Studying the Leverage Effect Based on High-Frequency Data

Yingying Li

Hong Kong University of Science and Technology

Joint work w/ **Yacine Ait-Sahalia** and **Jianqing Fan**

Leverage Effect

- Leverage Effect
  - Innovations to stock price and volatilities tend to be negatively correlated;

- Leverage Effect
  - Innovations to stock price and volatilities tend to be negatively correlated;
  - *Black (1976), Christie (1982), Nelson (1991), Engle and Ng (1993), Harvey & Shephard (1996), Bouchaud, Matacz & Potters (2001), Tauchen (2004, 2005), Yu (2005), Bollerslev, Litvinova & Tauchen (2006), Ait-Sahalia & Kimmel (2007), Bandi & Roberto (2009), Veraart & Veraart (2009)*$\cdots$

Heston Model

- The Heston Model:

$$dX_t = (\mu - \nu_t/2)dt + \sigma_t dB_t,$$

and

$$d\nu_t = \kappa(\alpha - \nu_t)dt + \gamma\nu_t^{1/2}dW_t,$$

where $\nu_t = \sigma_t^2$ and $Corr(B, W) = \rho$.

Heston Model

- The Heston Model:

$$dX_t = (\mu - \nu_t/2)dt + \sigma_t dB_t,$$

and

$$d\nu_t = \kappa(\alpha - \nu_t)dt + \gamma \nu_t^{1/2} dW_t,$$

where $\nu_t = \sigma_t^2$ and $Corr(B, W) = \rho$.
- Leverage: ρ.

Heston Model

- The Heston Model:

$$dX_t = (\mu - \nu_t/2)dt + \sigma_t dB_t,$$

  and

$$d\nu_t = \kappa(\alpha - \nu_t)dt + \gamma \nu_t^{1/2} dW_t,$$

  where $\nu_t = \sigma_t^2$ and $Corr(B, W) = \rho$.
- Leverage: ρ.
- Studies about ρ:
  - Chernov & Ghysels (2000), [S&P500 1985 - 1993, -0.018]
  - Pan (2002), [S&P500, 1989 - 1996, -0.57]
  - Jones (2003), [S&P100 1988 - 2000 -0.68]
  - Ait-Sahalia, Kimmel (2007),[S&P500 1990-2003, -0.767/-0.754]
  - ...

High Frequency Data

High Frequency Data

- Volatility Estimation

High Frequency Data

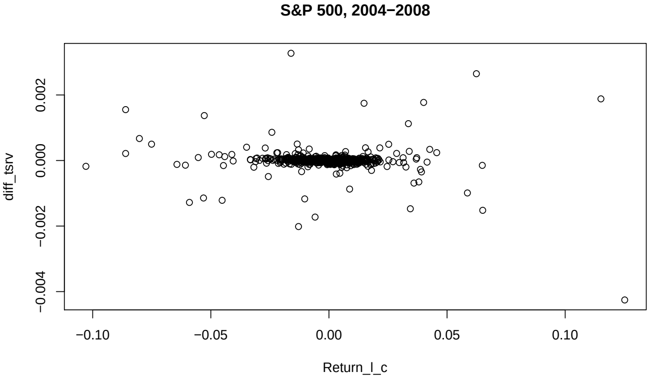
- Volatility Estimation
- Realized Volatiltiy (*Jacod and Protter (1998), Barndorff-Nielsen and Shephard (2002), Mykland and Zhang (2006)*)

High Frequency Data

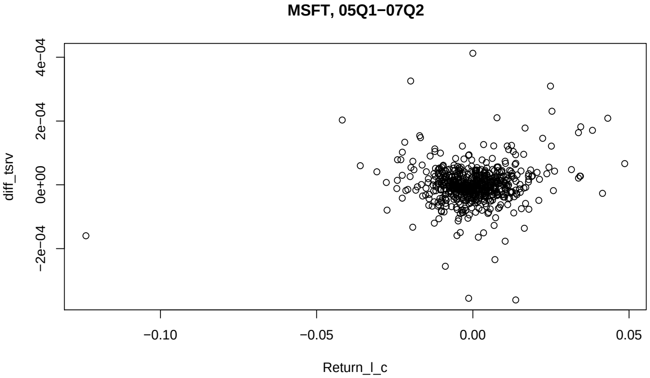
- Volatility Estimation
- Realized Volatiltiy (*Jacod and Protter (1998), Barndorff-Nielsen and Shephard (2002), Mykland and Zhang (2006)*)
- TSRV(*Zhang, Mykland and Ait-Sahalia (2006)*), MLE({ *Ait-Sahalia, Mykland and Zhang (2005),*}) MSRV (*Zhang(2006)*), Realized Kernels (Barndorff-Nielsen, Hansen, Lunde & Shephard (2008)), Pre-averaging Estimator (*Jacod, Li, Mykland, Podolskij & Vetter (2009)*), QMLE(*Xiu (2009)*) $\cdots$

High-Frequency Data, RV and TSRV

- Consider estimation of daily (integrated) volatility, based on observed process from time 0 to time 1(day).
- no market microstructure noise:
  - Observe $X_{t_i}, \quad , i = 0, \cdots, n$;
  - $RV = \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^2$;
  - $n^{1/2}(RV(X) - \int_0^1 \sigma_t^2 dt) \rightarrow_{\text{stably-}\mathcal{L}} \sqrt{2} \int_0^1 \sigma_t^2 dZ_t,$ where Z_t is a BM independent with B_t.
- with market microstructure noise:
  - Observe $Y_{t_i} = X_{t_i} + \epsilon_{t_i}, \quad , i = 0, \cdots, n$ $\epsilon_{t_i} \perp\!\!\!\perp X_{t_i}$, $E(\epsilon_{t_i}) = 0$ and $var(\epsilon_{t_i}) = \eta^2$.
  - $n^{1/6}(TSRV(Y) - \int_0^1 \sigma_t^2 dt) \rightarrow_{\text{s-}\mathcal{L}} \sqrt{\frac{4}{3}} \int_0^1 \sigma_t^2 dZ_t + N(0, 8\eta^4),$ where Z_t is a BM independent with B_t.


S&P 500, 2004−2008
diff_tsrv
0.002
0.000
−0.002
−0.004
−0.10
−0.05
0.00
0.05
0.10
Return_l_c

$$Corr = -0.118$$

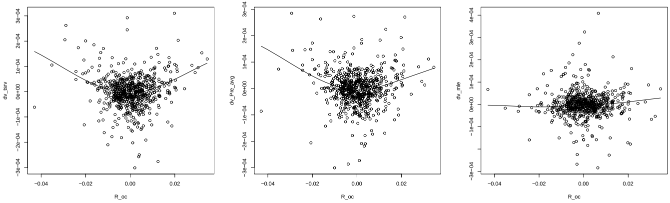

MSFT, 05Q1−07Q2
diff_tsrv
4e−04
2e−04
0e+00
−2e−04
−0.10
−0.05
0.00
0.05
Return_l_c

$Corr = 0.123$

Different Estimators

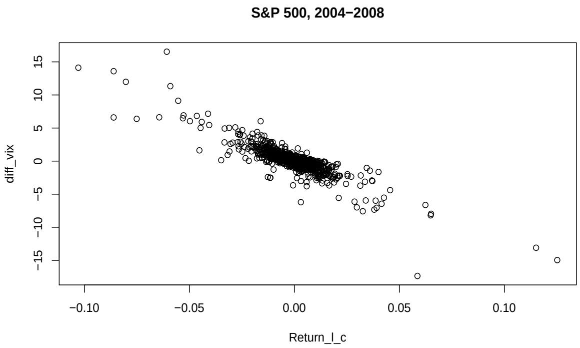
Data: MSFT 05Q1 - 07Q2

- TSRV, (*Zhang, Mykland and Ait-Sahalia (2006)*)
- Pre-Avg, (*Jacod, Li, Mykland, Podolskij & Vetter (2009)*)
- MLE (*Ait-Sahalia, Mykland and Zhang (2005)*)



- S& P 500 Return & VIX:

- S& P 500 Return & VIX:



-

- $Corr = -0.882$

The Heston Model:

$$dX_t = (\mu - \nu_t/2)dt + \sigma_t dB_t,$$

and

$$d\nu_t = \kappa(\alpha - \nu_t)dt + \gamma\nu_t^{1/2}dW_t,$$

where $\nu_t = \sigma_t^2$ and $Corr(B, W) = \rho$.

- $\rho = \lim_{\Delta \to 0} Corr(X_{t+\Delta} - X_t, \sigma_{t+\Delta}^2 - \sigma_t^2)$.

The Heston Model:

$$dX_t = (\mu - \nu_t/2)dt + \sigma_t dB_t,$$

and

$$d\nu_t = \kappa(\alpha - \nu_t)dt + \gamma\nu_t^{1/2} dW_t,$$

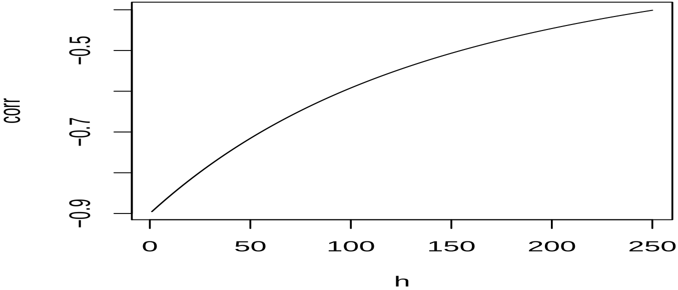
where $\nu_t = \sigma_t^2$ and $Corr(B, W) = \rho$.

- $\rho = \lim_{\Delta \to 0} Corr(X_{t+\Delta} - X_t, \sigma_{t+\Delta}^2 - \sigma_t^2)$.
- RV or TSRV, $\cdots$, estimators of $\int_s^{s+1} \sigma_t^2 dt$ (daily volatility of day s).

Theorem 1

$$Corr(\nu_{h+t} - \nu_t, X_{h+t} - X_t) = f_1(h, \kappa, \gamma, \alpha, \rho),$$

which has the following shape as a function of h.



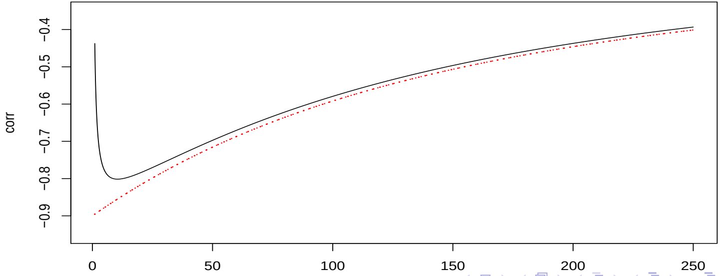
(ρ is taken to be -0.9, $\kappa = 5/252$, $\gamma = 0.05/252$, $\alpha = 0.04/252$; based on the exact corr function *vs* h. $h = 1, 2, \cdots, 250$.)

Theorem 2

Let $V_t = \int_{t-1}^{t} \nu_s ds = \int_{t-1}^{t} \sigma_s^2 ds$, we have,

$$Corr(V_{h+t} - V_t, X_{h+t} - X_t) = f_2(h, \kappa, \gamma, \alpha, \rho),$$

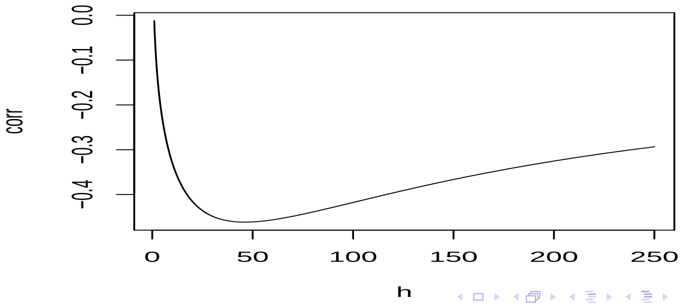
f_2 has the following shape when parameters are assigned to take these values: $\rho = -0.9$, $\kappa = 5/252$, $\gamma = 0.05/252$, $\alpha = 0.04/252$, $h = 1, 2, \cdots, 250$.



Theorem 3. When $\hat{V}$ is estimated by RV based on the latent X_t process,

$$Corr(\hat{V}_{h+t} - \hat{V}_t, X_{h+t} - X_t) = f_3(n, h, \kappa, \gamma, \alpha, \rho) + O(\frac{1}{n^2})$$

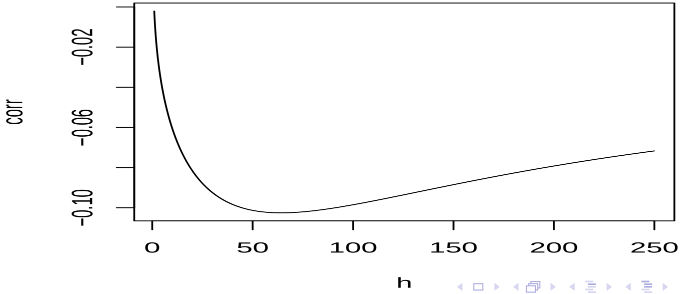
the relationship between f_3 and h is as the following (picture plotted assuming $n = 390$, equivalent to data observed every 1 minute).



Theorem 4. When $\hat{V}$ is estimated by *TSRV* based on the noisy Y_t process,

$$Corr(\hat{V}_{h+t} - \hat{V}_t, X_{h+t} - X_t) = f_4(n, h, \kappa, \gamma, \alpha, \rho) + O(\frac{1}{n^{2/3}})$$

the relationship between f_4 and h is as the following (picture plotted assuming $n = 390$, equivalent to data observed every 1 minute).



Simulation

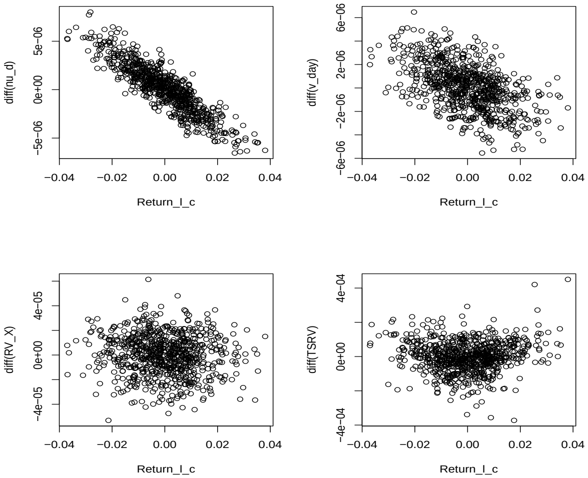
$$dX_t = (\mu - \nu_t/2)dt + \sigma_t dB_t,$$

and

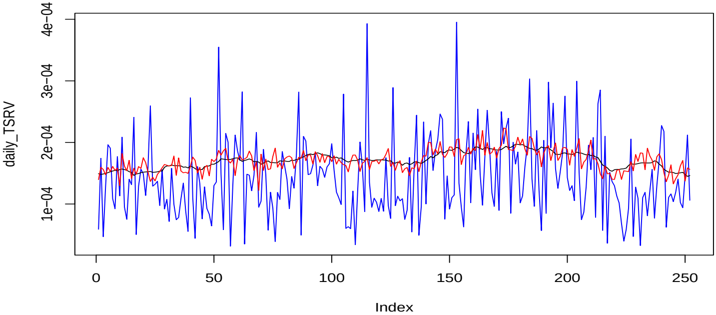
$$d\nu_t = \kappa(\alpha - \nu_t)dt + \gamma\nu_t^{1/2}dW_t,$$

- parameter values used: $\mu = 0.05/252, \kappa = 5/252, \alpha = 0.04/252, \gamma = 0.05/252, \rho = -0.9$.
- Market microstructure error $\epsilon_i^n \sim i.i.d.\ \mathcal{N}(0, 0.0005^2)$.
- Simulated the process from $t = 0$ to $T = 252 * 5$ (5 years' data).

$h = 1$:

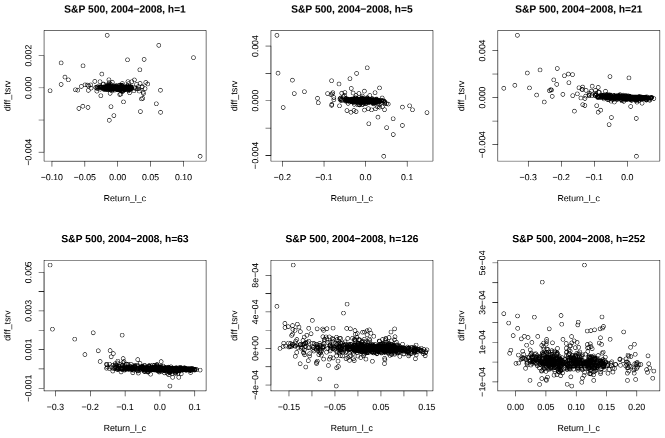


Cor: -0.893, -0.531, -0.087, -0.035



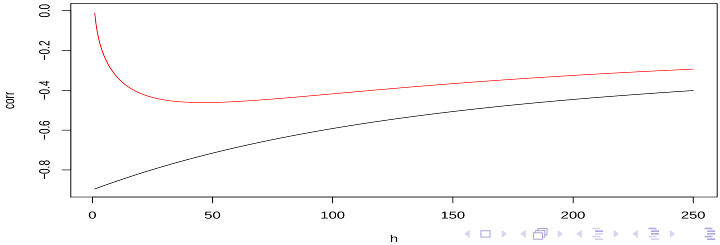
h	$TSRV_Y$	RV_X	V_{int}
1	-0.02649127	-0.0853894	-0.548445
5	-0.04109787	-0.2620538	-0.8247507
21	-0.08755033	-0.4442211	-0.8189264
63	-0.08929981	-0.4870836	-0.7201282
250	-0.06757756	-0.3359186	-0.4712475

S&P 500

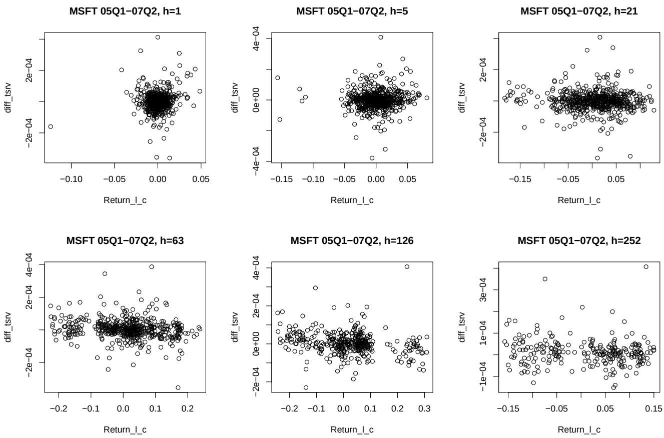

S&P 500, 2004−2008, h=1
S&P 500, 2004−2008, h=5
S&P 500, 2004−2008, h=21
S&P 500, 2004−2008, h=63
S&P 500, 2004−2008, h=126
S&P 500, 2004−2008, h=252
diff_tsrv
Return_l_c

h	TSRV	VIX
1	-0.1177373	-0.8816223
5	-0.4253013	-0.839663
21	-0.4350447	-0.8759314
63	-0.4186851	-0.6899913
126	-0.2623622	-0.3930312
252	-0.1685861	-0.1794003

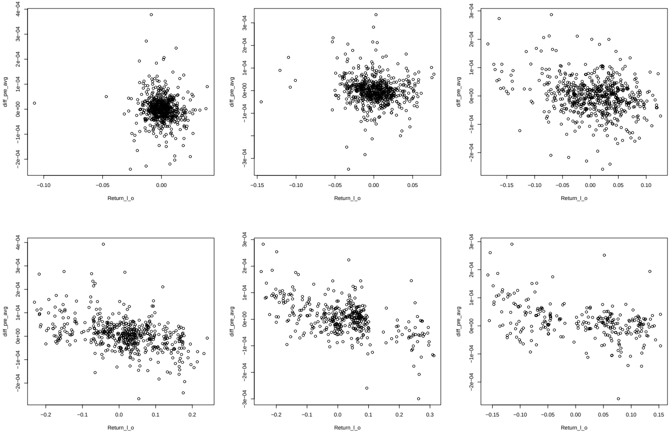
h	TSRV	VIX
1	-0.1177373	-0.8816223
5	-0.4253013	-0.839663
21	-0.4350447	-0.8759314
63	-0.4186851	-0.6899913
126	-0.2623622	-0.3930312
252	-0.1685861	-0.1794003



MSFT 05-07Q2, TSRV



MSFT 05-07Q2, Pre-Averaging



MSFT 05-07Q2

h	TSRV	Pre-Avg
1	0.0701774	-0.050773005
5	0.0003040	-0.1023419
21	-0.1408872	-0.2551622
63	-0.3207878	-0.4333282
126	-0.3904839	-0.5118337
252	-0.2522863	-0.3996102

Summary

- Leverage Puzzle
- Resolution
  - Naive ways are biased
  - Two sources of bias
  - Adding one dimension can be very helpful

Thank You!