Symbolic computation of elements of bifurcation in delay differential equations

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Outline

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1. Center manifolds (CM): Statement of the problem
2. Computation of CM for Hopf singularity
3. K-Asymptotic stability and Lyapunov constants

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Introduction

Consider the delay differential equation of the form

$$\dot{x}(t) = L(x_t) + g(x_t).$$

- x is a continuous function with $x(t) \in \mathbb{R}^n$,
- $x_t \in C := C([-r, 0]$ is the state of x at t, i.e.,

$$x_t(\theta) = x(t + \theta), \theta \in [-r, 0].$$

- $L : C \to \mathbb{R}^n$ is a bounded linear operator,
- $g \in C^\infty$: $g(0) = 0$ and $Dg(0) = 0$.

The linearized equation is given by

$$\dot{x}(t) = L(x_t).$$

Let

$$\sigma = \{\lambda \in \mathbb{C} : \det(\lambda I_{R^n} - L(e^{\lambda \cdot}))\}.$$

- If $\Re(\lambda) < 0 \ \forall \lambda \in \sigma$: Stability of the equilibrium of RFDE
- If $\Re(\lambda_0) > 0$ for some $\lambda_0 \in \sigma$: Instability of the equilibrium of RFDE

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- We are specially interested in cases where

$$(H) : \sigma \cap i\mathbb{R} \neq \varnothing$$

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We have

$$C = X_c \oplus X_s$$

with

- X_c: The generalized eigenspace associated with $\sigma(A) \cap i\mathbb{R}$.
- $X_s = \{\phi \in C :< \Psi, \phi >= 0\}$.
- Ψ: The basis for the dual subspace X_c^*.
- $$\langle \psi, \varphi \rangle = \psi(0)\varphi(0) - \int_{-r}^{0} \int_{0}^{\theta} \psi(s-\theta)\, d\eta(\theta)\, \varphi(s)\, ds,$$

  for $\varphi \in C$ and $\psi \in C^*$.

Definition

A center manifold is a graph of a function

$$h : X_c \to X_s, (W_c := Gr(h))$$

which is:

- Tangent to X_c at 0, $(i.e : h(0) = 0, Dh(0) = 0)$
- Locally invariant under the semi-flow generated by RFDE.

$$\begin{cases} \frac{d}{dt}z(t) = Bz(t) + \Psi(0)g(\Phi z(t) + h(z(t))) \\ z(0) = \xi, \qquad \xi \in \mathbb{R}^p \end{cases}$$

Information on $g(\Phi z(t) + h(z(t))) \Longleftrightarrow$ Information on h.

The knowledge of h is of primary importance !

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Existing works

- J. Carr.
  ODEs: "Applications of Center Manifold Theory".
  Springer-Verlag, New York, Heidelberg, Berlin, 1981.
- T. Faria et al. (1995, 2000):
  RFDEs: Normal forms for FDEs and FDEs with diffusion.
- M. Ait Babram et al.
  RFDEs:
  - (1997, 2000) Center manifolds for Hopf and Bogdanov singularities.
  - (2003): Normal forms, general case.

Center manifold calculation

$$h(\xi) = \sum_{k=2}^{m} \sum_{i=0}^{k} a_i^k \xi_1^{k-i} \xi_2^i + o(|\xi|^m) \text{ for } \xi \in V.$$

Theorem

Assume $\sigma \cap i\mathbb{R} = (\pm i\omega)$.
The coefficients $(a_i^k, i = 0, \cdots k, k \in \mathbb{N})$ are given in a unique way by:

- *For $k = 2$ and $j \in \{0, ..., 2\}$*

$$a_j^2(\theta) = (e^{\lambda_j \theta} \Delta^{-1}(\lambda_j) - \frac{1}{\lambda_j + i\omega} \phi_1(\theta)\psi_1(0) - \frac{1}{\lambda_j - i\omega} \phi_2(\theta)\psi_2(0)) M_j^1,$$

where $\lambda_j := -(2 - 2j)\omega$.

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- For $k > 2$ and $j \in \{0, ..., k\}$

$$a_j^k(\theta) = e^{-(k-2j)i\omega\theta} a_j^k(0) + S_j^{k-1}(\theta), \theta \in [-r, 0]$$

- $S_j^{k-1}(\theta) = \int_0^\theta e^{-i\omega(k-2j)(\theta-s)} F_j^{k-1}(s)ds, j \in \{0, ..., k\}$
- F_j^{k-1} is given by means of

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- For k even, or k odd and $j \notin \left\{\frac{k-1}{2}, \frac{k+1}{2}\right\}$

$$a_j^k(0) = \Delta^{-1}(-(k-2j)i\omega)\left[M_j^{k-1} - F_j^{k-1}(0) + LS_j^{k-1}\right]$$

- M_j^{k-1} is given by means of

$$((a_i^l, l = 2, ..., k-1), i = 0, ..., l)$$

Lemma

the matrix M given by

$$M = \begin{pmatrix} \Delta(-\omega i) & \psi_1^\top(0) \\ \left\langle \psi_1, e^{-\omega i.} \right\rangle & 0 \end{pmatrix}$$

is invertible

The vectors $a^k_{\frac{k-1}{2}}(0)$ *and* $a^k_{\frac{k+1}{2}}(0)$ *are given by*

- $$\begin{pmatrix} a^k_{\frac{k-1}{2}}(0) \\ 0 \end{pmatrix} = M^{-1} \begin{pmatrix} M^{k-1}_{\frac{k-1}{2}} - F^{k-1}_{\frac{k-1}{2}}(0) + LS^{k-1}_{\frac{k-1}{2}} \\ N^{k-1}_1 \end{pmatrix}$$

- $$\begin{pmatrix} a^k_{\frac{k+1}{2}}(0) \\ 0 \end{pmatrix} = \overline{a^k_{\frac{k-1}{2}}(0)}$$

when k is odd

Summary

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1. Involve only critical eigenvalues of the linearized equation
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k-asymptotic stability (Negrini, Salvadori and Bernfeld: 1979, 1980)

Consider the ODE

$$\frac{d}{dt}x(t) = Bx + H(x)$$

with

$$B = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

and

$$H(\xi) = \Psi(0)g(\Phi\xi + h(\xi))$$

Exchange of stability/Perturbations $\Leftrightarrow$ Hopf bifurcation

Definition

The zero equilibrium of the ODE is k-asymp. stable if

1. the zero of the system

$$\frac{d}{dt}u = Bu + H(u) + O\left(\|u\|^{k+1}\right)$$

is asym. stable $\forall O\left(\|.\|^{k+1}\right) \in C(B^2(0), \mathbb{R}^2)$.

2. k is the smallest integer such that property (1) is satisfied.

Relation with Lyapunov constant

Fix $k \in 2\mathbb{N}^*$. Then

- The equilibrium is k-asymp. stable.

$$\Updownarrow$$

- $\exists\ F : \mathbb{R}^2 \to \mathbb{R}^+ \in C^1$:

$$\dot{F}_{(1)}(\xi_1, \xi_2) = G_k \|(\xi_1, \xi_2)\|^{k+1} + O\left(\|(\xi_1, \xi_2)\|^{k+2}\right)$$

with $G_k < 0$.

Why G_k is interesting?

- Existence or non-existence of nontrivial periodic solutions
- Number of nontrivial periodic solutions
- Stability of periodic solutions

Computation of the Lyapunov constant

$$\xi_1^2 + \xi_2^2 = F_2 \longmapsto \rho_3 \longmapsto G_4$$

$$\rho_3 \longmapsto F_3$$

$$\vdots$$

$$F_{k-1}$$

For $k = 3$

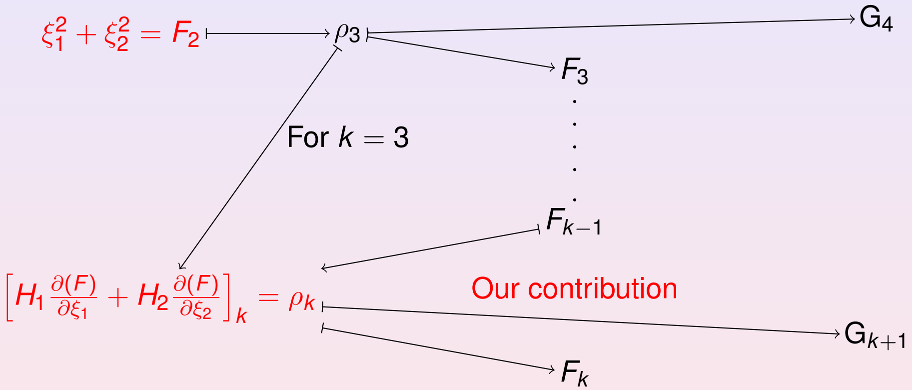
$$\left[H_1 \frac{\partial(F)}{\partial \xi_1} + H_2 \frac{\partial(F)}{\partial \xi_2}\right]_k = \rho_k$$

Our contribution

$$\rho_k \longmapsto G_{k+1}$$

$$\rho_k \longmapsto F_k$$

Computation of the Lyapunov constant



Let $\rho_k(\xi_1, \xi_2) = \sum_{j=0}^{k} p_j \xi_1^{k-j} \xi_2^{j}$

Proposition

Given an even integer $k > 2$, the Poincaré constant G_k of the reduced system is given in a unique way by

$$G_k = \frac{p_k + p_{k-2}/(k-1)\omega + \sum_{s=1}^{k/2-1} c_s d_s}{\frac{k}{2(k-1)\omega} + \sum_{s=1}^{k/2-1} c_s \frac{C_{s+1}^{k/2}}{(k-2s+1)\omega} + 1}$$

where

$$c_s = \frac{3 \times 5 \times 7 \cdots \times (2s+1)}{(k-1) \times (k-3) \cdots \times (k-2s+1)}$$

and

$$d_s = \frac{p_{k-2s-2}}{(k-2s-1)}, \forall s \in \{1, \cdots, \frac{k}{2} - 1\}.$$

Consider the following predator-prey system

$$\begin{aligned}\dot{x}(t) &= rx(t)(1 - \frac{x(t)}{K}) - \frac{x(t)y(t)}{a + x^2(t)}\\ \dot{y}(t) &= y(t)\left[\frac{\mu x(t-\tau)}{a + x^2(t-\tau)} - D\right],\end{aligned}$$

r, K, a, μ, D, and τ are positive constants.

- The system has an interior equilibrium (x_1, y_1) if $4aD^2 < \mu^2$,
- Let $C_1 = -\frac{\tau x_1}{a+x_1^2}$, $C_2 = \frac{\tau\mu y(a-x_1^2)}{(a+x_1^2)^2}$. Then

$$\Delta(\lambda) = \lambda^2 - C_1 C_2 e^{-\lambda}$$

has a pair of conjugate purely imaginary roots $\pm i\omega$

$$\Updownarrow$$

$$\sin(\omega) = 0, \omega^2 = -C_1 C_2.$$

The result of center manifold up to third order confirms the one established in

Ruan and D. Xiao.
Multiple bifurcations in a delayed predator-prey system with nonmonotonic functional response.
J. Diff. Eqns 2001, 176: 494-510.

Thank you for your attention