# Vast Volatility Matrix Estimation using High Frequency Data for Portfolio Selection

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# About this talk

- How to select sparsely optimal portfolio?
- How to use **high-frequency** data to shorten time horizon?
- How large the universe of assets can be handled?
- How does the estimation of **vast** covariance matrix impact on the allocation vector and portfolio risk?

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# Outline

1. Introduction
2. Portfolio selection with time-varying covariance.
3. Covariance estimation based high-frequency data
4. An empirical study
5. A simulation study

# Introduction

# Markowitz's Mean-variance analysis

**Portfolio allocation**: $\min_{\mathbf{w}^T\mathbf{1}=1, \mathbf{w}^T\mu=r_0} \mathbf{w}^T\Sigma\mathbf{w}$

Solution: $\mathbf{w} = c_1\,\Sigma^{-1}\mu + c_2\,\Sigma^{-1}\mathbf{1}$

- ★ Cornerstone of modern finance.
- ★ Too **sensitive** on input vectors and their estimation errors.
- ★ More severe for large portfolios: 2000 stocks involves **2 m** parameters! Error accumulation can be huge.
- Impact of dimensionality is large:

  Risk: $\mathbf{w}^T\hat{\Sigma}\mathbf{w}$. Allocation: $\hat{c}_1\hat{\Sigma}^{-1}\mathbf{1} + \hat{c}_2\hat{\Sigma}^{-1}\hat{\mu}$.

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# Exposure-constrained portfolio selection

**Portfolio allocation**: (Fan, et al, 08; DeMiguel et al, 08; Bordie et al, 08)

$$\min_{\mathbf{w}^T\mathbf{1}=1,\ \ \mathbf{A}\mathbf{w}=\mathbf{a}} \mathbf{w}^T\Sigma\mathbf{w}, \qquad \|\mathbf{w}\|_1 \le \mathbf{c}.$$

**Constraints**:

- expected return or sector exposures via **A**.
- **short positions**: $\mathbf{w}^- \le (c-1)/2$,

  since $w^+ + w^- \le c, \quad w^+ - w^- = 1.$

  $c = 1 \Longrightarrow$ no short-sale; $c = \infty \Longrightarrow$ Markowitz problem.

**Portfolio selection**: solution is usually sparse.

**Applicability**: Any coherent risk measures (Artzner et al, 1999)

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# Utility Approximations

Utility Approx.: Let $M(\mu,\Sigma) = \mathbf{w}^T\mu - \lambda\mathbf{w}^T\Sigma\mathbf{w}$ be expected utility.

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- No noise accumulation effect for moderate $c \le 3$, say.
- applicable to any number of assets $p$

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# Risk Approximation Theory

**Actual and Empirical risks**: $R(\mathbf{w}) = \mathbf{w}^T \Sigma \mathbf{w}, \qquad R_n(\mathbf{w}) = \mathbf{w}^T \hat{\Sigma} \mathbf{w}.$

■Theoretical and empirical allocation vector:

$$\mathbf{w}_{opt} = \mathrm{argmin}_{||\mathbf{w}||_1 \le c} R(\mathbf{w}), \qquad \hat{\mathbf{w}}_{opt} = \mathrm{argmin}_{||\mathbf{w}||_1 \le c} \mathrm{R}_n(\mathbf{w})$$

■Risks: $\sqrt{R(\mathbf{w}_{opt})}$ —oracle, $\sqrt{R_n(\hat{\mathbf{w}}_{opt})}$ —empirical; $\sqrt{R(\hat{\mathbf{w}}_{opt})}$ —actual risk of a selected portfolio.

**Theorem 1**: Let $a_n = |\hat{\Sigma} - \Sigma|_\infty$. Then, we have

$$\begin{aligned} |R(\hat{\mathbf{w}}_{opt}) - R(\mathbf{w}_{opt})| &\le 2a_n c^2 \\ |R(\hat{\mathbf{w}}_{opt}) - R_n(\hat{\mathbf{w}}_{opt})| &\le a_n c^2 \\ |R(\mathbf{w}_{opt}) - R_n(\hat{\mathbf{w}}_{opt})| &\le a_n c^2. \end{aligned}$$

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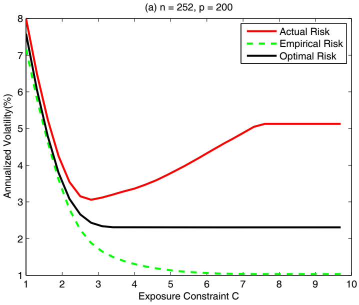
# Impact of dimensionality

## Actual vs Empirical risks

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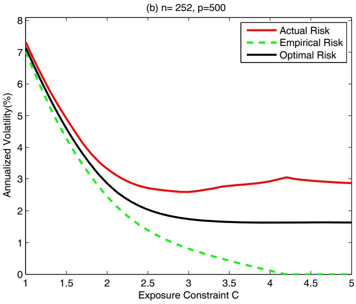


**Theorem 2**: If $\max_{i,j} P\{\sqrt{n}|\sigma_{ij} - \hat{\sigma}_{ij}| > x\} < \exp(-\mathbf{C}\mathbf{x}^{1/a})$ for large $x$,

$$|\Sigma - \hat{\Sigma}|_\infty = O_P\left(\frac{(\log \mathbf{p})^{\mathbf{a}}}{\sqrt{\mathbf{n}}}\right).$$

- Impact of dimensionality is limited.
- on inverse of tail.

# Portfolio Selection with dynamic covariance

# Time-dependent volatility matrix

**Return and Risk** with holding period $\tau$:

$$\textbf{Return} = \mathbf{w}^T \mathbf{R}_{t,\tau} = \mathbf{w}^T \int_t^{t+\tau} d\mathbf{X}_s, \qquad \textbf{risk} = \mathbf{w}^T \Sigma_{t,\tau} \mathbf{w},$$

where $\Sigma_{t,\tau} = E_t \int_t^{t+\tau} \mathbf{S}_u du$, allowing **stochastic** volatility and $\mathbf{S}_u = \left( \sigma_u^{(i)} \sigma_u^{(j)} \rho_u^{(i,j)} \right)$ is instantaneous cov matrix.

**Portfolio allocation and selection**:

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# Prediction of Covariance Matrix

Covariance matrix is predicted based on following approximations:

**short-horizon $\tau$** : $\frac{1}{\tau}\Sigma_{t,\tau} \approx \frac{1}{h}\int_{t-h}^{t} \mathbf{S}_u du$ (use of continuity)

**long-horizon $\tau$** : $\frac{1}{\tau}\Sigma_{t,\tau} \approx \frac{1}{h}E\int_{t-h}^{t} \mathbf{S}_u du$ (use of ergoticity)

- Even with observed $\mathbf{S}_u$ in the past, $\Sigma_{t,\tau}$ is at best approximated.
- Important to reduce the sensitivity of $\mathbf{w}$ on the prediction of $\Sigma_{t,\tau}$.
- Gross-exposure constraint is an effective method.

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# High- and low-frequency data

**Low frequency Data**: Daily data w/ $h = 252$ or $h = 512$ days.

- Estimated is the expected covariance matrix from $[t-h, t]$.
- Can be very different from $\Sigma_{t,\tau}$ next day or week.
- Not applicable to **short** holding period.
- Applicable to long holding period only when stationary.

**Use of high-frequency data**:

- ★ More data available for estimating covariance matrix
- ★ Shorten the time interval, reducing approximation errors
- ★ Adapts better local correlation.
- ★ Applicable to both **long- and short-term** holding periods

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# Covariance Estimation

# Using High-Frequency Data

# Style features

- Microstructure noise (Aït-Sahalia, Mykland, Zhang, RFS, 05);
- Nonsynchronized trading (Barndorff-Nielsen, Hansen, Lunde and Shephard, EconJ,08);
- Jumps in the data (Fan and Wang, 07; BNS, 04, 06, JFEC);
- Data cleaning (BNHLS, EconJ, 09)

# Integrated volatility: Diagonal elements

**Model**: $Y_{t_i} = X_{t_i} + \varepsilon_{t_i}$, $X_{t_i}$ —latent log-price, $\eta^2 = \mathrm{var}(\varepsilon)$

- Two-scale and Multi-scaled realized volatility. (AMZ, 05; Zhang, 07)
- Realized kernel method (BNHLS, JFEC 09, JEcon, 09)
- Wavelets (Fan and Wang, 07) and Bipower (BNS, 04, 06, JFEC)
- Quasi-MLE (Xiu, 09)
- Pre-averaging (smoothing) (Jacod, Li, Mykland, Podolskij, Vetter, 09).

# Sub-sampling

**Subsampling**: Use once every $K$ points

$$
\begin{aligned}
RV_{K,i} &= \sum_{j=1}^{n_s} (Y_{t_{i+jK}} - Y_{t_{i+(j-1)K}})^2, \qquad n_s = n/K, \quad \Theta = \int_{t-h}^{t} \sigma_u^2 du. \\
&= \Theta + 2n_s\eta^2 + \left[4n_s E\varepsilon^4 + \frac{2}{n_s}\int \sigma_t^4 dt\right]^{1/2} \cdot N(0,1),
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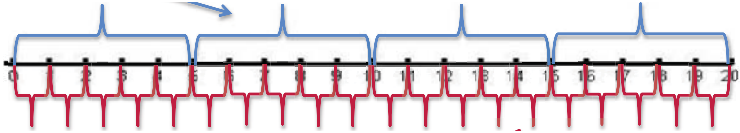


**Averaging** : 
$$
\begin{aligned}
[Y]^{(K)} &= \frac{1}{K}\sum_{i=0}^{K-1} R_{K,i} = \frac{1}{K}\sum_{i=1}^{n-K} (Y_{t_i+K} - Y_{t_i})^2 \\
&\approx \Theta + 2n_s\eta^2 + \left[\frac{4n_s}{K} E\varepsilon^4 + \frac{4}{3n_s}\int \sigma_t^4 dt\right]^{1/2} \cdot N(0,1)
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# Two-scale Realized Volatility

**TSRV**: $[Y]^{(K)} - [Y]^{(1)}/K \cdot \frac{n-K+1}{n}$

**Asymptotic normality** (AMZ, 05): with optimal choice $K = cn^{2/3}$,

$$n^{1/6}(TSRV - \Theta) \to \left[8c^{-2}\eta^4 + c\frac{4}{3}\int \sigma_t^4 dt\right]^{1/2} \cdot N(0,1).$$

**Theorem 3** (Concentration inequality): For large $x$ that satisfies $|x| \leq cn^{1/6}$,

$$P\{n^{1/6}|TSRV - \Theta| > x\} \leq 3\exp\{-Cx^2\}$$

- By Thm 2, diagonals be estimated uniformly with rate $O(\frac{(\log p)^{1/2}}{n_{\min}^{1/6}})$.

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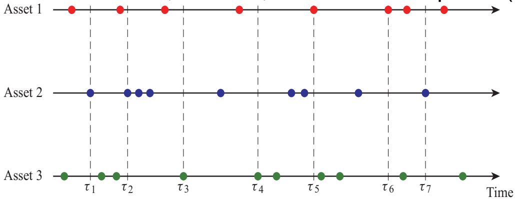
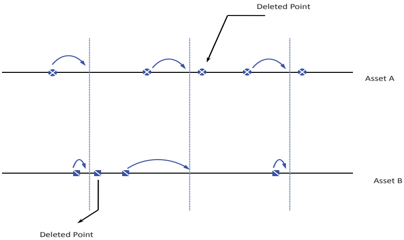
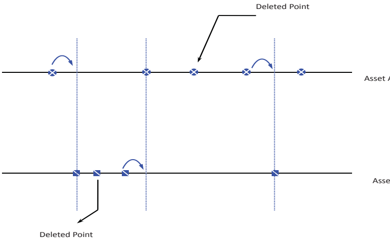
# Data Synchronization

**Refresh time**: Barndorff-Nielsen, Hansen, Lunde and Shephard (2008)




**Previous ticks and its generalization**: $\{\tau_i - \tau_{i-1}\}$ are i.i.d. $O_P(n^{-1})$, and at least 1 data for **each asset** in $(\tau_{i-1}, \tau_i]$.



# Estimation of integrated covariance

1. Two-Scale Realized Covariance (Zhang, 09):

$$\text{TSCV} = [Y_1, Y_2]^{(K)} - [Y_1, Y_2]^{(1)}/K \cdot \frac{\tilde{n} - K + 1}{\tilde{n}},$$

   where $\tilde{n}$ is no of synchronized data, and

   $[Y_1, Y_2]^{(K)} = \frac{1}{K}\sum_{i=K}^{\tilde{n}}(Y_{1,t_i} - Y_{1,t_{i-K}})(Y_{2,t_i} - Y_{2,t_{i-K}})$, **subsam cov**

2. Realized Covariance(BNHLS, 08): log-return $\mathbf{y}_t$

$$K(X) = \sum_{h=-H}^{H} k\left(\frac{h}{H+1}\right)\Gamma_h, \qquad \Gamma(h) = \sum_{j=|h|+1}^{n} \mathbf{y}_j \mathbf{y}'_{j-|h|}$$

3. QMLE (Aït-Sahalia, Fan and Xiu, 2010)

$$\widehat{\langle Y_1, Y_2 \rangle} = \frac{1}{4}\{\widehat{\langle Y_1 + Y_2, Y_1 + Y_2 \rangle}_{QMLE} - \widehat{\langle Y_1 - Y_2, Y_1 - Y_2 \rangle}_{QMLE}\}$$

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# A concentration inequality for TSCV

**Theorem 4**. For large $x$ that satisfies $|x| \leq c\tilde{n}^{1/6}$,

$$P\{\tilde{n}^{1/6}|\text{TSCV} - \int_0^1 \sigma_t^{Y_1}\sigma_t^{Y_2}\rho_t^{(Y_1,Y_2)}dt| > x\} \leq 3\exp\{-Cx^2\}.$$

**Conditions**:

1. **Log-price**: $dX_t^{(i)} = \sigma_t^{(i)} dB_t^{(i)}$ with $cor(B_t^{(i)}, B_t^{(j)}) = \rho_t^{(i,j)}$.
2. **Volatility**: $|\sigma_t^{(i)}| < C_\sigma$.
3. **Refresh time**: $\sup_j |\tau_j - \tau_{j-1}| \leq C_\Delta / n_1$
4. **Noise**: $\{\varepsilon_{t_j}^{Y_i}\}$ are independent, also independent of $X^{(i)}$.

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2. **Volatility**: $|\sigma_t^{(i)}| < C_\sigma$.
3. **Refresh time**: $\sup_j |\tau_j - \tau_{j-1}| \leq C_\Delta / n_1$
4. **Noise**: $\{\varepsilon_{t_j}^{Y_i}\}$ are independent, also independent of $X^{(i)}$.

# Applications to Portfolio Allocation

# Portfolio Optimization

**Portfolio allocation**: $\min_{\mathbf{w}^T\mathbf{1}=1,\ \ \|\mathbf{w}\|_1\le c} \mathbf{w}^T\widehat{\Sigma}\mathbf{w}$. The actual risk is no larger than $2|\hat{\Sigma}-\Sigma|_\infty c^2$ away from the oracle.

**Estimation of Covariance**

1. **Pairwise refresh**: Componentwise estimation, far more data, but $\hat{\Sigma}$ is **not** semi-positive:

$$|\hat{\Sigma}-\Sigma|_\infty = O\left(\frac{\sqrt{\log p}}{\bar{n}^{1/6}}\right), \qquad \bar{n} = \min_{i,j} n_{i,j}.$$

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# Projection of symmetric matrices

**Need of projection**: Programming algorithms require $\widehat{\Sigma} \geq 0$.

**Projection 1**: $\mathbf{A}_1^+ = \Gamma^T \text{diag}(\lambda_1^+, \cdots, \lambda_n^+)\Gamma$, for a symmetric matrix with SVD $\mathbf{A} = \Gamma^T \text{diag}(\lambda_1, \cdots, \lambda_n)\Gamma$.

**Projection 2**: $\mathbf{A}_2^+ = (\mathbf{A} - \lambda_{\min}^- I_p)/(1 - \lambda_{\min}^-)$, where $\lambda_{\min}^-$ is the negative part of the minimum eigenvalue.

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1. It appears projections **distort** more "pairwise refresh" method than "all refresh". Thus, the smaller componentwise estimation errors might not be materialized in implementation.

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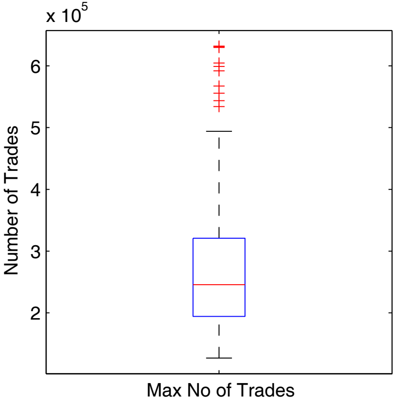
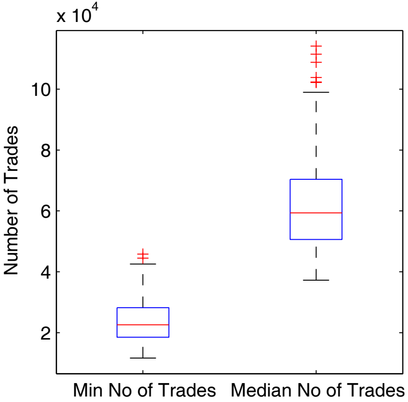
# An Empirical Study

# An empirical testing

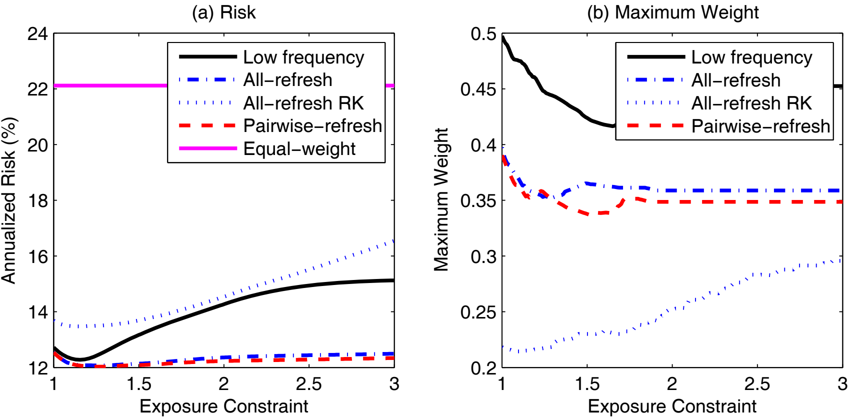
- 30 stocks from DJ Industrial components from 1/2/08–9/30/08 (Total trade: 2,307,004. Average trading: 76,900. Size: 13G)
- Holding period: $\tau = 1$ or 5 days and rebalanced
- testing period: 5/27/08 – 9/30/08 (90 days)
- Risk profile: Use **15 minutes returns** (total $26 * 90 = 2340$ returns), excluding overnight holding risks.
- High frequency $h = 10$ days; low frequency $h = 100$ days

# Summary of Trading Frequencies

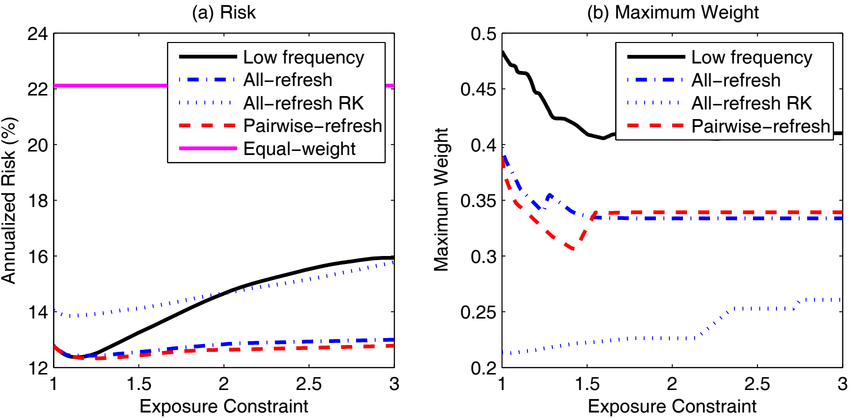


# An empirical result ($\tau = 1$)



# An empirical result ($\tau = 5$)



# A Simulation Study

# Stochastic models

**Log-prices** of $p$-stocks follow the **one-factor** model ($X_0^{(i)} = 1$):

$$dX_t^{(i)} = \mu^{(i)} dt + \rho^{(i)} \sigma_t^{(i)} dB_t^{(i)} + \sqrt{1 - (\rho^{(i)})^2} \sigma_t^{(i)} dW_t + \lambda^{(i)} dZ_t^{(i)},$$

the synchronized data highest freq (second) **—latent** (oracle) price.

**Stochastic volatility**: $\eta_t^{(i)} = \log \sigma_t^{(i)}$ follows Vasicek model (OU):

$$d\eta_t^{(i)} = \alpha^{(i)}(\beta_0^{(i)} - \eta_t^{(i)})dt + \beta_1^{(1)} dB_t^{(i)}.$$

**Choice of parameter**: $\rho^{(i)} = -0.7$, $\lambda^{(i)} = \exp(\beta_0^{(i)})$,

$$(\mu^{(i)}, \beta_0^{(i)}, \beta_1^{(i)}, \alpha^{(i)}) = (0.03, -1, .75, 1/40) \otimes \mathbf{U}^{(i)},$$

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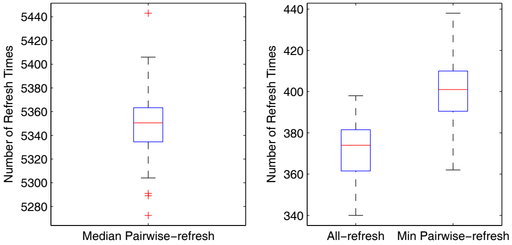
# Stochastic model (II)

**Trading frequency**: Poisson process with $\lambda_i = 0.02i \times 23400$

—no. of seconds / day.

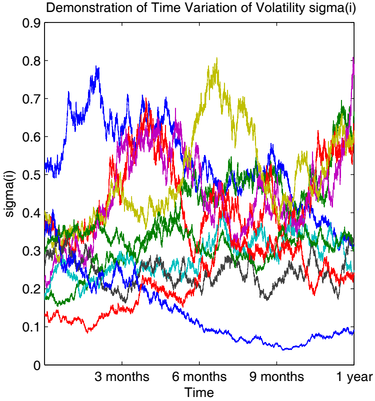
**Size of investment universe**: $p = 50$.

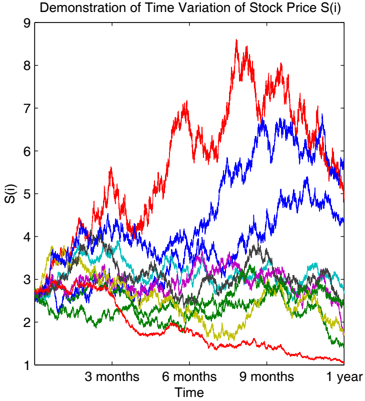



| all-fresh | pairwise-refresh | Ave of min pairwise-refresh |
|---|---|---|
| 375 | 5355 | 410 |

**Microstructural noise**: $Y_{t_{ij}}^{(i)} = X_{t_{ij}}^{(i)} + \mathbf{N(0, 0.0005^2)}$.

# Examples of realized volatilities and prices




- Varying volatility, but relatively calm.

# Risk approximation: In-sample evaluation

**Specific portfolios**: $w_1$ —equal weight, $w_2 = (1, 0, \cdots, 0)^T$,

$$w_3 = (1 + 2/p, -1, 1/p, \cdots, 1/p)^T, \qquad w_3 = (2, -1, 0, \cdots, 0)^T$$

**Evaluation**: Regard risk estimated by Latent price as the true risk.

Median and Robust Standard Deviation (RSD) of Risk

| | Latent | All-refresh TSRV | All-refresh RK | Pairwise TSRV |
|---|---|---|---|---|
| Portfolio | Median(RSD) | Median(RSD) | Median(RSD) | Median(RSD) |
| $w_1$ | 0.440(0.0032) | 0.387 (0.107) | 0.434 (0.024) | 0.419 (0.069) |
| $w_2$ | 0.591(0.0060) | 0.522 (0.125) | 0.623 (0.025) | 0.593 (0.128) |
| $w_3$ | 0.539(0.0044) | 0.469 (0.090) | 0.583 (0.025) | 0.520 (0.073) |
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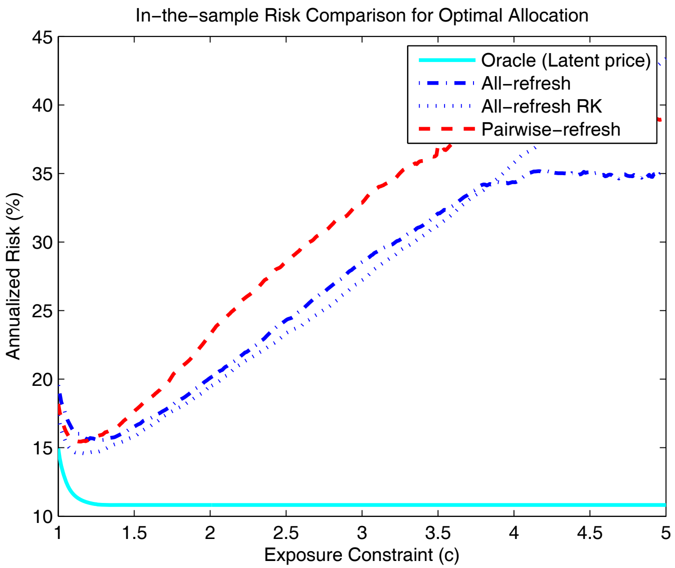
# Risk approximation error

| Median and RSD of Absolute Risk Difference from the Oracle (Latent) | | | |
|---|---|---|---|
| | All-refresh TSRV | All-refresh RK | Pairwise TSRV |
| Portfolio | Median(RSD) | Median(RSD) | Median(RSD) |
| $w_1$ | 0.0889 (0.0769) | 0.0183 (0.0153) | 0.0547 (0.0439) |
| $w_2$ | 0.1054 (0.0700) | 0.0344 (0.0272) | 0.0804 (0.0813) |
| $w_3$ | 0.0936 (0.0665) | 0.0437 (0.0300) | 0.0599 (0.0593) |
| $w_4$ | 0.1470 (0.1022) | 0.0794 (0.0393) | 0.1089 (0.0941) |
| **Median and RSD of $L_1$ Norm of Absolute Covariance Difference ($a_p$)** | | | |
| | All-refresh TSRV | All-refresh RK | Pairwise TSRV |
| Portfolio | Median(RSD) | Median(RSD) | Median(RSD) |
| | 0.2476 (0.1460) | 0.0603 (0.0270) | 0.1730 (0.0746) |

# Evaluation of portfolio allocation: In-sample risk ($\tau = 1$)



# Out-sample evaluation

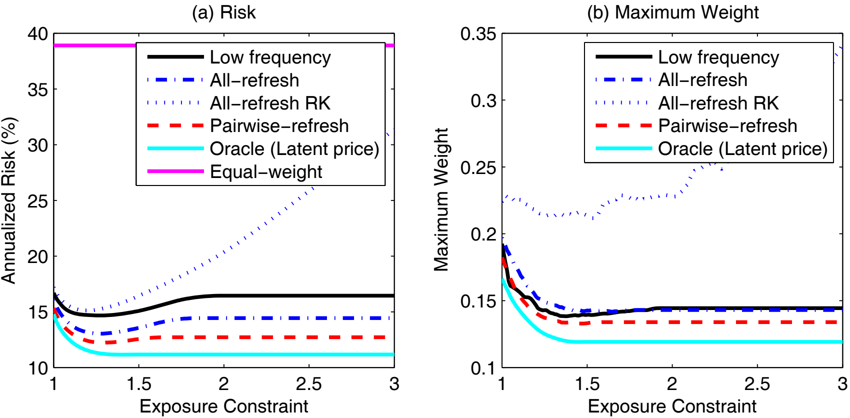
**Data**: Simulate 100 days high frequency data.

- Low-freq: past 100 days data;
- High-freq: past 10-day data

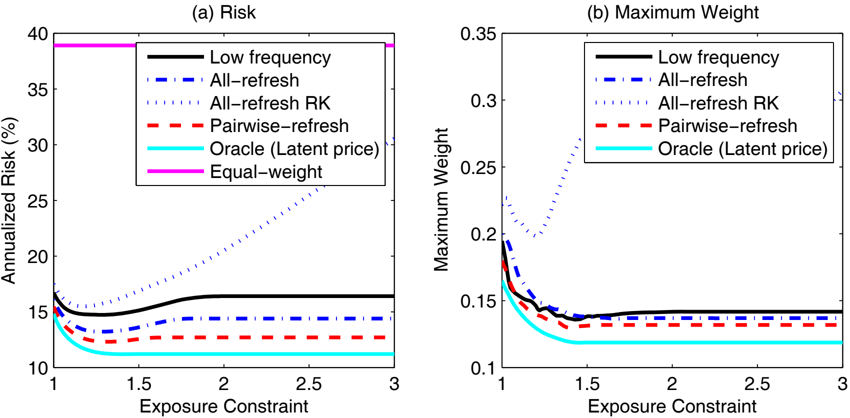
**Holding period**: holding period $\tau = 1$ or 5-days, rebalanced.

**Risk evaluation**: 15 minutes returns over 100 days (2600 returns).

# Out of sample performance ($\tau = 1$)



# Out of sample performance ($\tau = 5$)



# Conclusion

- Advocate portfolio selection with gross-exposure constraint.
- It is less sensitive to error of covariance estimation, and has little noise accumulation.
- Propose "all-fresh" and "pair-fresh" to estimates integrated covariance, derive the concentration inequalities, and demonstrate limited impact of portfolio size.
- Use of HF-data increases $n$, shortens time window, adapts to local covariation.
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# The End

*Thank*  *You*