The Dynamic Mixed Hitting-Time Model for Multiple Transaction Prices and Times (DMHT)

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Summary

DMHT $\rightarrow$ a Structural Dynamic Model for Durations

- Structural foundations for ACD (Autoregressive Durations Models) and extensions
- Structural because based on a Latent Multivariate Brownian Motion ($\rightarrow$ Extensive set of analytical expressions)
- Joint structural model for durations and prices
- Easy to accommodate multiple assets
- Applications in finance: paired trading, noise traders versus informed traders,…

Outline

1. Introduction
2. Single asset durations
3. Single asset prices
4. Likelihood formulas
5. Multiple assets
6. Preliminary empirics
7. Conclusion

1.Introduction

Problem: how to analyze events that occur at endogenous random times?

Duration models:
Direct modeling of the time elapsed between events
$\rightarrow$ need of a structural approach to:

1. Joint duration/price model with various levels of correlation between prices/durations

$\rightarrow$ can be estimated and interpreted w.r.t. noise traders/insiders

2. Possibility to study remaining durations at intermediate times
3. Multiple assets with multiple causality relationships (Renault-Werker(2010,JoE)) duration/volatility

4. Nesting and specification testing of popular models:

- Engle-Russell (1998) ACD
- Engle (2000) ACD-GARCH
- Engle-Russell (2004) ACM-ACD
- Bauwens-Veredas (2004) SCD
- Ghysels-Gourieroux-Jasiak (2004) SVD

Multiple asset dynamic mixed hitting time model (DMHT)

Observations (up to price discreteness issues):

$$(t_i)_{i=1}^{n} = \text{increasing sequence of random observation times (transaction times, etc.)}$$

$$\rightarrow \text{stopping times for a continuous-time filtration}: (\Im_t)_{t\geq 0}$$

$$Z_{t_i} = (Z_{t_i:k})_{k=1}^{p} = \text{p asset prices at time } t_i$$

$$= \Im_{\mathrm{t_i}} - measurable$$

2. Single asset durations:

Dynamic version of Abbring (2007)

mixed hitting time model

$$t_{i+1} = Inf\{t > t_i : W_t - W_{t_i} + \mu_{t_i}(t - t_i) = c_{t_i}\}$$

$$\mu_{t_i}, c_{t_i} = \Im_{t_i} - \text{measurable and} > 0.$$

$$(W_t) = Wiener\ \ process\ w.r.t.(\Im_t)$$

$$\Delta t_{i+1} = t_{i+1} - t_i \Big| \mathfrak{I}_{t_i} \approx IG\left(\frac{c_{t_i}}{\mu_{t_i}}, c_{t_i}^2 \right)$$

$$Inverse\,Gaussian\; IG(a_{t_i}, b_{t_i})$$

$$\rightarrow Laplace\,transform:$$

$$E_{t_i}\left\{\exp(-u\Delta t_{i+1})\right\} = \exp\left(\frac{b_{t_i}}{a_{t_i}} - \frac{b_{t_i}}{a_{t_i}} \sqrt{1 + 2\frac{a_{t_i}^2 u}{b_{t_i}}} \right)$$

$$= \exp\left(\mu_{t_i} c_{t_i} - \mu_{t_i} c_{t_i} \sqrt{1 + 2\frac{u}{\mu_{t_i}^2}} \right)$$

$$\Delta t_{i+1} = t_{i+1} - t_i \Big| \mathfrak{I}_{t_i} \approx IG\left(\frac{c_{t_i}}{\mu_{t_i}}, c_{t_i}^2 \right) = IG(a_{t_i}, b_{t_i})$$

$$\rightarrow E_{t_i}(\Delta t_{i+1}) = a_{t_i} = \frac{c_{t_i}}{\mu_{t_i}}$$

$$\rightarrow V_{t_i}(\Delta t_{i+1}) = \frac{a_{t_i}^3}{b_{t_i}} = \frac{c_{t_i}}{\mu_{t_i}^3}$$

$$\Delta t_{i+1} \times \lambda \Leftrightarrow a_{t_i}, b_{t_i} \times \lambda \Leftrightarrow c_{t_i} \times \sqrt{\lambda}, \mu_{t_i} \div \sqrt{\lambda}$$

R1. Finite moments because $\mu>0$

$\rightarrow$Alternative way to get finite moments (even with $\mu = 0$) = introducing two boundaries = exit time model instead of hitting time (former version)

$\rightarrow$ less tractable multivariate extensions

R2. Flexible speciation of conditional probability distribution of duration given past information

$\rightarrow$ Introducing mixture components into boundary c and drift μ.

Mixed Hitting Time Model

$\rightarrow$ flexibility on duration distributions:

$$\Delta t_{i+1} = t_{i+1} - t_i \Big| \Im_{t_i}, M_i, N_i \approx IG\left(\frac{\tilde{c}_{t_i}}{\tilde{\mu}_{t_i}}, \tilde{c}_{t_i}^{\,2} \right)$$

$$\tilde{c}_{t_i} = c_{t_i}(M_i), \tilde{\mu}_{t_i} = \mu_{t_i}(N_i)$$

$$M_i, N_i = i.i.d.\, mixing \text{ variables,}$$

$$\Big| (\Im_{t_i}, M_i, N_i) = mutually\ independent$$

$$\tilde{\psi}_{t_i} = E_{t_i}\left[\Delta t_{i+1} \middle| M_i, N_i\right] = \frac{\tilde{c}_{t_i}}{\tilde{\mu}_{t_i}}$$

$$\Rightarrow \frac{\Delta t_{i+1}}{\tilde{\psi}_{t_i}} \middle| \Im_{t_i}, M_i, N_i \approx IG\left(1, \tilde{\mu}_{t_i}\tilde{c}_{t_i}\right)$$

SCD model of Bauwens-Veredas (2004):

$$\frac{\Delta t_{i+1}}{\tilde{\psi}_{t_i}} \perp \Im_{t_i}$$

$$\Leftrightarrow Var_{t_i}\left(\frac{\Delta t_{i+1}}{\tilde{\psi}_{t_i}} \middle| M_i, N_i\right) = \left(\tilde{\mu}_{t_i}\tilde{c}_{t_i}\right)^{-1} \perp \Im_{t_i}$$

ACD model of Engle-Russell (1998)?

$$\frac{\Delta t_{i+1}}{E_{t_i}(\Delta t_{i+1})} = \frac{\Delta t_{i+1}}{\tilde{\psi}_{t_i}} \cdot \frac{\tilde{\psi}_{t_i}}{\psi_{t_i}} = \frac{\Delta t_{i+1}}{\tilde{\psi}_{t_i}} \cdot \frac{\tilde{\mu}_{t_i} \tilde{c}_{t_i}}{\tilde{\mu}_{t_i}^2 \psi_{t_i}} \perp \Im_{t_i}$$

ACD (within SCD) $\Leftrightarrow$ $\tilde{\mu}_{t_i} = \frac{1}{\sqrt{\psi_{t_i}}} N_i$

$$ACD \Leftrightarrow \tilde{\mu}_{t_i} = \mu_{t_i} N_i, \tilde{c}_{t_i} = c_{t_i} M_i, \mu_{t_i} c_{t_i} = 1,$$

$$\Rightarrow \frac{\tilde{\mu}_{t_i}}{E_{t_i}(\tilde{\mu}_{t_i})} \text{ and } \frac{\tilde{c}_{t_i}}{E_{t_i}(\tilde{c}_{t_i})} \perp \Im_{t_i}$$

3. Single Asset Prices

$$Z_t = \log - \mathit{price} \text{ at time } t \in [t_i, t_{i+1}]$$

$$Z_t - Z_{t_i} = \tilde{\nu}_{t_i}(t - t_i)$$

$$+ \tilde{\sigma}_{t_i}\left[\tilde{\rho}_{t_i}(W_t - W_{t_i}) + \sqrt{1 - \tilde{\rho}_{t_i}^2}(B_t - B_{t_i})\right]$$

$$(W_t, B_t) \text{ bivariate } \mathit{Wiener\ w.r.t.} (\mathfrak{I}_t)$$

$$\tilde{\nu}_{t_i}, \tilde{\sigma}_{t_i}, \tilde{\rho}_{t_i} = \mathfrak{I}_{t_i} - \mathit{measurable}$$

$$\mathit{functions\ of\ mixing} \text{ variables } M_i, N_i.$$

As Engle (2000), we characterize the (joint) distribution of future returns given future durations:

- Conditionally on $\Im_{t_i}, M_i, N_i \; and \; \Delta t_{i+1}$

The distribution of the price change $Z_{t_{i+1}} - Z_{t_i}$

is Gaussian with:

$$E_{t_i}\left\{Z_{t_{i+1}} - Z_{t_i} \middle| \Im_{t_i}, M_i, N_i, \Delta t_{i+1}\right\}$$

$$= \tilde{\nu}_{t_i} \Delta t_{i+1} + \tilde{\rho}_{t_i} \tilde{\sigma}_{t_i} (\tilde{c}_{t_i} - \tilde{\mu}_{t_i} \Delta t_{i+1})$$

$$Var_{t_i}\left\{Z_{t_{i+1}} - Z_{t_i} \middle| \Im_{t_i}, M_i, N_i, \Delta t_{i+1}\right\}$$

$$= (1 - \tilde{\rho}_{t_i}^2) \tilde{\sigma}_{t_i}^2 \Delta t_{i+1}$$

R1.Instantaneous causality volatility/duration

(Renault-Werker ,2010): $\rho_{t_i} \neq 0$

Contemporaneous Duration$\downarrow \Leftrightarrow$ Volatility$\uparrow$

$$\tilde{\nu}_{t_i}, \tilde{\rho}_{t_i}, \tilde{\sigma}_{t_i}, \tilde{\mu}_{t_i} = \nu_{t_i}, \rho_{t_i}, \sigma_{t_i}, \mu_{t_i}$$

$$\text{(no mixture component)}, \tilde{c}_{t_i} = c_{t_i} M_i$$

$$E_{t_i}\left\{ Z_{t_{i+1}} - Z_{t_i} \middle| \Delta t_{i+1} \right\} / \Delta t_{i+1}$$

$$= \nu_{t_i} + \rho_{t_i} \sigma_{t_i} \left(\frac{c_{t_i}}{\Delta t_{i+1}} E_{t_i}(M_i | \Delta t_{i+1}) - \mu_{t_i} \right)$$

$$Var_{t_i}\left\{ Z_{t_{i+1}} - Z_{t_i} \middle| \Delta t_{i+1} \right\} / \Delta t_{i+1}$$

$$= (1 - \rho_{t_i}^2) \sigma_{t_i}^2 + \rho_{t_i}^2 \sigma_{t_i}^2 \frac{c_{t_i}^2}{\Delta t_{i+1}^2} Var_{t_i}(M_i | \Delta t_{i+1})$$

R2. Distribution of price change = mixture of normal

$\Rightarrow$ (skewness, kurtosis, etc…)

- Mean, variance = affine functions of Δt

$\Rightarrow$Easy to deduce closed form formulas for the joint Laplace transform from Laplace transform of IG

$$E_{t_i}\left[\exp[-u\Delta t_{i+1} - v(Z_{t_{i+1}} - Z_{t_i})] \middle| M_i, N_i\right]$$

$\Rightarrow$ May be used for GMM inference

4. Likelihood Formulas

- 4.1. Durations:

$$\Delta t_{i+1} = t_{i+1} - t_i \Big| \mathfrak{I}_{t_i}, M_i, N_i \approx IG\left(\frac{\tilde{c}_{t_i}}{\tilde{\mu}_{t_i}}, \tilde{c}_{t_i}^{\,2} \right)$$

$$f_{\mu,c}(t) = \frac{c}{\sqrt{2\pi t^{3/2}}} \exp\left(-\frac{(c-\mu t)^2}{2t} \right)$$

$$\tilde{c}_{t_i} = c_{t_i} M_i, \tilde{\mu}_{t_i} = \mu_{t_i}$$

M_i $i.i.d.$ $mixing$ variables, $half-normal:$

$$f_M(m) = \frac{1}{\sigma_M}\sqrt{\frac{2}{\pi}} \exp\left(-\frac{m^2}{2\sigma_M^2} \right), E(M) = \sigma_M \sqrt{\frac{2}{\pi}}.$$

$$f(\Delta t_{i+1} = t \,|\, \Im_{t_i}) = \int_0^\infty f_{\mu t_i, mct_i}(t)\, f_M(m)\,dm$$

$$\rightarrow closed - form\ formula\ with:$$

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\,dt$$

$$E_{t_i}(\Delta t_{i+1}) = \frac{c_{t_i}}{\mu_{t_i}} E(M_i) \approx AR(1)\ Engle - Russell(1998)$$

$$\frac{c_{t_{i+1}}}{\mu_{t_i+1}} = \beta_0 + \beta_1 \Delta t_{i+1} + \beta_2 \frac{c_{t_i}}{\mu_{t_i}} \Rightarrow auto - correlation:$$

$$\beta_1 E(M_i) + \beta_2 = \beta_1 \sigma_M \sqrt{2/\pi} + \beta_2 < 1.$$

4.2. Price change given current duration:

$$Distribution\ of\ (Z_{t_{i+1}} - Z_{t_i})\big| M_i, \Delta t_{i+1}$$

$$= normal\ with\ mean:$$

$$\nu \Delta t_{i+1} + \rho\sigma(c_{t_i} M_i - \mu_{t_i} \Delta t_{i+1})$$

$$and\ \text{variance}: (1-\rho^2)\sigma^2 \Delta t_{i+1}.$$

$$Bayes\ \ formula\ \ for\ \ f(M_i = m \big| \Delta t_{i+1} = t)$$

$$\rightarrow closed - \ form\ \ formula\ (with\ erf\ \ function)\ for$$

$$Distribution\ of\ (Z_{t_{i+1}} - Z_{t_i})\big| \Delta t_{i+1}$$

Two empirical issues:

1.Intra-day seasonality:

$\rightarrow$Model applied to durations after diurnal adjustment:

- *Hasbrouck (1999)*
- *Andersen, Dobrev,Schaumburg (2008)*

$$\Delta\tilde{t}_{i+1} = \frac{\Delta t_{i+1}}{\phi(t_i)}$$

$$\phi(t_i) = \text{function of } t_{cal}$$

$$\text{and } 3 \text{ unknown parameters } \chi_1, \chi_2, \chi_3.$$

2. Discretely observed prices:

$\rightarrow$We assume that the unobserved fundamental price is in the interval $[0.95,1.05]\times$observed midquote

$\rightarrow$ Ordered Probit model
$\approx$Autoregressive Conditional Multinomial Model in Engle-Russell(2004)

$\rightarrow$ Additional complexity without qualitative difference in estimation results

5. Multiple Assets

- K assets $\rightarrow$ K -dimensional Brownian motions1
  + 1 common Wiener defining times:

$$t_{i+1} = Inf\{t > t_i : W_t - W_{t_i} + \tilde{\mu}_{t_i}(t - t_i) = Min_{1 \le k \le K} \tilde{c}_{t_i:k}\}$$

$$Min_{1 \le k \le K} \tilde{c}_{t_i:k} = \tilde{c}_{t_i:l} \Leftrightarrow Transaction\ on\ asset\ l$$

- Internal consistency property when adding an asset

6. PRELIMINARY EMPIRICS

- PBG (Pepsi Bottling Group) traded on NYSE, February 2008, 20 trading days
- 42858 observed trade-to-trade durations (remove zero durations)
- Average duration : 10.9 seconds
- Transaction date $\rightarrow$ price = midquote (average of bid and ask quotes)
- Half of prices changes = 0
- Price $\in [\$33.82, \$36.37]$

Parameter	$\beta 1$	$\beta 2$	μ	$\sigma(M)$	ρ
Estimate	0.126	0.917	0.409	0.443	0.019
	(0.0074)	(0.0041)	(0.0050)	(0.0314)	(0.0065)

•$P[M<1]=97.6\% \rightarrow$ large fraction of very short durations (while fitting some long durations)

•Persistence: $\beta 1.E(M)+\beta 2$ = 0.962

•SCD/ACD restriction = strongly rejected.

•$\rho>0 \rightarrow$ correlation price/duration

$$E_{t_i}\left\{Z_{t_{i+1}} - Z_{t_i} \middle| \Delta t_{i+1}\right\} / \Delta t_{i+1}$$

$$= \nu_{t_i} + \rho_{t_i}\sigma_{t_i}\frac{\mu_{t_i}}{\Delta t_{i+1}}\left(\frac{c_{t_i}}{\mu_{t_i}}E_{t_i}(M_i \mid \Delta t_{i+1}) - \Delta t_{i+1}\right)$$

$$\rho_{t_i} > 0$$

$\Rightarrow$Expected returns are lower for longer durations:

Engle (2000), Engle-Russell(2004)

Diamond-Verrechia (1987):

Bad news but short selling constraints $\Rightarrow$ no trade

7. CONCLUSION

- Structural version of SCD/ACD model

- Useful theoretical basis for modeling random times between events, including non ACD extensions

- Models the complete distribution of durations, at any point of time

- Joint with multivariate prices models

•Work in progress on mixture models to capture Granger/instantaneous causality relationships between volatility and duration

= parametric complement to:

Renault-Werker (2010) = semiparametric (GMM)

Li-Mykland-Renault-Zhang-Zheng (2009)
= model free (realized variance)