Density Approximations for Multivariate Affine Jump-Diffusions

with Applications to Econometrics and Option Pricing

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- Modeling decision: Affine Markov process $X := (X_t)_{t\geq 0, X_0 = x_0 \in \mathcal{D}}$ with characteristic function

$$\mathbb{E}\left[e^{uX_t} | X_0 = x_0\right] = e^{\phi(t,u)+\psi(t,u)\cdot x_0}, \quad u \in i\mathbb{R}^d$$

- ϕ and ψ solve system of Riccati equations
- Construction around characteristic function (rather then distribution function). What about likelihood-based inference?
- Theory on $\mathcal{D} = \mathbb{R}^m_+ \times \mathbb{R}^n$,$\mathcal{D} = \mathbb{S}^+_d \times \mathbb{R}^n$. Affine models are members of the polynomial family. In this family polynomial moments of order k map to polynomials of order $\leq k$ in x_0

Goals

- Develop approximations to transition densities that:
  - avoid the need for Fourier inversion
  - work for *any* fixed time horizon $t > 0$
  - converge uniformly and pointwise to the true unknown transition density
  - are available in closed-form
  - are tractable enough so that we can integrate over them in closed-form
- This will be very beneficial for applications in:
  - finance: option pricing, credit risk,...
  - econometrics (likelihood inference), computational statistics

Existence of (Derivatives of) Densities

- For existence of transition densities we can investigate asymptotic behavior of the characteristic function. By Fourier theory

$$g(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iux} e^{\phi(t,iu)+\psi(t,iu)x_0} du$$

if $\left| e^{\phi(t,iu)+\psi(t,iu)x_0} \right| \in \mathcal{L}^1$

- We have results for
  - (multidimensional) CBI processes
  - integrated affine processes
  - Heston
  - ...

- Parametric restrictions impose boundary non-attainment of $\mathcal{D}$

Weighted L^2s Spaces: Basics I

- Let g be a density function
  - exists, but not given in tractable/closed form
  - With moments of arbitrary order, $\mu_k := \int \xi^k g(\xi) d\xi < \infty$
- Desire: Approximate g in nice (closed) form
- Restrictions: Cannot apply nonlinear transformations (Aït-Sahalia (2002))
- Approximation Choice: $\mathcal{L}^2$ expansion $\longrightarrow$ true conditional moments will appear in the coefficients of the expansion

Weighted L^2s Spaces: Basics II

- Pick some weight function w such that
- $g/w \in \mathcal{L}^2_w$, where

$$\mathcal{L}^2_w := \left\{ f \mid \int_{\mathcal{D}} |f(\xi)|^2 w(\xi) d\xi < \infty \right\}$$

- When w has exponential tails polynomials are dense in L^2_w (use, e.g., Carleman condition).
- Integrability condition ($g/w \in \mathcal{L}^2_w$) boils down to existence of exponential moments, since we need

$$\int_{\mathcal{D}} g(\xi)^2 e^{a|\xi|} d\xi \leq C \int_{\mathcal{D}} g(\xi) e^{a|\xi|} d\xi < \infty$$

Weighted L^2s Spaces: Basics III

- Let H_k be an orthonormal basis of polynomials and $c_k := \int H_k(\xi)g(\xi)d\xi$. Then (Hilbert space theory)

$$g = w \cdot \sum c_k H_k, \text{ and define} \tag{1}$$

$$g^{(J)} := w \cdot \sum_{0 \le k \le J} c_k H_k \tag{2}$$

  Note: c_k linear combination of μ_k

- Intuitively: The closer w to g the smaller the contribution from correcting polynomials
- Nice: For each series truncation (J) in (2), the approximation integrates to one
- Unfortunately (2) is not a density, because it may become negative

Beyond $\mathcal{L}^2$ Convergence

- Sobolev Embedding Theorem gives us

$$\|g^{(J)} - g\|_\infty \to 0, \text{ for } J \to \infty$$

if we can show that $D^\alpha g \in \mathcal{L}^2_{\frac{1}{w}}$ for $\alpha > \frac{d}{2}$. Convergence depends on dimensionality of state vector

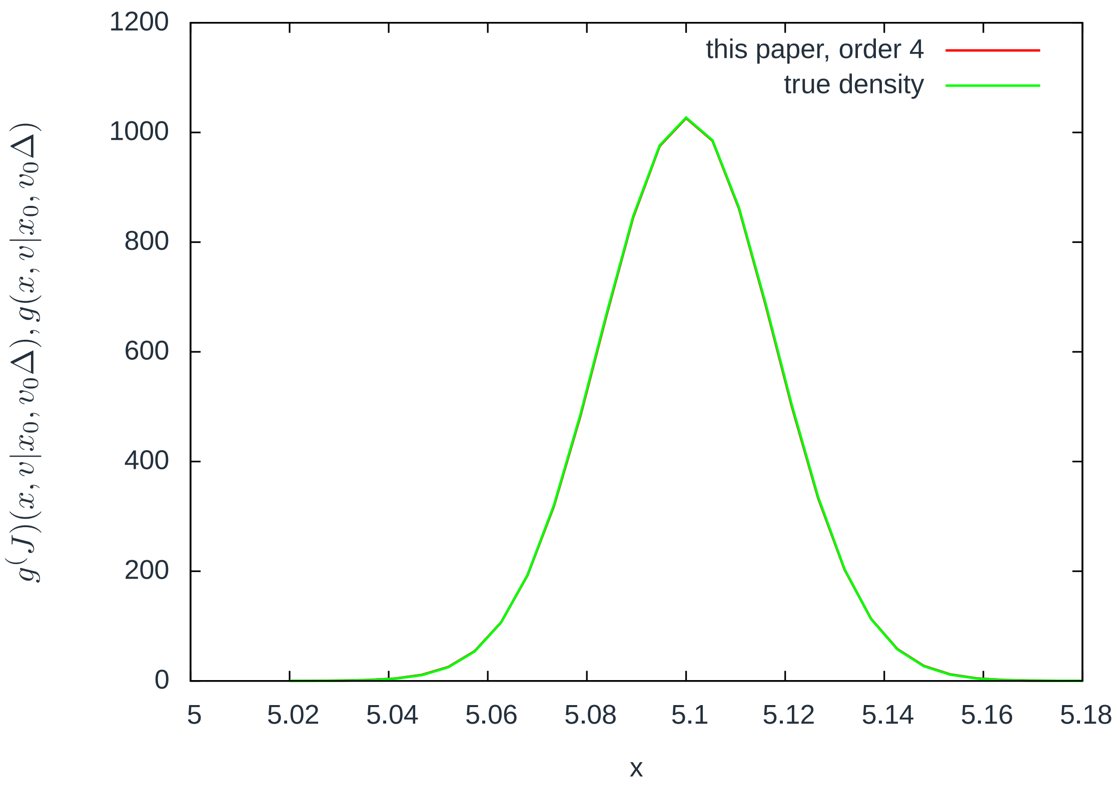
- We can get a handle on $D^\alpha g$ through integration by parts formula

$$\int_{\mathcal{D}} Dg(\xi)\xi^\alpha d\xi = -\int_{\mathcal{D}} g(\xi) D\xi^\alpha d\xi = -\alpha_i \mu_{\alpha-1}$$

- We can approximate derivatives of the density function (w.r.t. forward variable) the same way as the original density function

Heston Density Picture

- Picture of Heston transition density with order 4 expansion and true density with $\Delta = 1/52, \rho = -0.8, \kappa\theta = 0.04$ and $\kappa = 1, \mu = 0.05, \sigma = 0.3, x_0 = 5.1, v_0 = v = 0.045$



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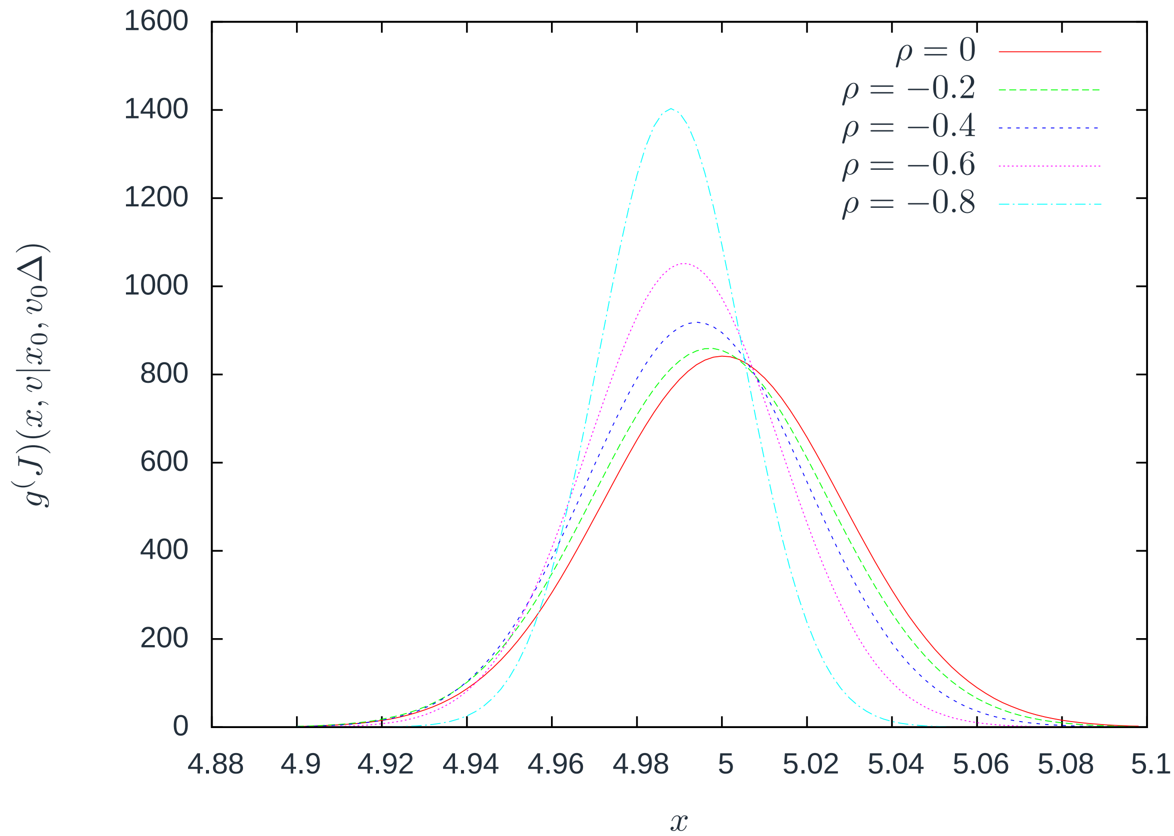


Likelihood Inference

- Exact simulations from Heston model and CBI jump-diffusion

$$dY_t = (\kappa\theta - \kappa Y_t)dt + \sigma\sqrt{Y_t}dW_t + EdP_t,$$

  where E is exponential with mean μ and P_t a Poisson counter with intensity l

- 200 datasets for Heston and CBI model using reasonable parameters

- Reestimate the model under Bayesian and Frequentist methodology and assess posterior distribution, respectively sampling distribution of parameters

Bayesian Inference

- Kolmogoroff-Smirnoff test for equality of posterior densities. The numbers are p-values (H_0 is equality)

	Heston Model			CBI	
KS	Order 2	Order 4	KS	Order 2	Order 4
$\kappa\theta_V$	0.0199	0.6654	$\kappa\theta_V$	0.0038	0.2360
κ_V	0.0000	0.0693	κ_V	0.0396	0.7658
σ_V	0.0001	0.2574	σ_V	0.0000	0.6754
$\kappa\theta_X$	0.0018	0.0348	l	0.0000	0.0571
ρ	0.0003	0.3291	μ	0.0000	0.0049

- Posterior distribution from order 4 expansion can not statistically be distinguished from posterior from true density for most parameters

- We can construct density directly around $X_t|v_0, x_0$
- Natural standardization $Y := \frac{X-\mu}{\zeta}$, where $\mu := \mathbb{E}\left[X_t|x_0, v_0\right]$, and $\zeta := \sqrt{\mathbb{V}\left[X_t|x_0, v_0\right]}$
- Choose $\mathcal{L}^2$ weight w with support on $\mathbb{R}$. For example bilateral Gamma distribution, or double exponential
- Gaussian weight is not an admissible choice, but easy to work with and ok for low orders. European Options prices under stochastic (multi-)volatility jump-diffusion models in closed form

- Recall that the price of a European call option with maturity t and strike price K is given by :

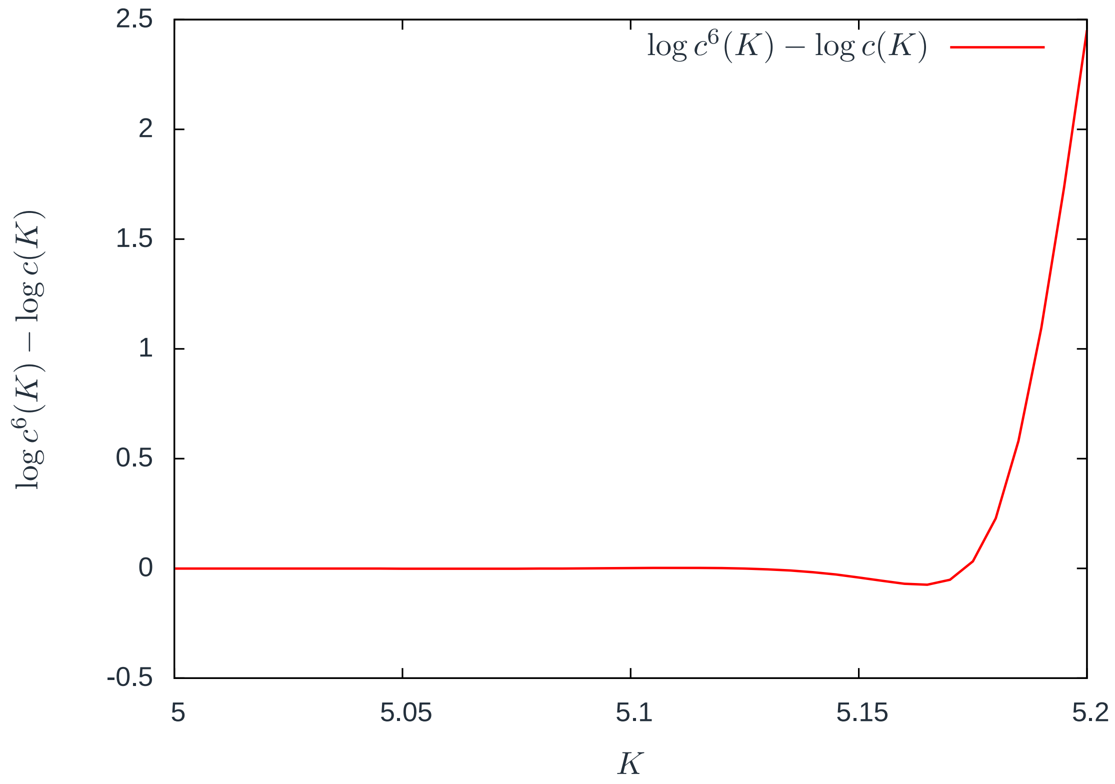
$$C(t,K) := e^{-rt}\mathbb{E}\left[\left(e^{X_t} - K\right)^+ |x_0, v_0\right]$$

$$= e^{-rt}\left(\underbrace{\int_{\log K}^{\infty} e^x p(x|x_0,v_0)dx}_{\text{HA}} - K\underbrace{\int_{\log K}^{\infty} p(x|x_0,v_0)dx}_{\text{HB}}\right)$$

- Compute HA and HB using (J) order density approximation $g^{(J)}$. If w admits closed-form computation of $\int_C^\infty e^{ax}x^k w(x)dx$ then we get closed-from approximation for option prices

Picture of Heston Option Prices

- Same parameterization as in likelihood inference example before. Figure shows the log difference of Heston call option prices using our expansions (in closed form!) to true option prices (Carr/Madan) for various strike prices K with spot=5.1



- We develop approximations to transition densities using polynomial expansion methods. The approximations are tailored for affine models and exploit their properties to a great extent
- Approximations in this paper are applicable to multivariate models with jumps
- It works equally well for reducible and irreducible models alike
- Quality of approximations independent of time interval between observations
- Expansions are performed on the correct state space of the processes
- Expansions integrate to unity by construction and hence are suited for Bayesian inference