sigma_t^2} \xrightarrow{\mathbb{P}} \mathbb{E}\left(e^{-u\sigma_t^2}\right)$$

However volatility is latent!

Construction of the Realized Laplace Transform

We will use the high-frequency data to solve the latency problem: if we sample frequently enough the volatility will be approximately constant $\Longrightarrow$ the high-frequency return will be

$$\sigma \times Z_i \times \sqrt{\Delta},$$

where σ is the unknown level of (locally constant) volatility, Δ is the length of the high-frequency interval, and $\{Z_i\}$ is a sequence of independent standard normal variables

$\Longrightarrow$ by using fill-in asymptotics and averaging "locally"

$$\frac{1}{n}\sum_i \cos\left(\Delta^{-1/2} \times \sigma \times Z_i \times \sqrt{\Delta}\right),$$

we can recover the characteristic function of the normal innovation with variance σ^2, i.e. $e^{-u^2\sigma^2/2}$.

Construction of the Realized Laplace Transform

Recognizing that volatility changes and integrating over time we have cancelation of errors which allows us to estimate **on a given path**

$$\int_t^{t+1} e^{-u^2\sigma_s^2/2} ds$$

at the standard $\sqrt{n}$-rate.

From here, by using **long-span asymptotics**, we can estimate

$$\mathbb{E}\left(e^{-u^2\sigma_t^2/2}\right),$$

which, when viewed as a function in $u^2/2$, is the Laplace transform of σ_t^2.

Construction of the Realized Laplace Transform

The above discussion was based on the premise that by sampling frequently returns **look like Gaussian**.

However, returns even locally might be non-Gaussian, e.g., the VIX index data. A more general assumption is that returns are **locally Stable** with index β ($\beta = 2$ nests the locally Gaussian case).

In this case the high-frequency returns will be approximately

$$\sigma \times Z_i \times \Delta^{1/\beta},$$

for Z_i being stable random variable and our analysis will lead us to recover more generally

$$\mathbb{E}\left(e^{-u^{\beta}\sigma_t^{\beta} Z_{\beta}}\right),$$

for an appropriate constant Z_{β}.

Construction of the Realized Laplace Transform

- Formally, the discretely-observed process is denoted with X
- The observation times are: $0, \Delta_n, \ldots, [T/\Delta_n]$
- Most of the asymptotic results will be for the case $\Delta_n \downarrow 0$ and $T \uparrow \infty$

The **Realized Laplace Transform** is defined as

$$V_T(X, \Delta_n, \beta, u) = \sum_{i=1}^{[T/\Delta_n]} \Delta_n \cos(u \Delta_n^{-1/\beta} \Delta_i^n X), \quad \Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}.$$

Construction of the Realized Laplace Transform

This is different from two classical approaches based on trigonometric transformations:

1. Fourier transform of the price increments

$$\frac{1}{n}\sum_j x_j e^{-i\,u\,j/n}$$

   Malliavin and Mancino (2008)

2. Empirical characteristic function of the price increments

$$\frac{1}{n}\sum_j e^{-iux_j}$$

where $x_j = \Delta_j^n X$. The empirical characteristic function for a *fixed* grid is used for method of moments estimation when the density is not available in convenient closed form but the characteristic function is.

Assumptions

The process X is defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) **Jump-Diffusion**

$$dX_t = \alpha_t dt + \sigma_t dW_t + \int_{\mathbb{R}} \delta(t-, x)\mu(dt, dx),$$

where α_t and σ_t are càdlàg processes; W_t is a Brownian motion; μ is a homogenous Poisson measure with compensator (Lévy measure) $\nu(x)dx$; $\delta(t, x) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is càdlàg in t;

(b) **Pure-Jump**

$$dX_t = \alpha_t dt + \int_{\mathbb{R}} \sigma_{t-}\kappa(x)\tilde{\mu}(dt, dx) + \int_{\mathbb{R}} \sigma_{t-}\kappa'(x)\mu(dt, dx),$$

where α_t and σ_t are càdlàg processes; κ is a symmetric function with bounded support with $\kappa(x) = x$ in a neighborhood of 0 and $\kappa'(x) = x - \kappa(x)$; $\tilde{\mu}(dt, dx) = \mu(dt, dx) - dt\nu(x)dx$.

Assumptions

Assumption A. The Lévy measure of μ satisfies:

(a) ***Jump-Diffusion***

$$\int_0^t \int_{\mathbb{R}} (|\delta(s,x)|^p \wedge 1) ds \nu(x) dx < \infty, \quad \int_{\mathbb{R}} |\delta(t,x)| \nu(x) dx < \infty,$$

for every $t > 0$ and every $p > \beta'$, where $0 \leq \beta' < 1$ is some constant.

(b) ***Pure-Jump***

$$\nu(x) = \frac{A}{|x|^{\beta+1}} + \nu'(x), \quad A > 0, \quad \beta \in (1,2), \quad \int_{\mathbb{R}} |x| \nu(x) dx < \infty,$$

where there exists $x_0 > 0$ such that for $|x| \leq x_0$ we have $|\nu'(x)| \leq \frac{K}{|x|^{\beta'+1}}$ for some $\beta' < 1$ and a constant $K \geq 0$.

Plus additional regularity conditions (See paper).

Asymptotic Results: Increasing Span and Estimated Activity

Theorem 1. *Suppose there exists an estimator of β, denoted with $\widehat{\beta}$ and assumptions A, B and C-u for some $u > 0$ hold.*

(a) If $\widehat{\beta} - \beta = o_p\left(\frac{\Delta_n^\alpha}{\sqrt{T}}\right)$ for some $\alpha > 0$, then we have

$$\sqrt{T}\left(\frac{1}{T}V_T(X, \Delta_n, \widehat{\beta}, u) - \frac{1}{T}V_T(X, \Delta_n, \beta, u)\right) = o_p\left(\frac{1}{\sqrt{T}}\right).$$

Asymptotic Results: Increasing Span and Estimated Activity

(b) *If we have in addition assumption* B', $\widehat{\beta}$ *uses only information before the beginning of the sample or an initial part of the sample with a fixed time-span (i.e., one that does not grow with* T*), and further* $\widehat{\beta} - \beta = O_p(\Delta_n^{\alpha})$ *for* $0 < \alpha < (1 - \beta'/\beta) \vee (2 - 2/\beta)$ *and* $\alpha \leq 1/2$*, then we have*

$$\sqrt{T}\left(\frac{1}{T}V_T(X, \Delta_n, \widehat{\beta}, u) - \frac{1}{T}V_T(X, \Delta_n, \beta, u)\right)$$
$$- \frac{\sqrt{T}\log(\Delta_n)\mathbb{E}(G_\beta(uZ_\beta\sigma_t))}{\beta^2}(\widehat{\beta} - \beta) \xrightarrow{\mathbb{P}} 0,$$

where $G_\beta(x) = \beta x^\beta e^{-x^\beta}$ *for* $x > 0$.

(c) *Under the conditions of part (b), a consistent estimator for* $\mathbb{E}(G_\beta(uZ_\beta\sigma_t))$ *is given by*

$$\widehat{G}_\beta = \frac{\Delta_n}{T}\sum_{i=1}^{[T/\Delta n]}\left(u\Delta_n^{-1/\widehat{\beta}}\Delta_i^n X\right)\sin\left(u\Delta_n^{-1/\widehat{\beta}}\Delta_i^n X\right) \xrightarrow{\mathbb{P}} \mathbb{E}(G_\beta(uZ_\beta\sigma_t)).$$

Asymptotic Results: Increasing Span and Estimated Activity

Comments:

- The estimation β does not need long span, it requires only high-frequency
- An example of estimator $\widehat{\beta}$ with $\sqrt{1/\Delta_n}$ rate of convergence is the one proposed in Todorov and Tauchen (2009):

$$\widehat{\beta} = \frac{\ln(2)\, p^*}{\ln(2) + \ln[\Phi_t(X, p^*, 2\Delta_n)] - \ln[\Phi_t(X, p^*, \Delta_n)]}$$

where p^* is optimally chosen from a first-step estimation of the activity and the power variation $\Phi_T(X, p, \Delta_n)$ is defined as

$$\Phi_t(X, p, \Delta_n) = \sum_{i=1}^{[t/\Delta n]} |\Delta_i^n X|^p.$$

Monte Carlo

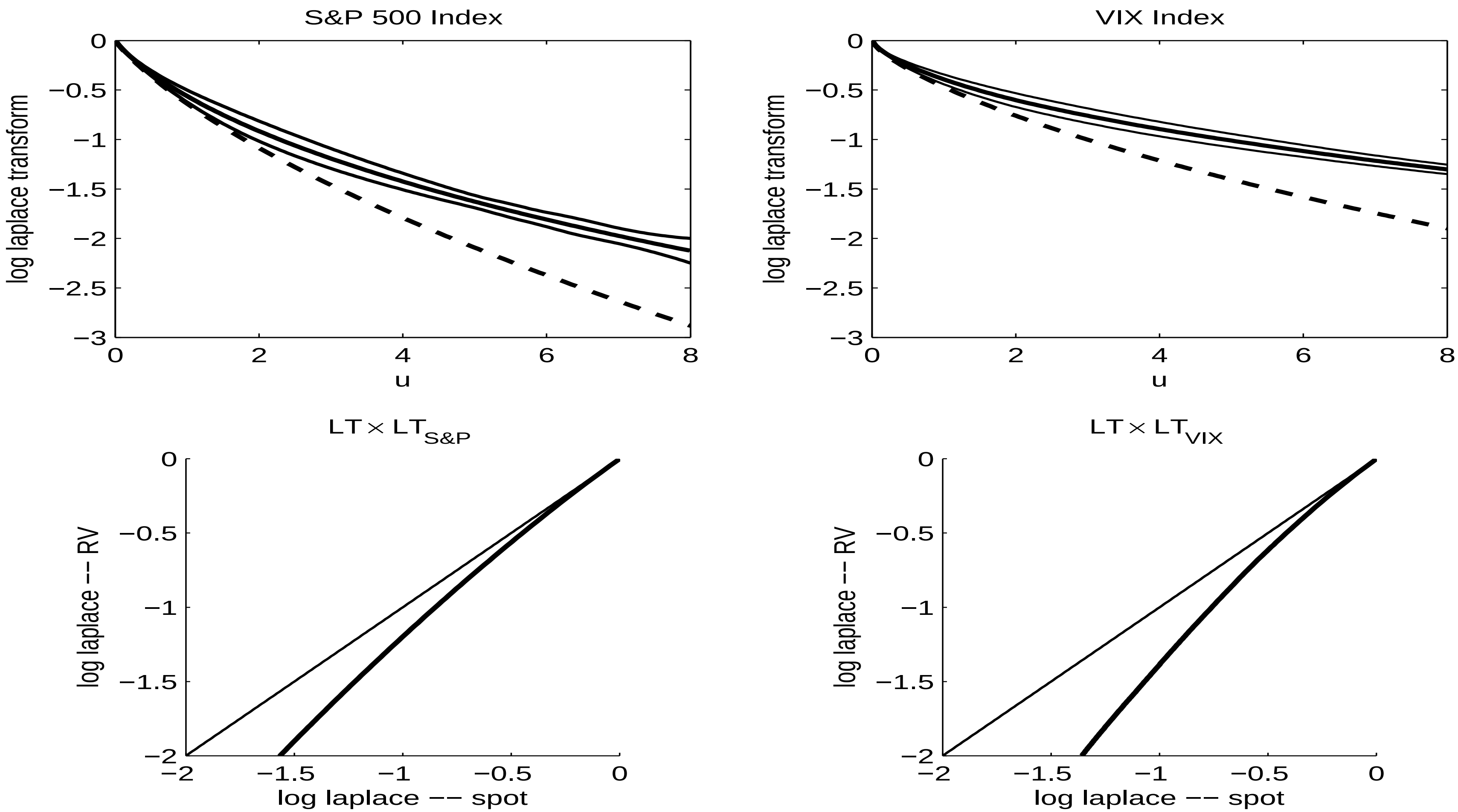
See Paper

Empirical Application

We use two data sets:

- 1-minute level data on the S&P 500 futures index, January 1, 1990, to December 31, 2008, yielding $1,900,000$ 1-minute log returns

- 5-minute observations on the S&P volatility index, the VIX index, from September 22, 2003, to December 31, 2008, for a total of $93,324$ returns

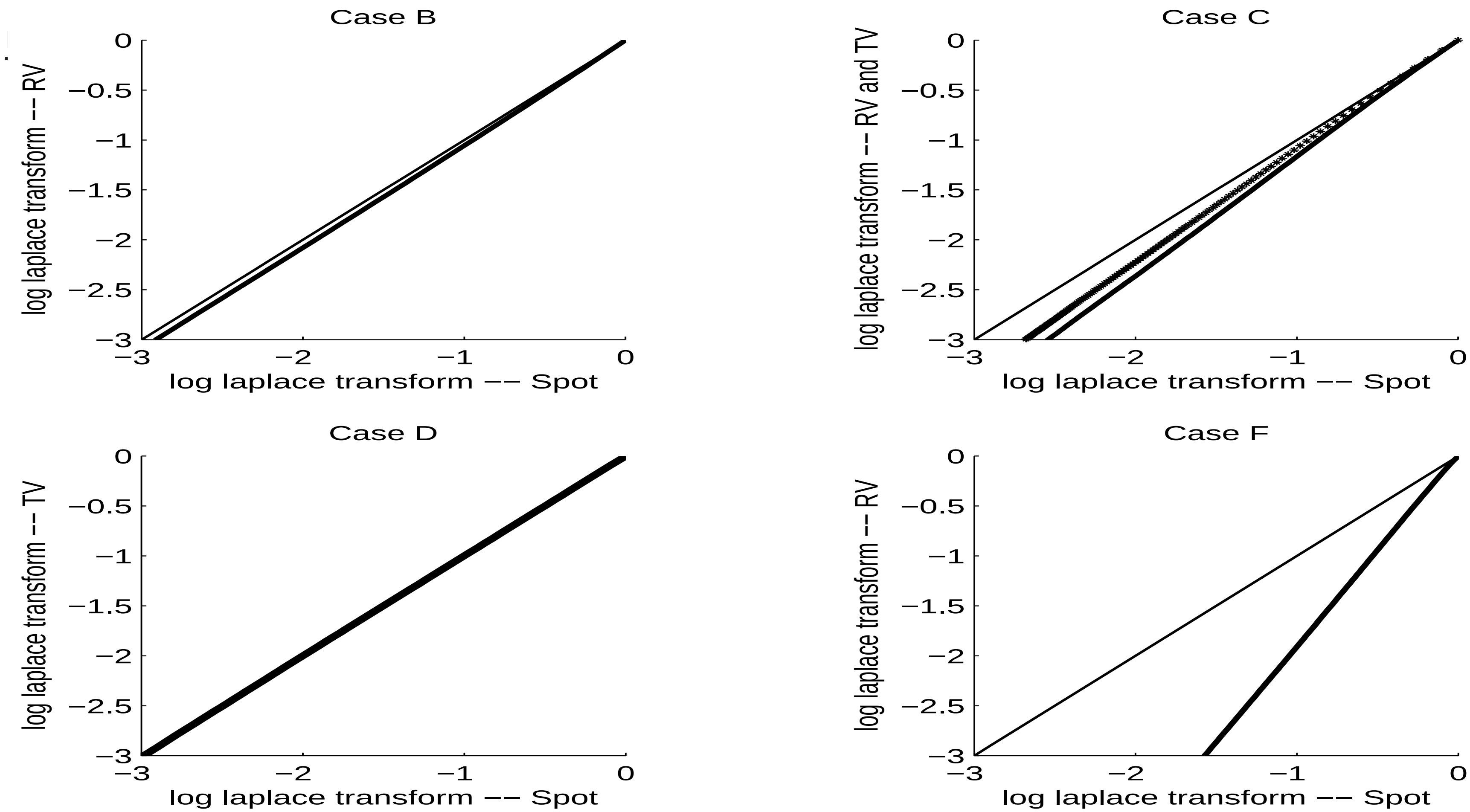


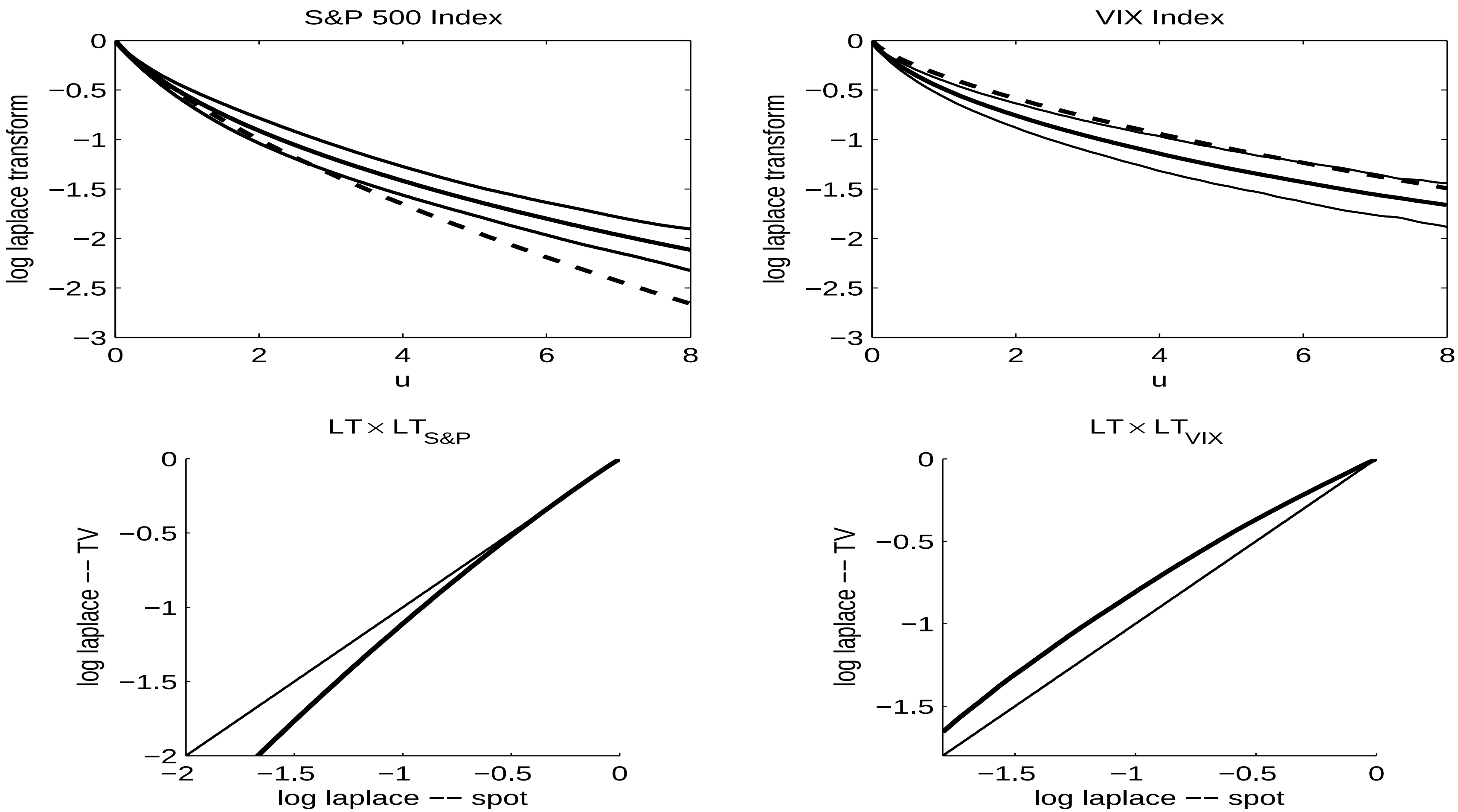
S&P 500 Index
VIX Index
log laplace transform
u
0
−0.5
−1
−1.5
−2
−2.5
−3
2
4
6
8
LT × LT$_{S\&P}$
LT × LT$_{VIX}$
log laplace −− RV
log laplace −− spot

Empirical Application

What is driving the wedge between the Laplace transforms of RV and spot volatility?

- price jumps?
- short-term volatility spikes?
- wrong beta?


Case B
Case C
Case D
Case F
log laplace transform -- RV
log laplace transform -- RV and TV
log laplace transform -- TV
log laplace transform -- RV
log laplace transform — Spot
0
−0.5
−1
−1.5
−2
−2.5
−3

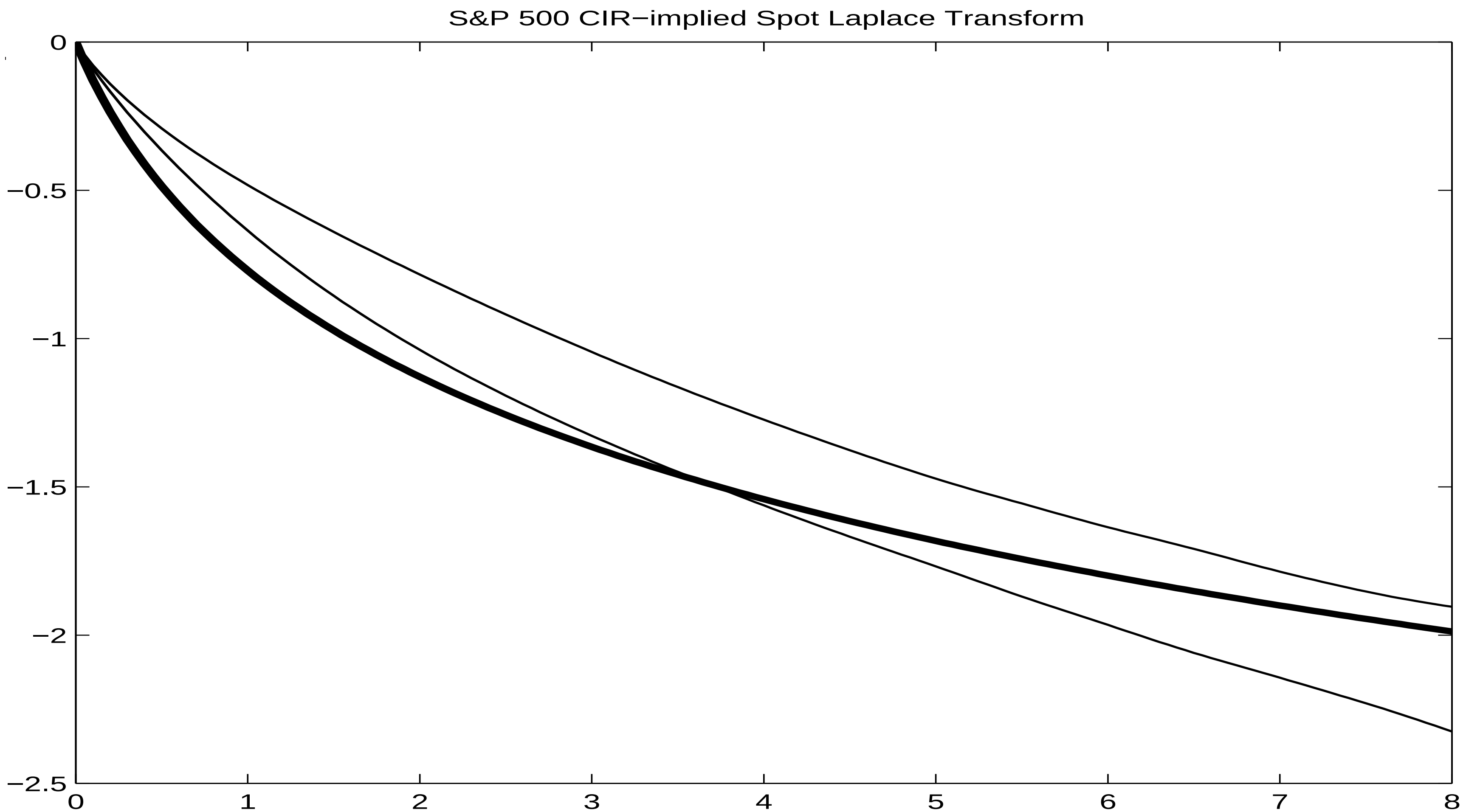

S&P 500 Index
VIX Index
log laplace transform
u
LT × LT_S&P
LT × LT_VIX
log laplace -- TV
log laplace -- spot

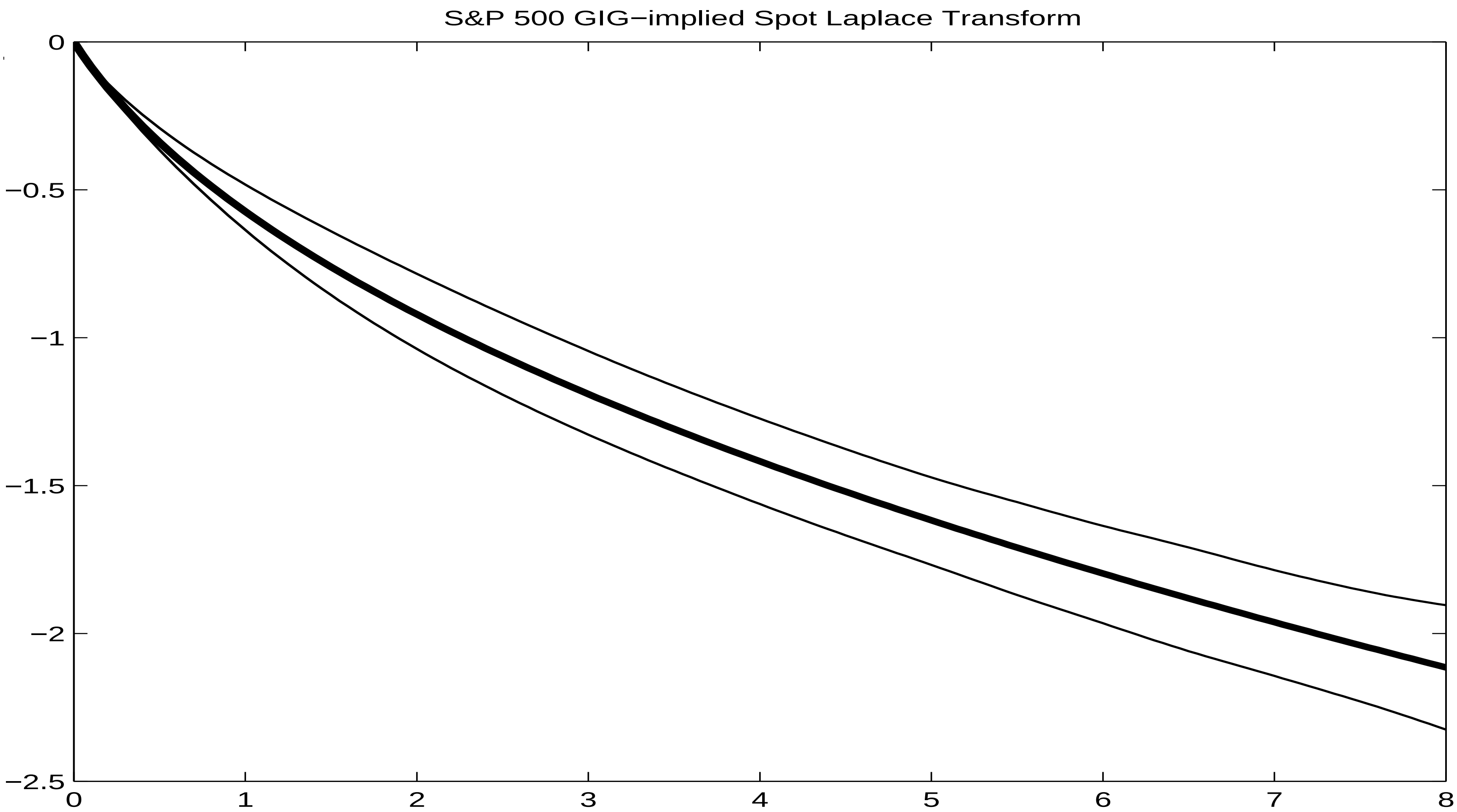
Empirical Application

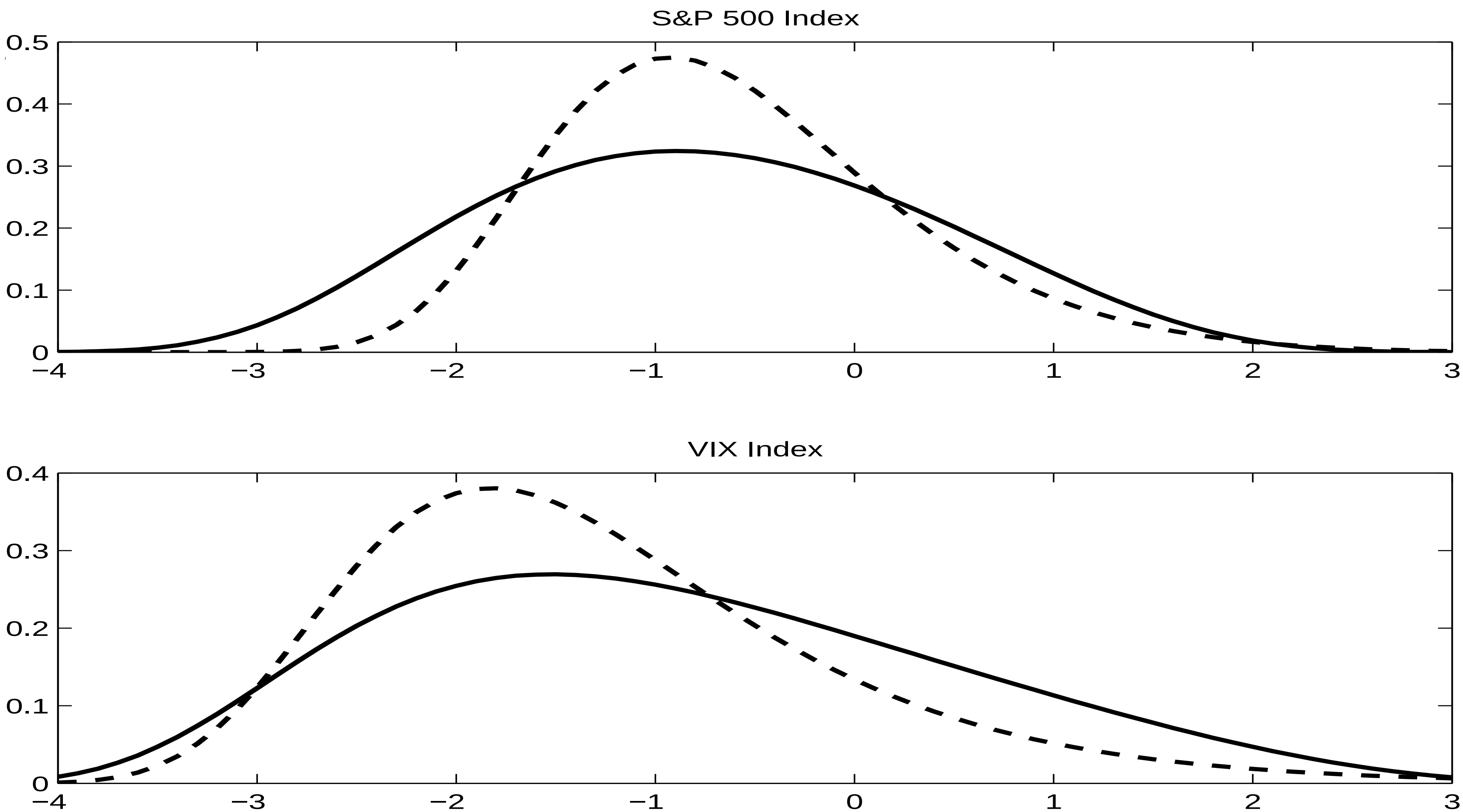
What is the marginal distribution of volatility?

- the Gamma as implied by square-root or CIR model (see **Case A** above) for volatility ?
- the Generalized Inverse Gaussian or **GIG** model (see Barndorff-Nielsen and Shephard (2001)?

Calibrate the parameters by fitting the model-implied Laplace transform to the estimated Laplace transform.


S&P 500 CIR−implied Spot Laplace Transform
0
−0.5
−1
−1.5
−2
−2.5
0
1
2
3
4
5
6
7
8


S&P 500 GIG−implied Spot Laplace Transform
0
−0.5
−1
−1.5
−2
−2.5
0
1
2
3
4
5
6
7
8


S&P 500 Index
0.5
0.4
0.3
0.2
0.1
0
−4
−3
−2
−1
0
1
2
3
VIX Index
0.4
0.3
0.2
0.1
0
−4
−3
−2
−1
0
1
2
3

Conclusions

- We propose a jump-robust estimator for volatility from high-frequency data: **Realized Laplace Transform**.
- It provides a direct estimate for the Laplace transform of the spot volatility and hence fully identifies its marginal distribution.

Future work:

- Develop estimation of the marginal distribution by minimizing distance on the space of Laplace transforms
- Extend the analysis to the case of conditional Laplace transform:

$$V_1(X,\Delta_n,\beta,u)\left(V_k(X,\Delta_n,\beta,u)-V_{k-1}(X,\Delta_n,\beta,u)\right)$$
$$\xrightarrow{\mathbb{P}} \int_0^1\int_{k-1}^k e^{-|Z_\beta u\sigma_t|^\beta}e^{-|Z_\beta u\sigma_s|^\beta}dsdt.$$