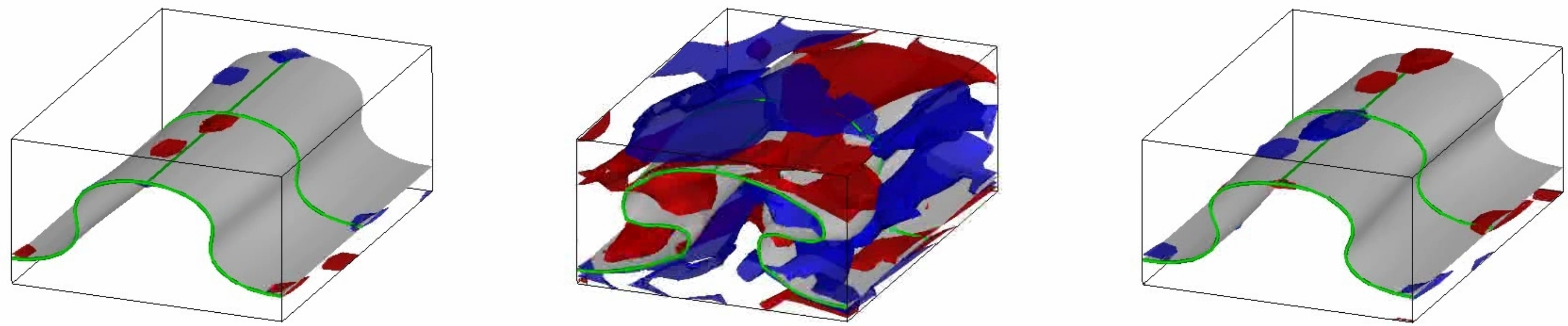
The tangled edge of turbulence in bursting Couette flow

Genta Kawahara
Matsumura Atsushi

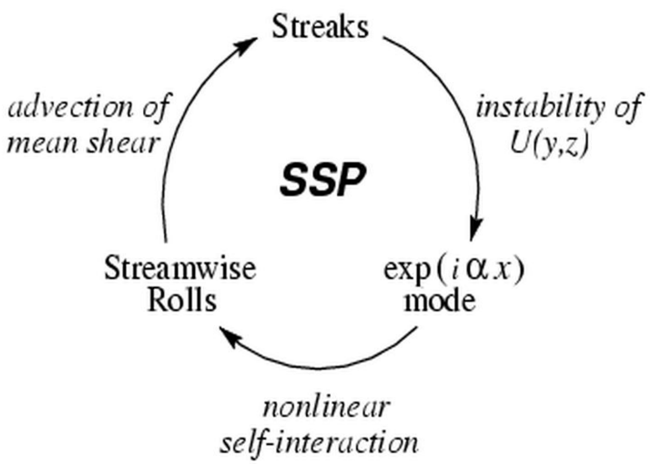
大阪大学 OSAKA UNIVERSITY
UOIT CHALLENGE INNOVATE CONNECT

Lennaert van Veen

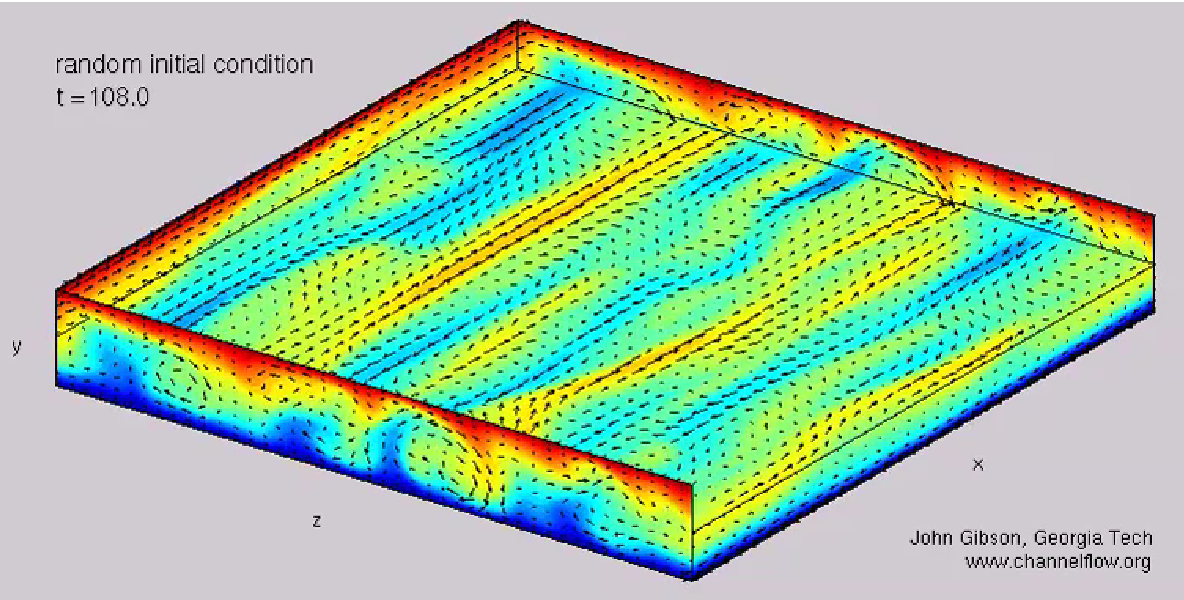


We study plane Couette flow at $Re = 400$ in the minimal flow unit. The aim is to understand the global dynamics of bursting and subcritical transition.

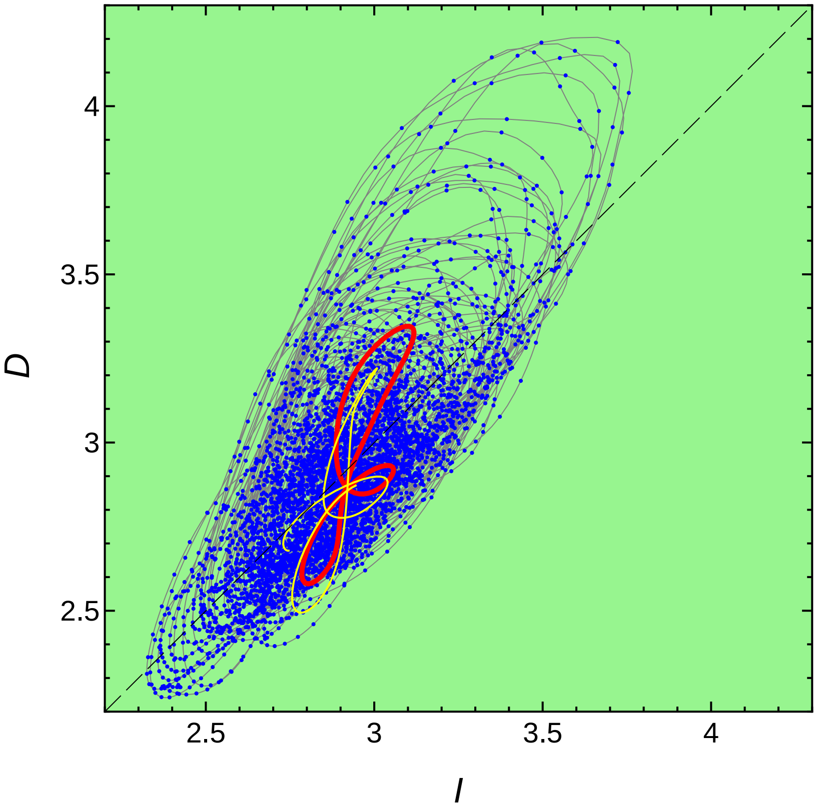
From a *dynamical systems* point of view we are interested in equilibria, time-periodic solutions and invariant manifolds.

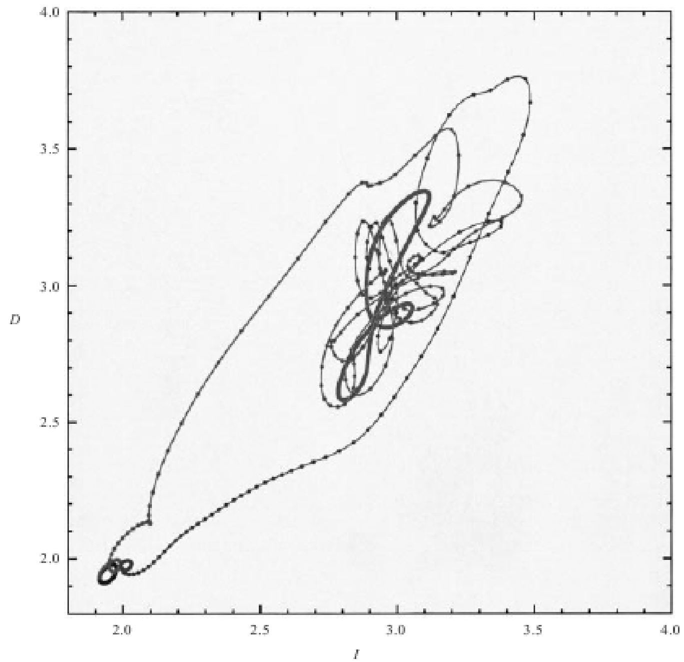


Fabian Waleffe, *Physics of Fluids*, **9**, 1997



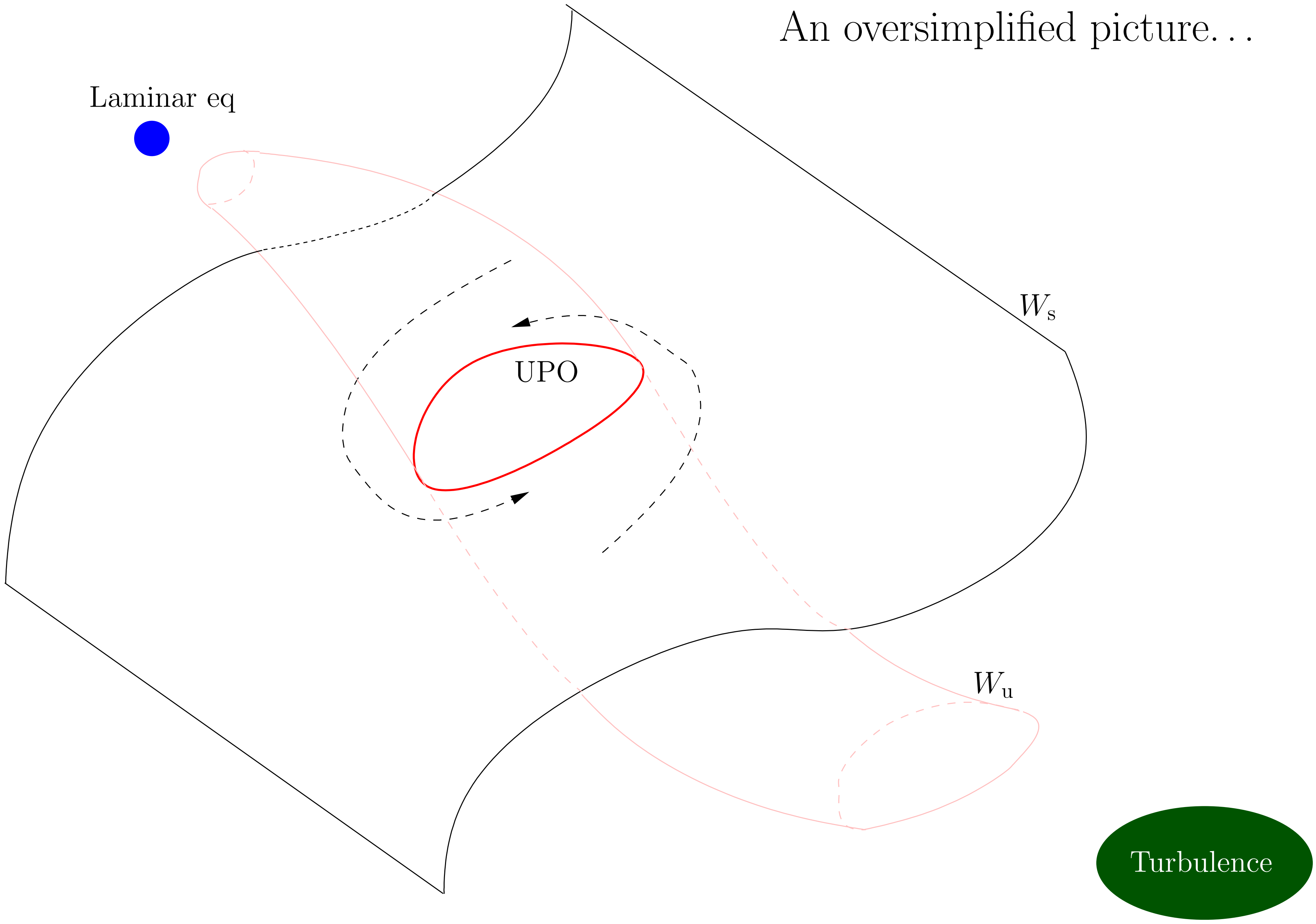
Kawahara & Kida computed a periodic solution close to the laminar state. Its stable manifold separates the phase space. See also work by Viswanath, Gibson, Schneider, . . .

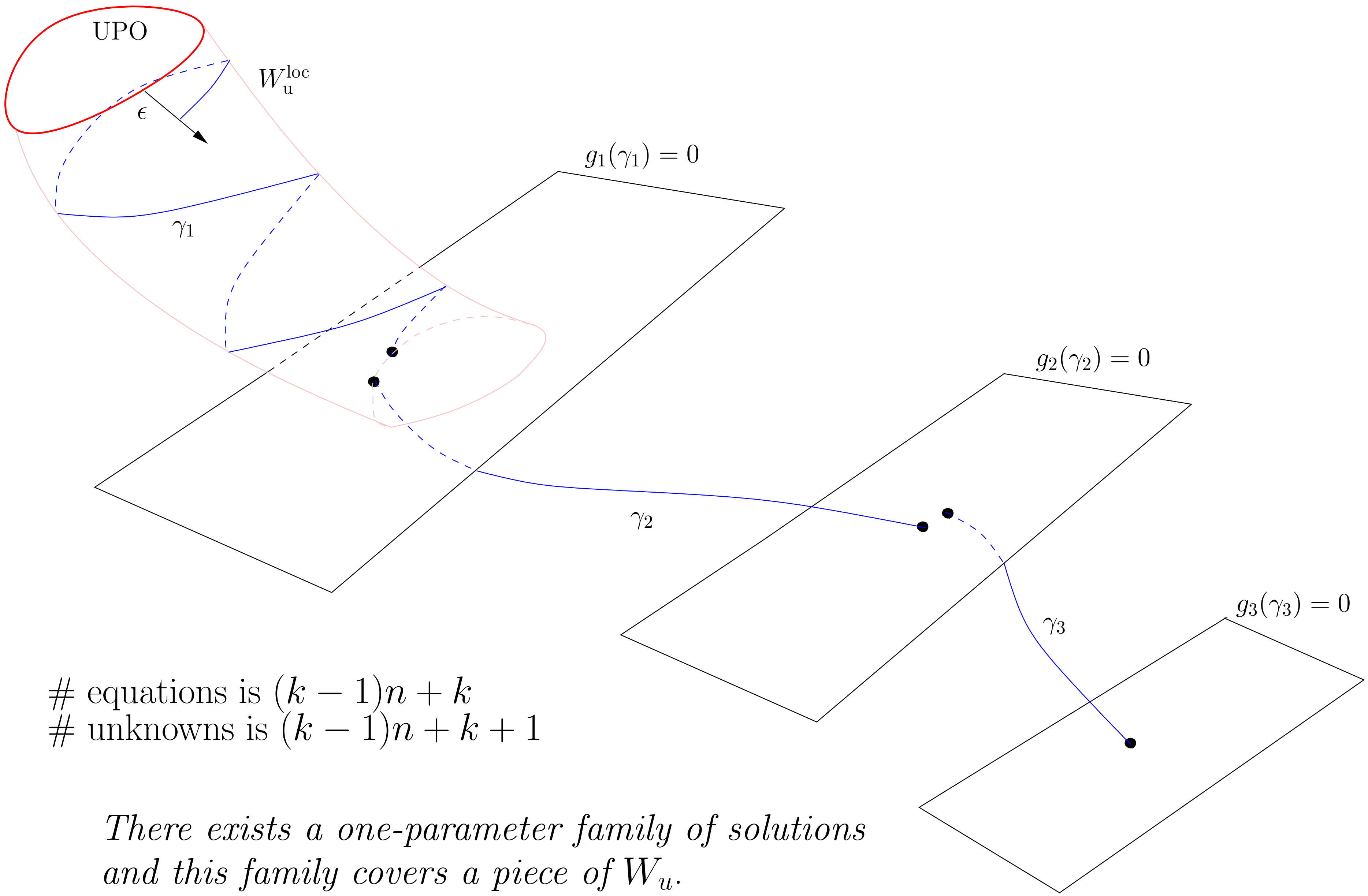




A quiescent and a turbulent periodic orbit, projected on energy input rate and energy disspipation rate. Kawahara & Kida, JFM 2001.

An oversimplified picture. . .





\# equations is $(k-1)n + k$
\# unknowns is $(k-1)n + k + 1$

There exists a one-parameter family of solutions and this family covers a piece of W_u.

We can solve the *arclength continuation* equations

$$\mathbf{F}(\mathbf{X}) = \mathbf{0} \quad \text{where } \mathbf{F}: \ \mathbb{R}^{N+1} \to \mathbb{R}^{N}$$

by a prediction–correction method. In every step we must solve

$$\mathcal{A}\,\mathrm{d}\mathbf{X} = \left(\begin{array}{cccc|c} \mathbb{I}_n & & & & \\ -J_1 & \mathbb{I}_n & & & \\ & -J_2 & \mathbb{I}_n & & \mathbf{A} \\ & & \ddots & \ddots & \\ \hline & & \mathbf{B} & & \mathbf{C} \end{array}\right) \mathrm{d}\mathbf{X} = \left(\begin{array}{c} -F_1(\mathbf{X}) \\ \cdot \\ \cdot \\ \cdot \\ -F_{(k-1)n}(\mathbf{X}) \\ \hline -F_{(k-1)n+1}(\mathbf{X}) \\ \vdots \\ -F_{(k-1)n+k}(\mathbf{X}) \\ 0 \end{array}\right)$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are sparse. The last row in the matrix is $\boldsymbol{T} = \dot{\boldsymbol{X}}$.

Multiple shooting Newton-Krylov continuation of BVP

1. Find an initial solution by forward integration starting from $\gamma(0) = \bar{\mathbf{x}} + \epsilon_0 \mathbf{u}_1$. Set $\mathbf{T} = (0, \ldots, 0, 1)^t$.
2. Prediction: $\mathbf{z}_{i+1}^0 = \mathbf{z}_i + \Delta s\, \mathbf{T}_i$.
3. Correction: approximate the solution to

$$\mathcal{A}\,\delta\mathbf{z}^j = \begin{pmatrix} D\mathbf{F} \\ \mathbf{T}_i^t \end{pmatrix} \delta\mathbf{z}^j = - \begin{pmatrix} \mathbf{F}(\mathbf{z}_{i+1}^j) \\ 0 \end{pmatrix}$$

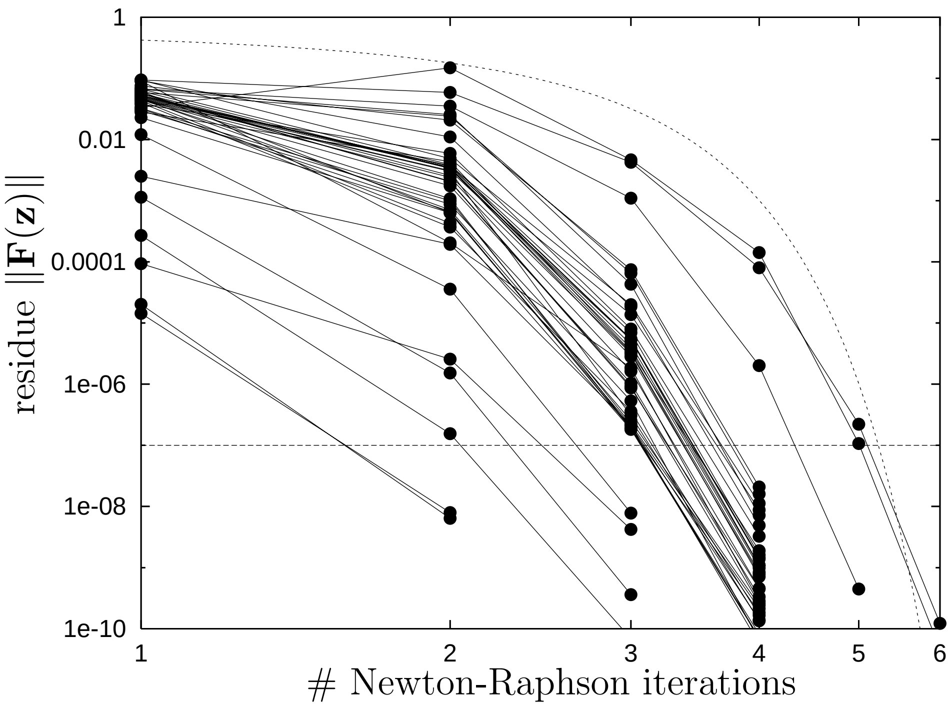
   by GMRES iterations up to tolerance d and update $\mathbf{z}_{i+1}^{j+1} = \mathbf{z}_{i+1}^j + \delta\mathbf{z}^j$ until a Newton-Raphson convergence criterion is met. Then set $\mathbf{z}_{i+1} = \mathbf{z}_{i+1}^j$.
4. Control step size Δs.
5. Compute $\mathbf{T}$ by finite differences.
6. Repeat 2.-5. for $i = 1, 2, \ldots, i_{\max}$.

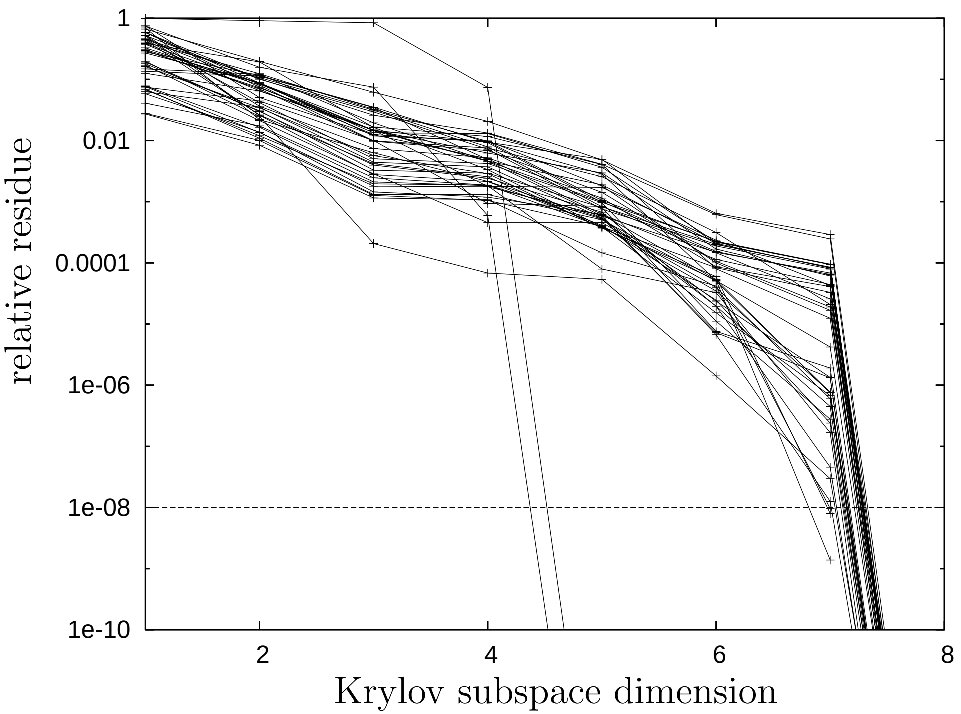
Lemma

Matrix $\mathcal{A}$ has eigenvalue $\lambda_0 = 1$ with algebraic multiplicity at least $(k-1)(n-1)$ and geometric multiplicity at least $(n-1)$

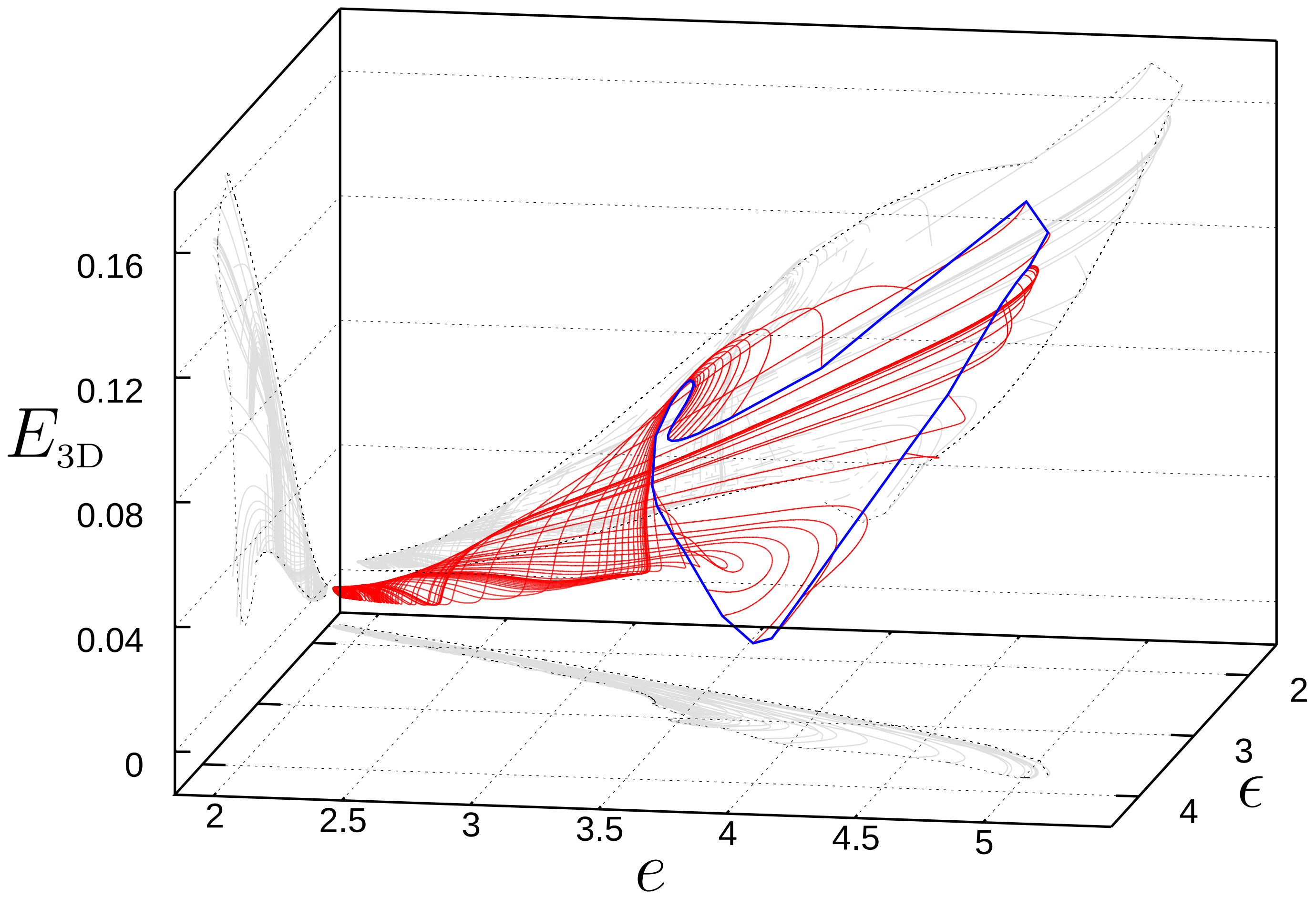
Proposition

Assume that all eigenvalues of $\mathcal{A}$ other than $\lambda_0 = 1$ are simple. Then the number of GMRES iterations necessary is at most $(3k-1)$ with exact arithmetic.

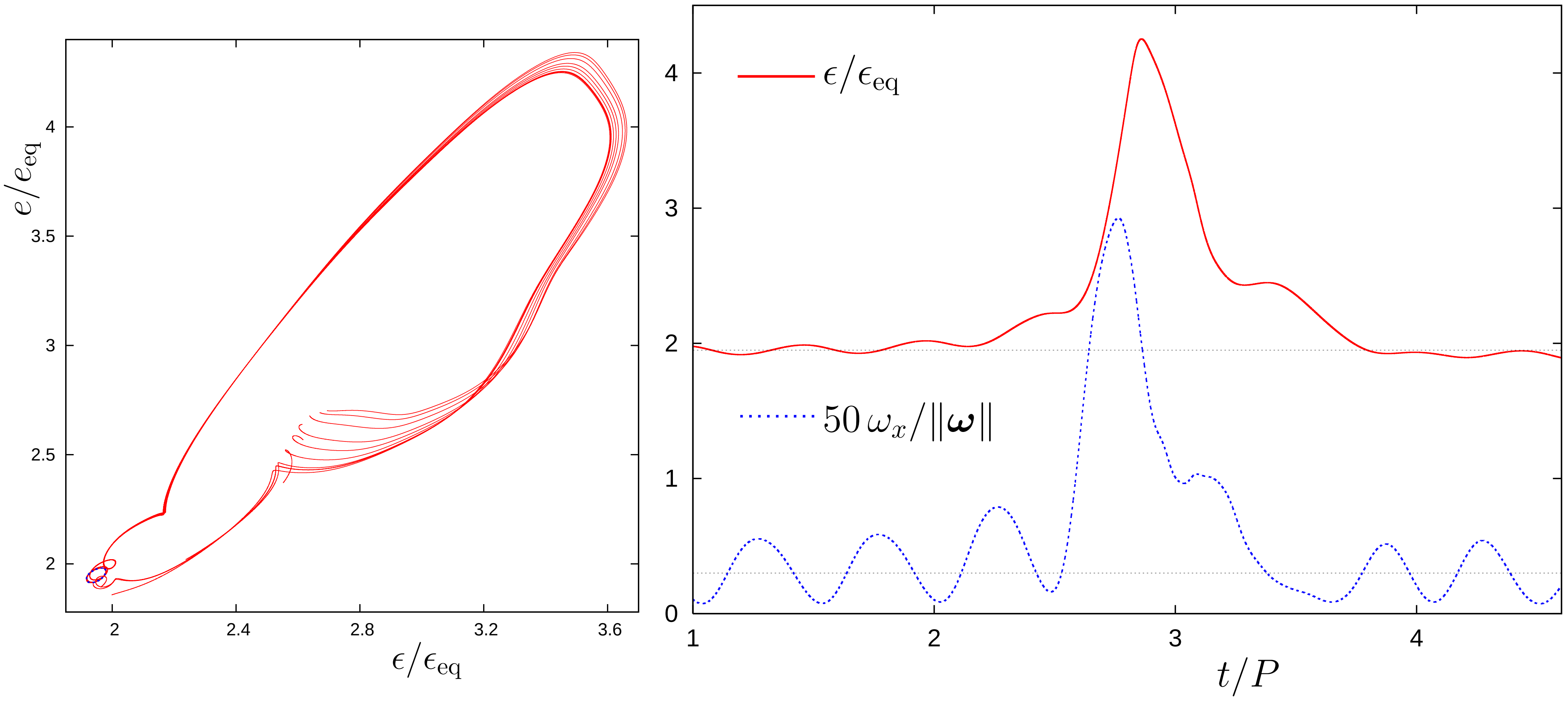


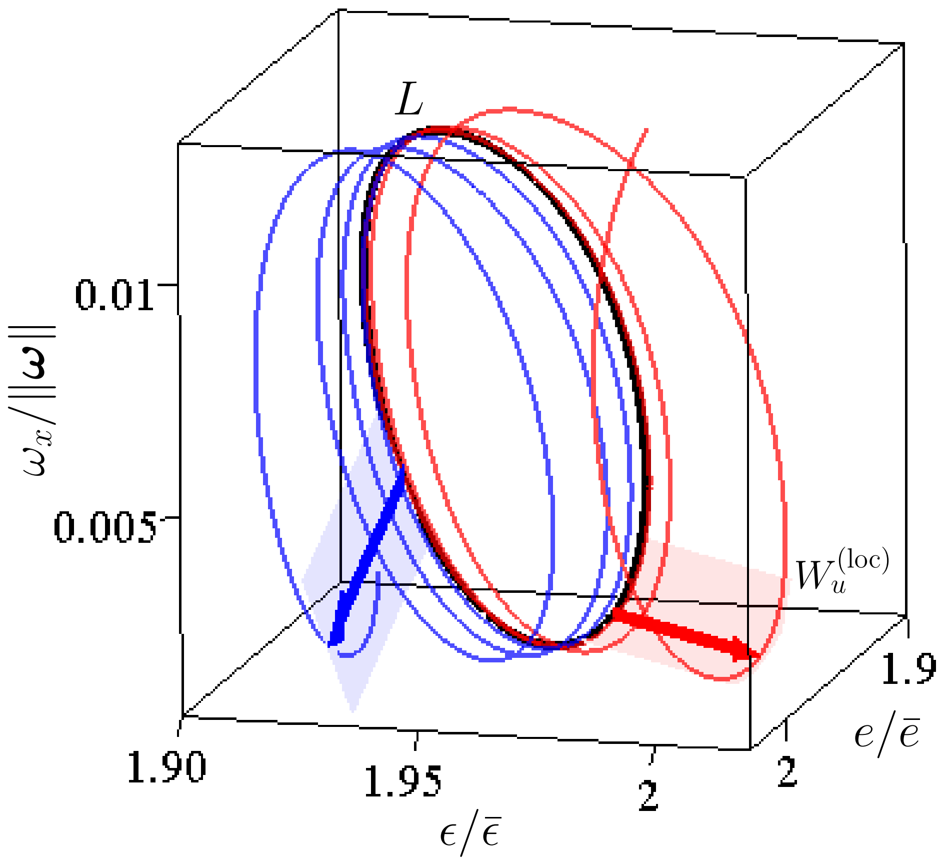


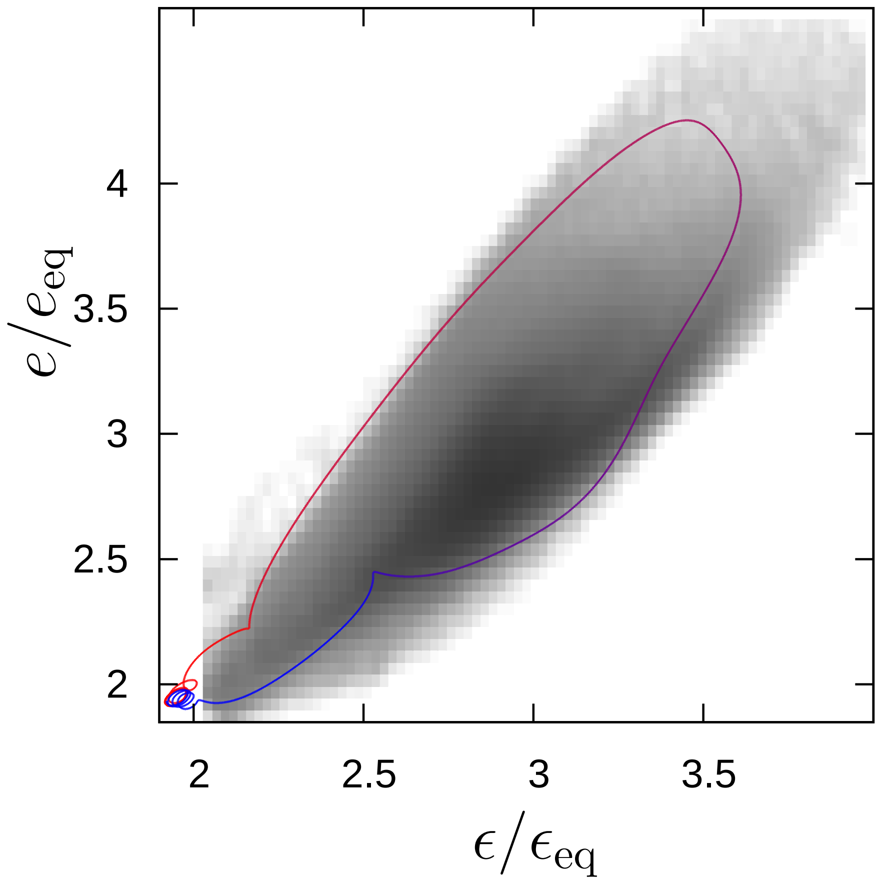
As expected, the “local” unstable manifold looks like a cylinder:



A piece of the global unstable manifold:




L
$W_u^{(\mathrm{loc})}$
0.01
0.005
$\omega_x/\|\boldsymbol{\omega}\|$
1.90
1.95
2
$\epsilon/\bar{\epsilon}$
1.9
2
$e/\bar{e}$


4
3.5
3
2.5
2
e/e_{eq}
2
2.5
3
3.5
$\epsilon/\epsilon_{\mathrm{eq}}$

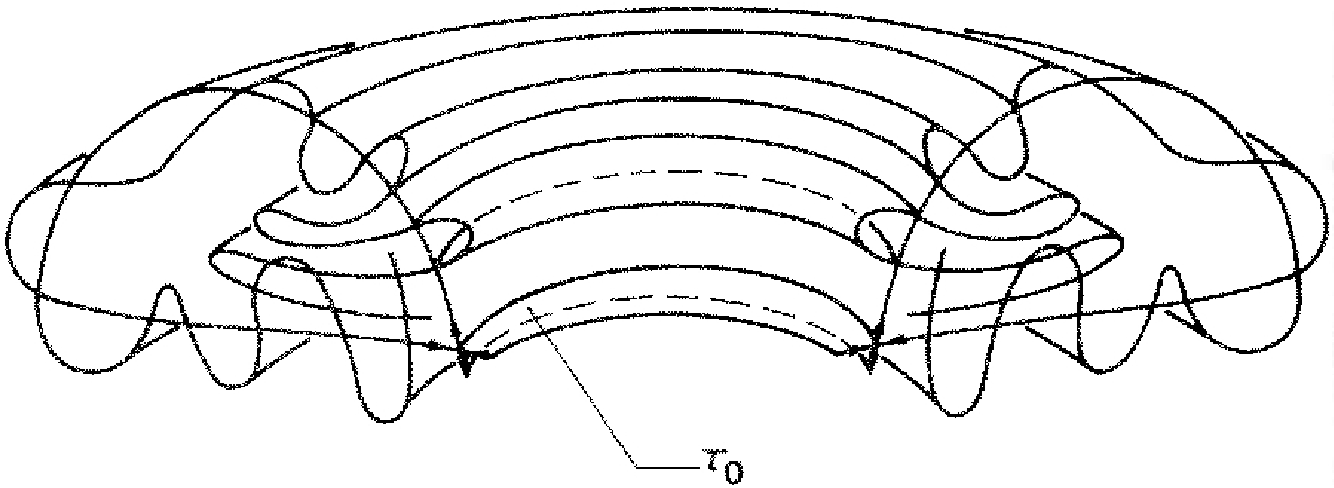


Figure 3.4.7. The Homoclinic Torus Tangle, Cut Away Half View.

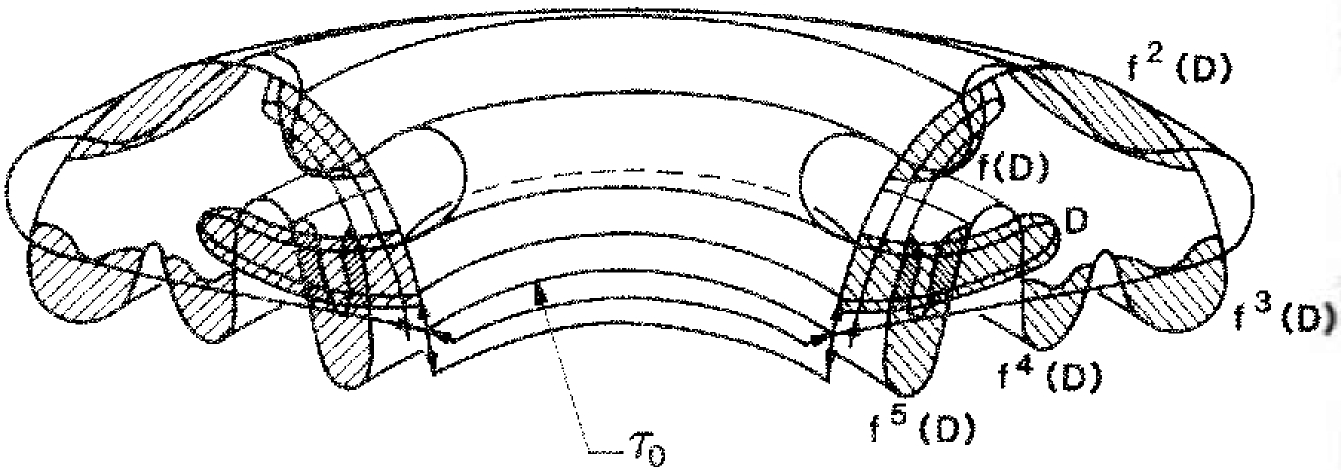


Figure 3.4.8. The Region D and Its Iterates, Cut Away Half View.

1) Strong evidence for the existence of an orbit homoclinic to the “edge state”.
2) The global geometry of the (un)stable manifold will be quite complex.
3) The homoclinic orbit might serve as a global target for control.