The backward bifurcation in compartmental models for West Nile Virus

Hui Wan

LAMPS

Department of Mathematics and Statistics

York University

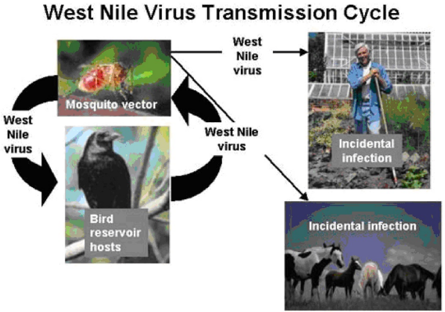
Joint work with: Huaiping Zhu

Outline

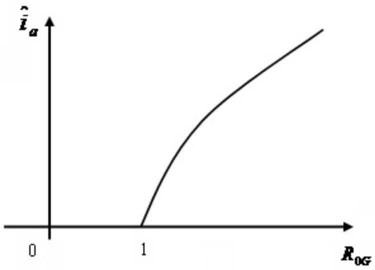
- Introduction
- Review and comparison
- Dynamical analysis of Cruz-Pacheco's Model
- Simulations
- Conclusion

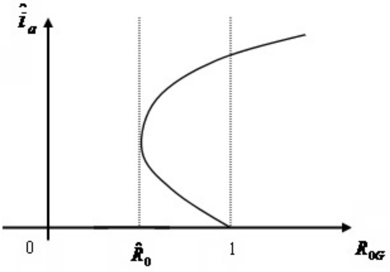
West Nile virus:

- West Nile virus (WNV) is a mosquito-borne disease.
- The life cycle of the virus circulates between mosquitoes and birds.
- The virus is transmitted to humans by mosquitoes, but, cannot be transmitted back to mosquitoes due to the low concentration of the virus in human body.



Forward bifurcation and Backward bifurcation:





Review and Comparison

Model	Mosquitos	Birds	Backward bifurcation
Wonham et al. 2004	Constant	Constant	No
Bowman et al. 2005	Logistic	Logistic	Yes [7]
Lord and Day 2001	Logistic	Logistic	?
Cruz-Pacheco et al. 2005	Constant	Logistic	?

Question:

- If the backward bifurcation can occur for the model of Cruz-Pacheco et al. and Lord and Day ?
- When will the backward bifurcation happen in the compartmental models for WNV ?

State variables:

State variables	Mosquito	Bird
Susceptible	S_V	S_R
Infectious	I_V	I_R
Recovered		R_R
Total adults	N_V	N_R

Parameters:

Parameters in the models	Vector	Reservoir
Per capita birth rate	b_V	
Recruitment rate		Λ_R
Proportion of births that are infected	p	
Natural death rate	μ_V	μ_R
Disease-induced death rate		α_R
Recovery rate		γ_R
Biting rate	b	
Transmission probability (from vectors to birds)		β_R
Transmission probability (from birds to vectors)	β_V	

Model:

$$
\begin{cases}
\dfrac{dS_R}{dt} = \Lambda_R - \dfrac{b\beta_R}{N_R} I_V S_R - \mu_R S_R, \\
\dfrac{dI_R}{dt} = \dfrac{b\beta_R}{N_R} I_V S_R - (\gamma_R + \mu_R + \alpha_R) I_R, \\
\dfrac{dR_R}{dt} = \gamma_R I_R - \mu_R R_R, \\
\dfrac{dS_V}{dt} = \mu_V S_V + (1-p)\mu_V I_V - b\beta_V \dfrac{I_R}{N_R} S_V - \mu_V S_V, \\
\dfrac{dI_V}{dt} = p\mu_V I_V + b\beta_V \dfrac{I_R}{N_R} S_V - \mu_V I_V,
\end{cases} \tag{1}
$$

where $S_V + I_V = N_V$ and $S_R + I_R + R_R = N_R$ are the total number of mosquitoes and birds respectively.

Variable change:

$$
\begin{aligned}
s_a &= \frac{S_R}{\Lambda_R/\mu_R}, \quad i_a = \frac{I_R}{\Lambda_R/\mu_R}, \\
r_a &= \frac{R_R}{\Lambda_R/\mu_R}, \quad n_a = \frac{N_R}{\Lambda_R/\mu_R}, \\
s_v &= \frac{S_V}{N_V}, \qquad i_v = \frac{I_V}{N_V}.
\end{aligned} \tag{2}
$$

Dynamical analysis of the Cruz-Pacheco's Model

$$
\begin{cases}
\dfrac{ds_a}{dt} = \mu_R - \dfrac{b\beta_R m}{n_a} i_v s_a - \mu_R s_a, \\
\dfrac{di_a}{dt} = \dfrac{b\beta_R m}{n_a} i_v s_a - (\gamma_R + \mu_R + \alpha_R) i_a, \\
\dfrac{di_v}{dt} = \dfrac{b\beta_V}{n_a} i_a (1 - i_v) - (1 - p)\mu_V i_v, \\
\dfrac{dn_a}{dt} = \mu_R - \mu_R n_a - \alpha_R i_a.
\end{cases}
\tag{3}
$$

$m = \frac{N_V}{\Lambda_R/\mu_R}$: the ratio of the number of mosquitoes and birds at the disease-free equilibrium.

Existence of equilibria:

The disease-free equilibrium $P_0(1, 0, 0, 1)$ always exists.

For any positive equilibrium $\hat{P}(\hat{s}_a, \hat{i}_a, \hat{i}_v, \hat{n}_a)$, its coordinates satisfy the following relations

$$\hat{s}_a = \frac{\mu_R - (\gamma_R + \mu_R + \alpha_R)\hat{i}_a}{\mu_R}, \tag{4}$$

$$\hat{n}_a = \frac{\mu_R - \alpha_R \hat{i}_a}{\mu_R}, \tag{5}$$

$$\hat{i}_v = \frac{\mu_R b \beta_V \hat{i}_a}{(b\beta_V \mu_R - \alpha_R(1-p)\mu_V)\hat{i}_a + (1-p)\mu_V \mu_R}. \tag{6}$$

Dynamical analysis of the Cruz-Pacheco's Model

i_a-coordinate must be a positive root of the quadratic polynomial

$$\Gamma(i_a) = Ai_a^2 + Bi_a + C, \tag{7}$$

in the interval $\hat{i}_a \in \left(0, \frac{\mu_R}{\mu_R+\gamma_R+\alpha_R}\right)$, where

$$\begin{aligned} A &= [b\beta_V\mu_R - \alpha_R(1-p)\mu_V]\frac{\alpha_R}{\mu_R}, && (8) \\ B &= 2\alpha_R(1-p)\mu_V - b\beta_V\mu_R - mb^2\beta_R\beta_V, && (9) \\ C &= \mu_R(1-p)\mu_V(\mathbb{R}_{0G}^2 - 1), && (10) \end{aligned}$$

and $\mathbb{R}_{0G}$ is the basic reproduction number.

Dynamical analysis of the Cruz-Pacheco's Model

Theorem

For existence of equilibria of the model (3), we have the following:

1. *The boundary equilibrium, the disease free equilibrium $P_0(1,0,0,1)$ always exists.*
2. *If $\mathbb{R}_{0G} > 1$, there exists a unique positive equilibrium P^*.*
3. *If $\mathbb{R}_{0G} = 1$, then there exists a positive equilibrium provided $A < 0$ and $B > 0$, otherwise there is no positive equilibrium.*
4. *If $\mathbb{R}_{0G} < 1$ and*
    1. *if $A \geqslant 0$, there is no positive equilibrium;*
    2. *if $A < 0$, the system (3) has two positive equilibria P_1 and P_2 if and only if $\Delta > 0$ and $0 < \frac{-B}{2A} < \frac{\mu_R}{\mu_R+\gamma_R+\alpha_R}$. And these two equilibria coalesce if and only if $0 < \frac{-B}{2A} < \frac{\mu_R}{\mu_R+\gamma_R+\alpha_R}$ and $\Delta = 0$; otherwise there is no positive equilibrium.*

Stability of equilibria:

Theorem

The disease-free equilibrium P_0 is stable iff $\mathbb{R}_{0G} < 1$.
For positive equilibria,

1. *if $\mathbb{R}_{0G} \geq 1$, then the unique positive equilibrium P^* is locally asymptotically stable.*
2. *if $\mathbb{R}_{0G} < 1$ and conditions*

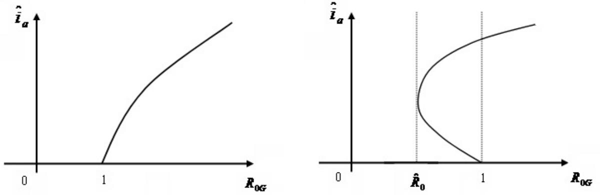
$$A < 0, \quad 0 < -\frac{B}{2A} < \frac{\mu_R}{\mu_R + \gamma_R + \alpha_R}, \quad \textit{and} \quad \Delta = B^2 - 4AC \geqslant 0. \tag{11}$$

are satisfied, then the positive equilibrium P_1 is a saddle point and P_2 is locally asymptotically stable.

Backward bifurcation:

When $\mathbb{R}_{0G} < 1$, (α_R: disease-induced death rate, γ_R: recovery rate)

- with the assumption $\alpha_R \geq \gamma_R$, model (1) has no positive equilibrium which implies the forward bifurcation occurs at $\mathbb{R}_{0G} = 1$ [4].
- with the assumption $\alpha_R > \gamma_R$, by our analysis, there can be at most two positive equilibria which implies the backward bifurcation occurs at $\mathbb{R}_{0G} = 1$.



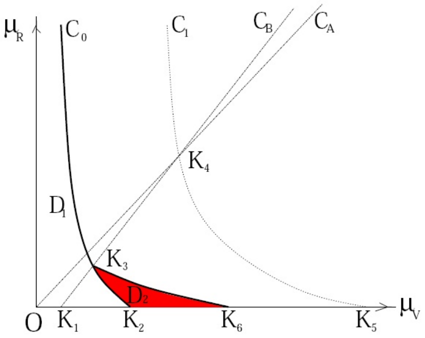
Backward bifurcation diagram:

We describe the backward bifurcation condition

$$\mathbb{R}_{0G} < 1, \quad A < 0, \quad 0 < -\frac{B}{2A} < \frac{\mu_R}{\mu_R + \gamma_R + \alpha_R} \quad \text{and} \quad \Delta > 0. \quad (12)$$

in the (μ_V, μ_R)-plane.

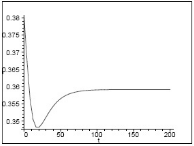
Dynamical analysis of the Cruz-Pacheco's Model



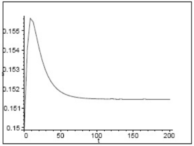
$C_0 : \mathbb{R}_{0G} = 1,\ C_1 : 2\mu_R A + (\mu_R + \gamma_R + \alpha_R)B = 0,$
$C_A : A = 0,\ C_B : B = 0.$

Simulation

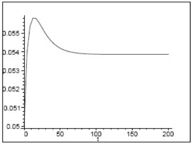
Let $(\mu_V, \mu_R) = (0.6, 0.13)$. Note that $\mathbb{R}_{0G} = 0.9959 < 1$. The initial data is $(s_a(0), i_a(0), i_v(0), n_a(0)) = (0.38, 0.15, 0.05, 0.58)$.



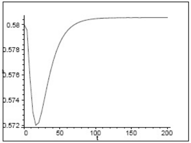
(a) The time course of s_a



(b) The time course of i_a



(c) The time course of i_v

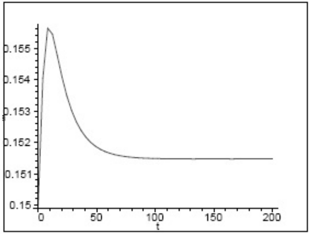


(d) The time course of n_a

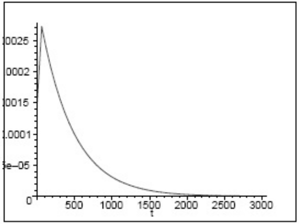
Although the basic reproduction number smaller than unity, the disease can still spread persistently.

Simulation

The time course of i_a with the same parameter values but different initial states.



(a) $(s_a(0), i_a(0), i_v(0), n_a(0)) = (0.38, 0.15, 0.05, 0.58)$.



(b) $(s_a(0), i_a(0), i_v(0), n_a(0)) = (0.989, 0.00015, 0.00011, 0.999)$.

In the case backward bifurcation occurs, different initial states may induce different transmission results.

Conclusion

Model	Mosquitos	Birds	Backward bifurcation
Wonham et al. 2004	Constant	Constant	No
Bowman et al. 2005	Logistic	Logistic	Yes [7]
Cruz-Pacheco et al. 2005	Constant	Logistic	Yes
Lord and Day 2001	Logistic	Logistic	Yes

(For the model of Lord and Day, it is non-autonomous, if we ignore the seasonal effect and do not distinguish birds as juvenile and adult and consider all parameters as constant, one can follow the analysis in Section 3 to analyze the simplified version of the model and prove the existence of the backward bifurcation.)

Conclusion

- The backward bifurcation is an intrinsic property for compartmental model of WNV if bird population satisfies logistic type growth and disease-induced death rate is big enough.

Conclusion

- The backward bifurcation is an intrinsic property for compartmental model of WNV if bird population satisfies logistic type growth and disease-induced death rate is big enough.
- If the backward bifurcation occurs,
  the basic reproduction number is not enough to describe whether the disease will prevail of not. In this case, a subthreshold condition value $\hat{\mathbb{R}}_0$ is needed and we should also pay more attention to the initial states.

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