

Nonparametric leverage effects

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Related Papers

- Nonparametric stochastic volatility (2008)
- **Nonparametric leverage effects** (2009)
- Infinitesimal cross-moments and return/variance cojumps (*in progress*)

Papers downloadable from my home page.



Why is the leverage effect called this way?

Leverage effect: observed **negative** correlation between stock returns and their volatility.

In a Modigliani-Miller economy:

$$V = D + E$$

Stock volatility:

$$\sigma_S = \sigma_V \left(1 + \frac{D}{E} \right)$$

Leverage effect: changes in the debt-to-equity ratio affect market volatility.



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A baseline model for leverage effect (continued)

- **Caution:** Leverage effect has been shown (e.g. Figleski and Wang 2000, among others) not to be due to leverage only.

$$\partial\sigma_S = -\sigma_V \frac{D}{E^2} \partial E = - \left(\frac{\sigma_S - \sigma_V}{E} \right) \partial E \Rightarrow$$

$$\frac{\partial\sigma_S}{\partial E} = - \left(\frac{\sigma_S - \sigma_V}{E} \right) < 0.$$

- The magnitude of the leverage effect depends on volatility itself.



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Our contribution

- We investigate the dependence of leverage effect on volatility
- We use a continuous-time framework and allow for **jumps** both in price and in volatility
- We use *spot volatility* as the fundamental state variable of the system
- We provide nonparametric leverage estimators in this setting.



Inverse problems

- Suppose you observe a variable in the form of a time series of n observations over a time span T :

$$X_0, X_{1T/n}, X_{2T/n}, X_{3T/n}, \dots, X_T$$

- Assume that the dynamics of X_t is driven by a **stochastic differential equation** (continuous time).
- Inverse problem: infer the coefficients of the stochastic differential equation (including leverage effect)
- The asymptotics is based on

$$\Delta_{n,T} = T/n \rightarrow 0$$

as

$$n \rightarrow \infty,$$

$$T \rightarrow \infty$$



Jumps

Jumps will be modelled as Poisson processes:

$$J_t = \sum_{i=1}^{N_T} c_{\tau_i} = \int_0^T \lambda_s \int_Y c(y, s) dy ds$$

- N_t is the counting process
- τ_i jump instants
- c_{τ_i} are the jump sizes
- λ is the jump intensity
- We do not have infinite activity jumps in our setting.



Volatility estimation: nonparametric estimation

Estimation of $\sigma^2(X_t)$:

- **Univariate setting:**

No jumps, Florens-Zmirou(1993), Bandi and Phillips (2003).

With jumps: Bandi and Nguyen (2003) or Mancini and Renò (2009)

- **Multivariate setting:** (no jumps) Bandi and Moloche (2008).

- **Stochastic volatility:** (no jumps)

Renò (2006) (with realized volatility)

Renò (2008) (Fourier method)

Comte, Genon-Catalot, and Rohzenolc (2008)



A simple example: Florens-Zmirou (1993)

The model is:

$$dX_t = \mu_t dt + \sigma(X_t) dW_t$$

The estimator is of Nadaraya-Watson type:

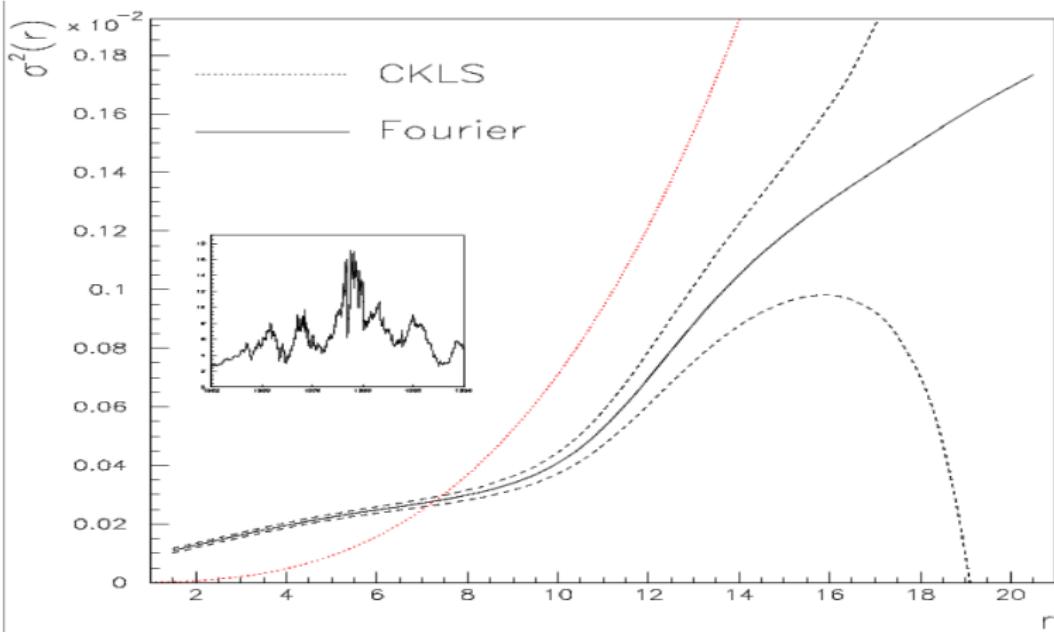
$$\hat{\sigma}^2(x) = \frac{n \sum_{t=1}^{n-1} K\left(\frac{X_t - X}{h}\right) (X_{t+1} - X_t)^2}{T \sum_{t=1}^n K\left(\frac{X_t - x}{h}\right)}.$$

Asymptotic properties are fully assessed in Florens-Zmirou (1993).



Parametric vs. Nonparametric

Jiang (1998) data set and estimates.



The continuous-time model

The continuous-time model we consider is:

$$\begin{aligned} r_{t,t+dt} &= d \log(p_t) = \mu_t dt + \sigma_t dW_t^r + dJ_t^r, \\ d\xi(\sigma_t^2) &= m_t dt + \Lambda(\sigma_t^2) dW_t^\sigma + dJ_t^\sigma, \end{aligned}$$

where the state of the economy at time t is summarized by σ_t^2 and

- $\{W_t^r, W_t^\sigma\} = \left\{ \rho(\sigma_t^2) dW_t^1 + \sqrt{1 - \rho^2(\sigma_t^2)} dW_t^2, dW_1 \right\}$
- $\{J_t^r, J_t^\sigma\}$ is a bi-dimensional compound Poisson process with volatility-dependent intensities.

We ask for **recurrence** of the volatility process.



Recurrence and local time

- Recurrence is defined as: every value of the state variable is crossed infinitely often.
- Recurrence is weaker than stationarity
- The fundamental econometric object in this context is the Local Time of volatility (the single state variable):

$$L_{\sigma^2}(T, x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^T \mathbf{1}_{[x, x+\varepsilon]}(\sigma_s^2) \frac{\partial \xi^{-1}(\xi(\sigma_s^2))}{\partial \xi} \Lambda^2(\sigma_s^2) ds \quad a.s.$$

- $L_{\sigma^2}(T, x)$ defines the amount of time, in information units or in units of the continuous component of the process' quadratic variation, that σ_t^2 spends in a small right neighborhood of x between time 0 and time T



Estimation of local time

Local time estimator:

$$\widehat{L}_{\sigma^2}(T, x) = \frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K} \left(\frac{\tilde{\sigma}_{iT/n}^2 - x}{h_{n,T}} \right)$$

(in the stationary case) = $T \cdot \widehat{f}(x)$

Where $\widehat{f}(x)$ is the stationary density.

In the stationary case, $L_{\sigma^2}(T, x) \propto T$

In the more general recurrent case,

$$L_{\sigma^2}(T, x) \propto v(T) \rightarrow \infty$$

(at a lower speed than T).



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Infinitesimal moments

Definition:

$$\vartheta_k(\sigma^2) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \mathbf{E} \left[\left(\xi(\sigma_{t+\Delta}^2) - \xi(\sigma_t^2) \right)^k \middle| \sigma_t^2 = \sigma^2 \right], \quad k = 2, 3, 4,$$

Estimator:

$$\hat{\vartheta}_k(\sigma^2) = \frac{\sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - \sigma^2}{h_{n,T}} \right) \left(\xi(\sigma_{(i+1)T/n}^2) - \xi(\sigma_{iT/n}^2) \right)^k}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - \sigma^2}{h_{n,T}} \right)} \quad k = 2, \dots$$

We develop a theory for infinitesimal moments from which we draw the quantities of interest.

For the moment, we assume volatility is observed



Infinitesimal covariance

Definition:

$$\vartheta_{1,1}(\sigma^2) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \mathbf{E} \left[(\log p_{t+\Delta} - \log p_t) (\xi(\sigma_{t+\Delta}^2) - \xi(\sigma_t^2)) \mid \sigma_t^2 = \sigma^2 \right],$$

Estimator:

$$\widehat{\vartheta}_{1,1}(\sigma^2) = \frac{\sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - \sigma^2}{h_{n,T}} \right) (\log p_{(i+1)T/n} - \log p_{iT/n}) (\xi(\sigma_{(i+1)T/n}^2) - \xi(\sigma_{iT/n}^2))}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - \sigma^2}{h_{n,T}} \right)},$$

The first leverage estimator we analyze is:

$$\widehat{\rho}(\sigma^2) = \frac{\widehat{\vartheta}_{1,1}(\sigma^2)}{\sigma \sqrt{\widehat{\Lambda}^2(\sigma^2)}}$$



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Asymptotic theory: assumptions on bandwidths

$v(T)$ is a regularly-varying function at infinity. Let, also, $h_{n,T} \rightarrow 0$ with $\Delta_{n,T} \rightarrow 0$ for $n, T \rightarrow \infty$. Let, also, $\phi_{n,T} \rightarrow 0$ and $k \rightarrow \infty$.

3.1 $\frac{v(T)}{h_{n,T}} \left(\Delta_{n,T} \log \frac{1}{\Delta_{n,T}} \right)^{1/2} \rightarrow 0.$

3.2 $\frac{h_{n,T}^5 v(T)}{\Delta_{n,T}} \rightarrow C$, where C is a suitable constant.

(in absence of jumps in volatility)

3.3 $h_{n,T} v(T) \rightarrow \infty$

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Asymptotic theory: The continuous case

In absence of jumps, $\widehat{\vartheta}_2(\sigma^2) \xrightarrow{p} \Lambda^2(\sigma)$.

Theorem 1. Assume $J^r = J^\sigma = 0$. Then $\widehat{\rho}(\sigma^2) \xrightarrow{p} \rho(\sigma^2)$ and

$$\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,\sigma^2)}{\Delta_{n,T}}} \left\{ \widehat{\rho}(\sigma^2) - \rho(\sigma^2) - \widetilde{\Gamma}_\rho(\sigma^2) \right\} \Rightarrow \mathbf{N} \left(0, \mathbf{K}_2 \left[1 - \frac{1}{2} \rho^2(\sigma^2) \right] \right),$$

where

$$\widetilde{\Gamma}_\rho(\sigma^2) = \frac{1}{\sigma\sqrt{\vartheta_2}} \Gamma_{\vartheta_{1,1}} - \frac{\vartheta_{1,1}(\sigma^2)}{2\sigma\sqrt{\vartheta_2^3(\sigma^2)}} \Gamma_{\vartheta_2}.$$



Jumps in volatility

Generalized Duffie, Pan and Singleton model:

$$\begin{aligned} dX_t &= \mu(\sigma_t^2)dt + \sigma_t dW_t^r \\ d\sigma_t^2 &= m(\sigma_t^2)dt + \Lambda(\sigma_t^2)dW_t^\sigma + dJ_t^\sigma, \\ dJ_t^\sigma &= \xi^\sigma dN_t^\sigma, \quad \xi^\sigma \sim \exp(\mu_\xi). \end{aligned}$$

In infinitesimal moments

$$\begin{aligned} \theta^1(x) &= m_{\sigma^2}(x) + \mu_\xi \lambda_{\sigma^2}(x) \\ \theta^2(x) &= \Lambda_{\sigma^2}^2(x) + 2\mu_\xi^2 \lambda_{\sigma^2}(x) \\ \theta^3(x) &= 6\mu_\xi^3 \lambda_{\sigma^2}(x) \\ \theta^4(x) &= 24\mu_\xi^4 \lambda_{\sigma^2}(x) \end{aligned}$$

Nonparametric estimators

$$\begin{aligned} \widehat{\mu}_\xi &= \frac{1}{n} \sum_{i=1}^n \frac{\widehat{\theta}^4(\tilde{\sigma}_{iT/n}^2)}{4\widehat{\theta}^3(\tilde{\sigma}_{iT/n}^2)}, \\ \widehat{\lambda}_{\sigma^2}(x) &= \frac{\widehat{\theta}^4(x)}{24\widehat{\mu}_\xi^4}, \\ \widehat{\Lambda}_{\sigma^2}^2(x) &= \widehat{\theta}^2(x) - 2\widehat{\mu}_\xi \widehat{\lambda}_{\sigma^2}(x), \\ \widehat{m}_{\sigma^2}(x) &= \widehat{\theta}^1(x) - \widehat{\mu}_\xi \widehat{\lambda}_{\sigma^2}(x). \end{aligned}$$



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Leverage with jumps in volatility

Thus, in general,

$$\tilde{\rho}(\sigma^2) = \frac{\widehat{\vartheta}_{1,1}(\sigma^2)}{\left(\sigma^2 \cdot f(\widehat{\vartheta}_2, \widehat{\vartheta}_3, \widehat{\vartheta}_4)(\sigma^2)\right)^{\frac{1}{2}}},$$

where $f(\widehat{\vartheta}_2, \widehat{\vartheta}_3, \widehat{\vartheta}_4)(\cdot)$ is a specific function of the infinitesimal moments.

For the Generalized DPS model:

$$f(\widehat{\vartheta}_2, \widehat{\vartheta}_3, \widehat{\vartheta}_4)(\sigma^2) = \widehat{\Lambda}^2(x) = \widehat{\vartheta}_2(\sigma^2) - \frac{\widehat{\vartheta}_4(\sigma^2)}{12\widehat{\mu}_\sigma^2}$$

$$\widehat{\mu}_\sigma^2 = \left(\frac{1}{\bar{n}} \sum_{i=1}^{\bar{n}} \frac{\widehat{\vartheta}_4(\sigma_{iT/n}^2)}{4\widehat{\vartheta}_3(\sigma_{iT/n}^2)} \right)^2$$



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Independent jumps in both price and volatility

Theorem 4.

$$\begin{aligned} & \sqrt{h_{n,T} \widehat{\bar{L}}_{\sigma^2}(T, \sigma^2)} \left\{ \tilde{\rho}(\sigma^2) - \rho(\sigma^2) \right\} \\ \Rightarrow & \mathbf{N} \left(0, \mathbf{K}_2 \frac{\rho^2(\sigma^2)}{4\Lambda^4(\sigma^2)} \lambda_\sigma(\sigma^2) \mathbf{E} \left(\left((c_\sigma)^2 - \frac{1}{12\mu_\sigma^2} (c_\sigma)^4 \right)^2 \right) \right) \end{aligned}$$

Note that the rate is slower than in the case of $J^\sigma = 0$.



The general case: generalized leverage

When there are dependent jumps, we have three intensities:

$$\lambda^r, \quad \lambda^\sigma, \quad \lambda_{r,\sigma}^{\parallel}$$

In this case:

$$\tilde{\rho}(\sigma^2) \xrightarrow{p} \Xi(\sigma^2) = \rho(\sigma^2) + \frac{\lambda_{r,\sigma}^{\parallel}(\sigma^2)}{\sigma \Lambda(\sigma^2)} \mathbf{E}[c_r c_\sigma].$$

- leverage is composed of two terms
- the leverage due to the jump part is proportional to the covariance of jumps sizes and to the frequency of common jumps
- the two parts can be identified separately



Re-establishing consistency

- Mancini and Gobbi (2009): use a threshold function and estimate the cross variation as

$$C_t = \sum \Delta X_1 \cdot \Delta X_2$$

- Our approach: use cross-moments,

$$\vartheta_{p_1, p_2}(\sigma^2) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \mathbf{E} \left[(\log p_{t+\Delta} - \log p_t)^{p_1} \left(\xi(\sigma_{t+\Delta}^2) - \xi(\sigma_t^2) \right)^{p_2} \middle| \sigma_t^2 = \sigma^2 \right],$$

The intuition is as follows:

the cross-moments of order higher than $p_1 = 1, p_2 = 1$ depend solely on the features of the co-jumps and may therefore be used to identify $\lambda_{r,\sigma}^{\parallel}$ and $\mathbf{E}[c_r^{\parallel} c_{\sigma}^{\parallel}]$.



Cojumps estimation: methodology

Lay down the moments of price and volatility:

$$\left\{ \begin{array}{l} \vartheta_{1,0}(\cdot) = \mu(\cdot) \\ \vartheta_{2,0}(\cdot) = \cdot + (\lambda_r^*(\cdot) + \lambda_{r,\sigma}^{\parallel}(\cdot))\sigma_{J,r}^2 \\ \vartheta_{4,0}(\cdot) = 3(\lambda_r^*(\cdot) + \lambda_{r,\sigma}^{\parallel}(\cdot))\sigma_{J,r}^4 \\ \vartheta_{6,0}(\cdot) = 15(\lambda_r^*(\cdot) + \lambda_{r,\sigma}^{\parallel}(\cdot))\sigma_{J,r}^6 \end{array} \right. \quad \left\{ \begin{array}{l} \vartheta_{0,1}(\cdot) = m(\cdot) \\ \vartheta_{0,2}(\cdot) = \Lambda^2(\cdot) + (\lambda_\sigma^*(\cdot) + \lambda_{r,\sigma}^{\parallel}(\cdot))\sigma_{J,\sigma}^2 \\ \vartheta_{0,4}(\cdot) = 3(\lambda_\sigma^*(\cdot) + \lambda_{r,\sigma}^{\parallel}(\cdot))\sigma_{J,\sigma}^4 \\ \vartheta_{0,6}(\cdot) = 15(\lambda_\sigma^*(\cdot) + \lambda_{r,\sigma}^{\parallel}(\cdot))\sigma_{J,\sigma}^6 \end{array} \right.$$

Then lay down the cross-moments for identification:

$$\left\{ \begin{array}{l} \vartheta_{1,1}(\cdot) = \rho(\cdot)\sqrt{\Lambda}(\cdot) + \lambda_{r,\sigma^2}^{\parallel}(\cdot)\rho_J\sigma_{J,r}\sigma_{J,\sigma} \\ \vartheta_{2,2}(\cdot) = \lambda_{r,\sigma^2}^{\parallel}(\cdot)\sigma_{J,r}^2\sigma_{J,\sigma}^2(1+2\rho_J^2) \\ \vartheta_{3,1}(\cdot) = 3\lambda_{r,\sigma^2}^{\parallel}(\cdot)\rho_J\sigma_{J,r}^3\sigma_{J,\sigma} \\ \vartheta_{1,3}(\cdot) = 3\lambda_{r,\sigma^2}^{\parallel}(\cdot)\rho_J\sigma_{J,r}\sigma_{J,\sigma}^3 \end{array} \right.$$



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Integrated volatility estimation

Volatility is not observable!

Estimation of integrated volatility

$$\int_0^t \sigma_s^2 ds$$

- **Continuous diffusions:** realized volatility (Andersen and Bollerslev, 1998 and many others), alternative estimators.
- **In presence of jumps:** bipower variation (Barndorff-Nielsen and Shephard 2004), threshold estimation (Mancini, 2007), both (Corsi, Pirino and Renò, 2009)
- **With microstructure noise:** two scales (Zhang, Mykland and Ait-Sahalia, 2005), kernel estimator (Barndorff-Nielsen et al., 2008).

This list is admittedly fairly incomplete!



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A theory of spot variance estimation

Given an estimator $\widehat{V}_{iT/n}$ of integrated volatility $\int_{iT/n}^{iT/n+\phi_{n,T}} \sigma_s^2 ds$, built with k observations in the interval $[iT/n, iT/n + \phi_{n,T}]$, we define our spot variance estimator as:

$$\tilde{\sigma}_{iT/n}^2 = \frac{\widehat{V}_{iT/n}}{\phi_{n,T}}$$

We can show for many integrated volatility estimators in the literature that:

$$\phi_{n,T}^\beta k^\alpha \left\{ \tilde{\sigma}_{iT/n}^2 - \sigma_{iT/n}^2 \right\} \xrightarrow[k \rightarrow \infty, \phi_{n,T} \rightarrow 0]{} MN \left(0, a \left(\sigma_{iT/n}^4 \right)^\eta + b \right)$$

Basic example: with realized volatility:

$$\beta = 0, \quad \alpha = \frac{1}{2}, \quad a = 2, \quad \eta = 1, \quad b = 0$$

Related literature: Mykland and Zhang (2008), Fan and Wang (2008), Kristensen (2008), Mattiussi and Renò (2009).

Integrated variance estimators

Other estimators:

$dJ_t^r = 0$, without microstructure noise. $dJ_t^r = 0$, with microstructure noise.

① Bipower variation

(Barndorff-Nielsen and Shephard, 2004, 2005): $\alpha = \frac{1}{2}$, $\beta = 0$, $a \approx 2.6$, $b = 0$, and $\eta = 1$.

② Realized range

(Christensen and Podolskij, 2007): $\alpha = \frac{1}{2}$, $\beta = 0$, $a \approx 0.4$, $b = 0$, and $\eta = 1$.

③ Fourier estimator

(Malliavin and Mancino, 2002): same as realized variance.

① Two-scale estimator

(Zhang et al., 2005): If $q = \tau k^{2/3}$, then

$$\beta = 1, \alpha = \frac{1}{6}, a = 0, \text{ and } b = \left(\frac{8}{\tau^2}\right) (\mathbf{E}(\varepsilon^2))^2.$$

② Realized kernels

(Barndorff-Nielsen et al., 2006):
If $q = \tau k^{2/3}$, then $\beta = 1, \alpha = \frac{1}{6}, a = 0, \text{ and } b = 4 (\mathbf{E}(\varepsilon^2))^2 \frac{1}{\tau^2} \{g'(0)^2 + g'(1)^2\}$.



Estimating volatility in presence of jumps

We can provide exact asymptotic results for the following two estimators:

Threshold realized variance (Mancini, 2009):

$$TRV = \sum_{i=1}^{n-1} |\Delta X_i|^2 \longrightarrow \int \sigma_s^2 ds$$

Threshold bipower variation (Corsi, Pirino and Renò, 2009):

$$V^{TBP} = \frac{\pi}{2} \sum_{i=1}^{n-1} |\Delta X_i| |\Delta X_{i+1}| \longrightarrow \int \sigma_s^2 ds$$

where

$$\Delta X_i = \Delta X_i \cdot I_{\{|\Delta X_i| \leq \vartheta_t\}}$$

and

$$\vartheta_t (\Delta \log(1/\Delta))^{-\frac{1}{2}} \longrightarrow \infty$$



Estimation: data

- S&P 500 futures, high frequency data from 1982 to 2009
- 6,675 trading days
- Volatility estimation with *threshold bipower variation* (Corsi, Pirino and Renò, 2009)
- Small sample corrections:

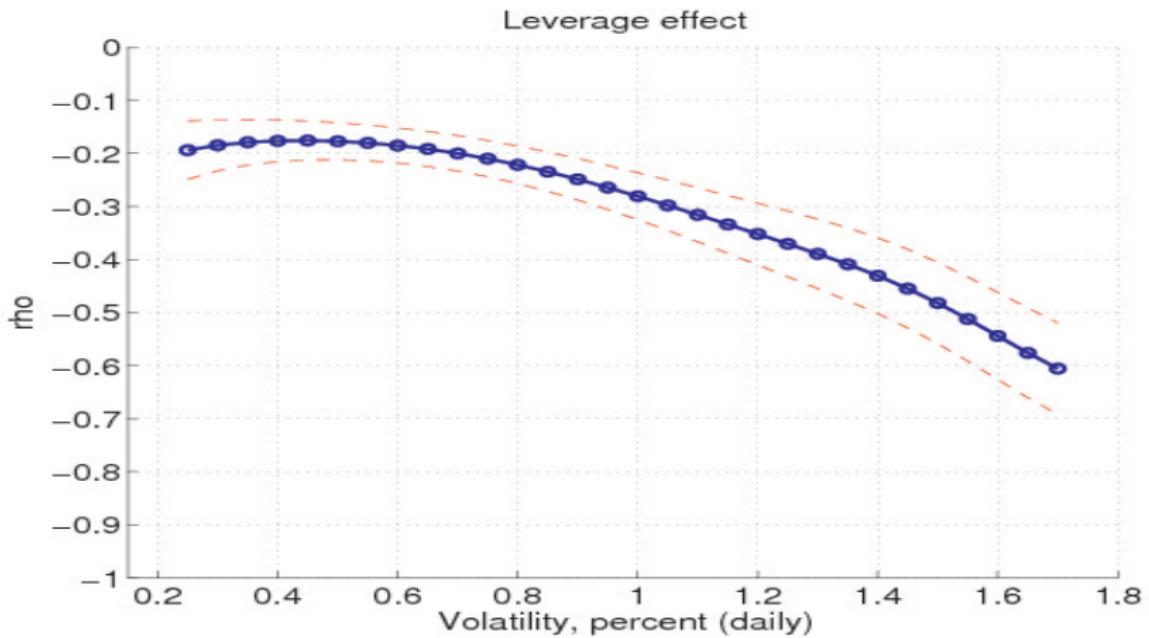
$$\vartheta_2(\sigma) \approx \Lambda^2(\sigma) + 2\mu_\sigma^2 \lambda_\sigma(\sigma),$$

$$\vartheta_3(\sigma) \approx 6\mu_\sigma^3 \lambda_\sigma(\sigma) + 3\vartheta_1(\sigma)\vartheta_2(\sigma)\Delta,$$

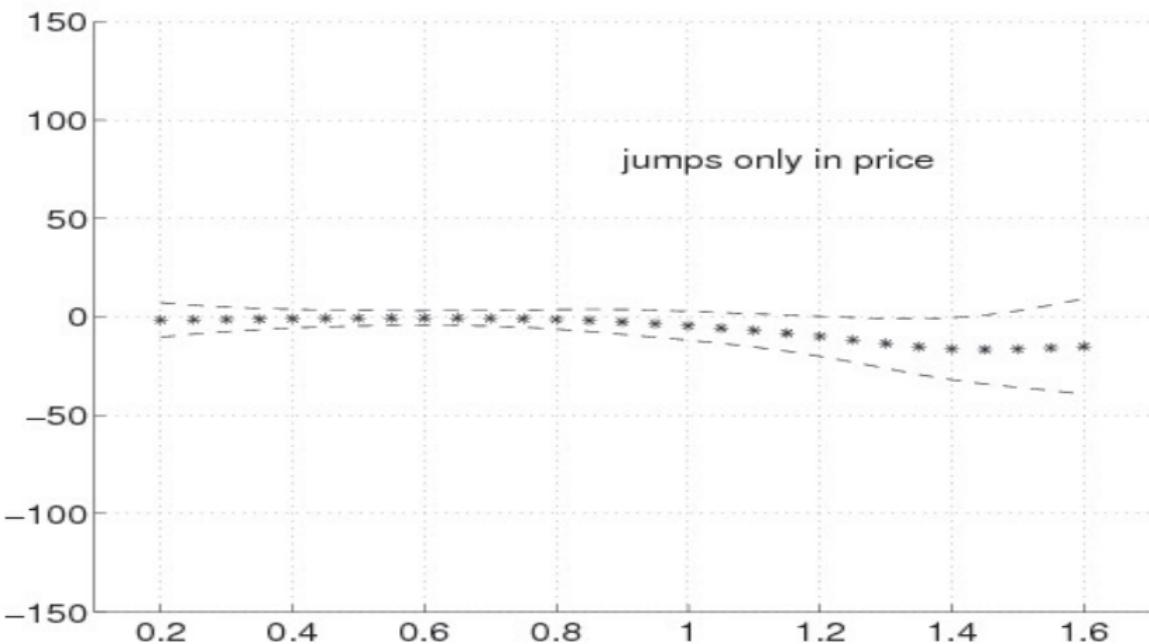
$$\vartheta_4(\sigma) \approx 24\mu_\sigma^4 \lambda_\sigma(\sigma) + \left[3(\vartheta_2(\sigma))^2 + 4\vartheta_1(\sigma)\vartheta_3(\sigma) \right] \Delta.$$



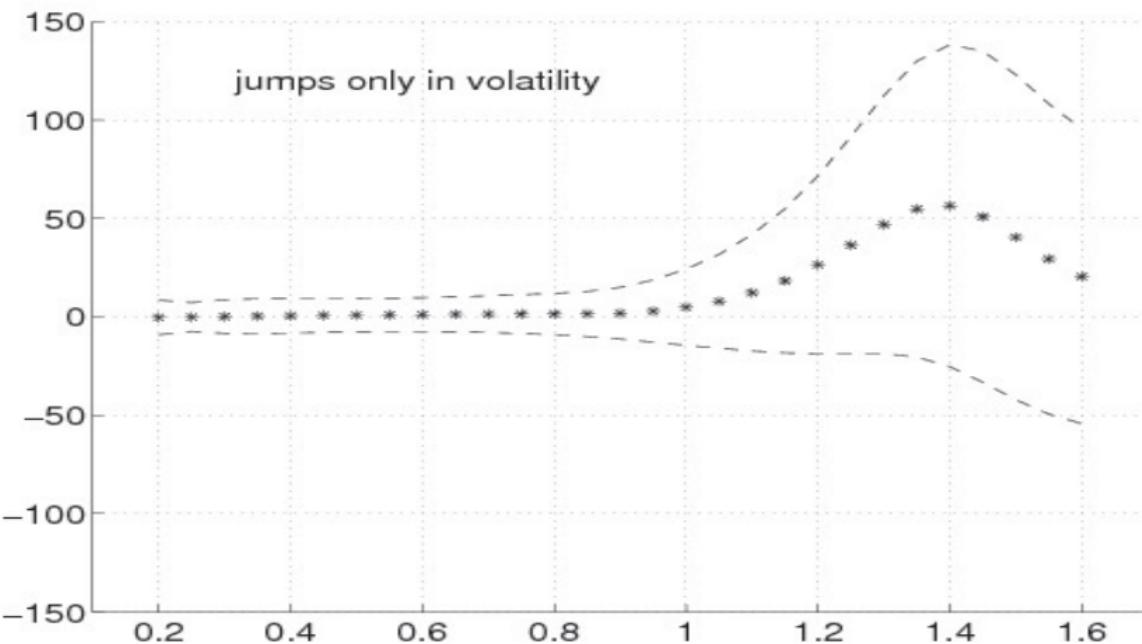
Estimation: volatility of volatility



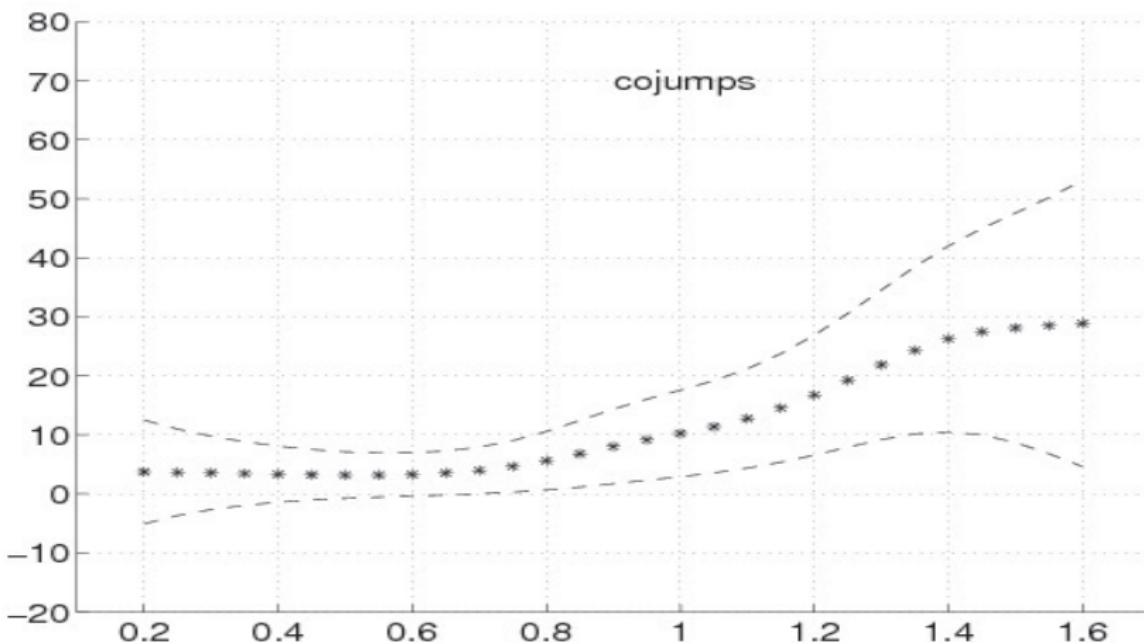
Estimation:jumps in price



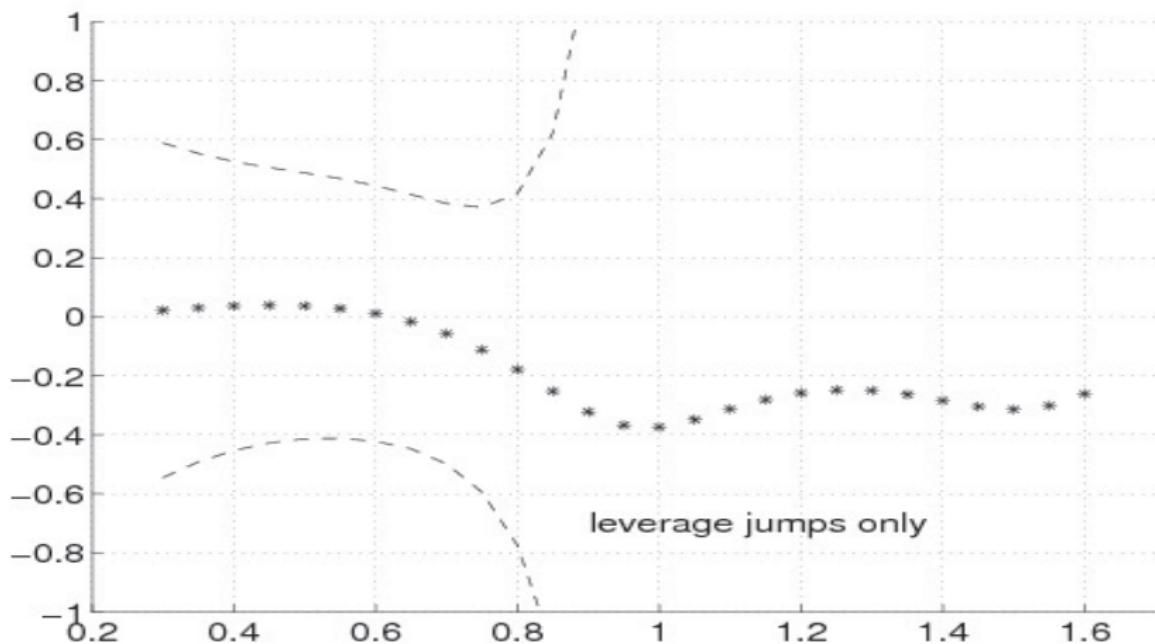
Estimation:jumps in volatility



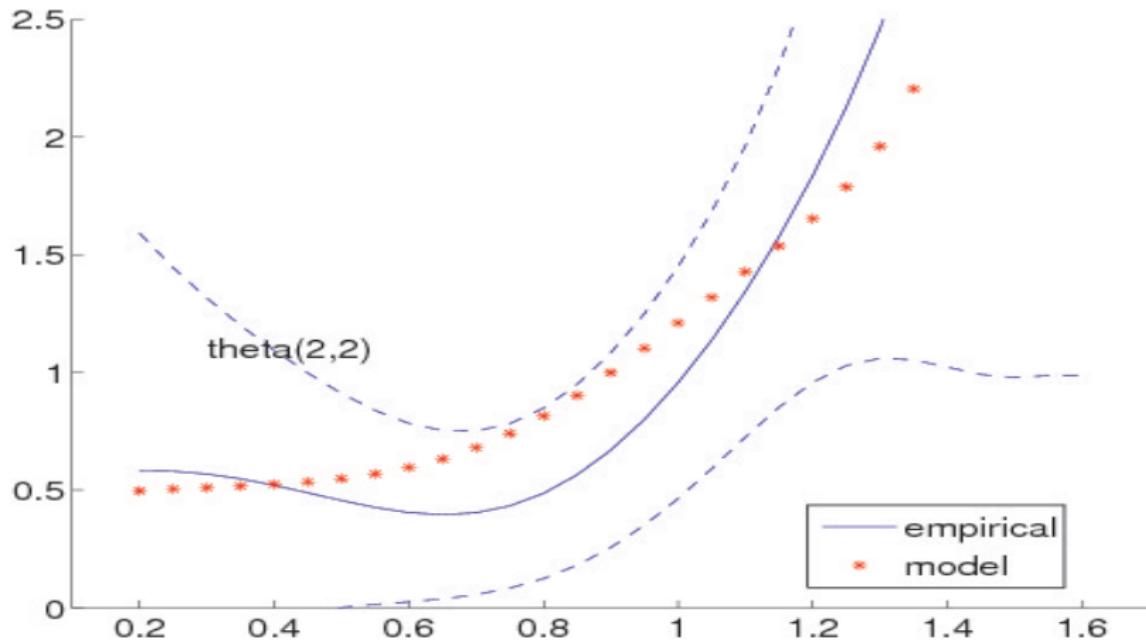
Estimation: cojumps



Estimation: correlation of jumps



Overall fit test: Overidentifying restrictions



Conclusions in a nutshell

- We study nonparametric estimation methods for stochastic volatility models in presence of bivariate jumps
- We provide full asymptotic theory for moment estimators and for the driving functions under recurrence of the data generating process
- We deline a theory for spot volatility estimation
- We provide Monte Carlo evidence of unbiasdness in small samples and we estimate the model on real data.
- We reveal a dependence of leverage effect on volatility, which is compatible with economic theory
- We estimate nonlinear leverage effect, the number of jumps in price and volatility, the number of cojumps and the correlation between jump sizes



Related literature

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- Comte, F., Genon-Catalot, V. and Y. Rozenholc (2007). Nonparametric estimation for a stochastic volatility model. *Finance and Stochastics*, forthcoming
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