Coating Flows on Slowly Rotating Cylinders

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Thank you for the invitation!

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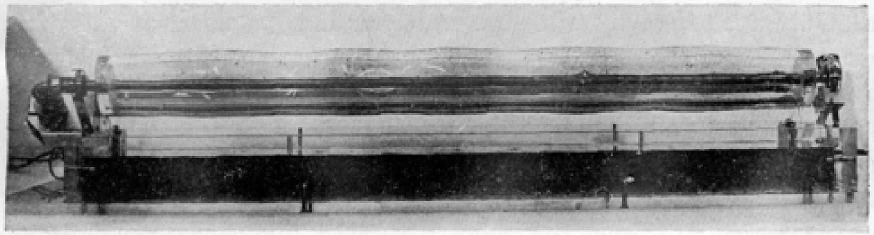
Warning: this talk does not cite all the work that it should cite. Our preprint is more responsible: "Nonnegative solutions for a long-wave unstable thin film equation with convection" with M. Chugunova and R.M. Taranets, to appear in the SIAM Journal on Mathematical Analysis.

Coating and rimming flows

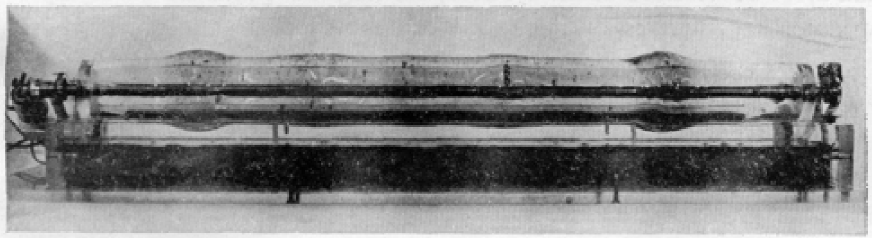
Consider a horizontal cylinder, rotating about its axis. If there is a fluid on the outside of the cylinder, this is called a coating flow. If the fluid is on the inside of the cylinder, this is called a rimming flow.

"It is a matter of common experience that if a knife is dipped in honey and then held horizontally, the honey will drain off; but that the honey may be retained on the knife by simply rotating it about its length. The question arises: what is the maximum load of honey that can be supported per unit length of knife for a given rotation rate?" — H.K. Moffatt, Journal de Méchanique 16(1977)5:651–673.

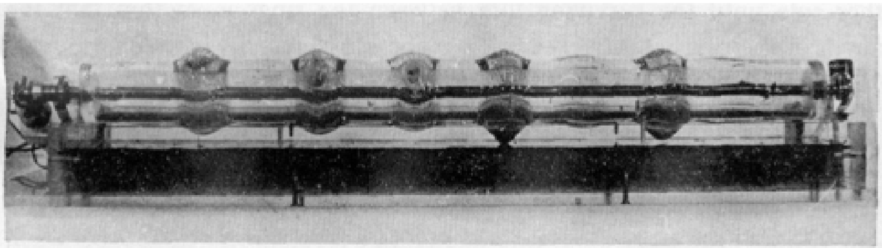
Coating experiments



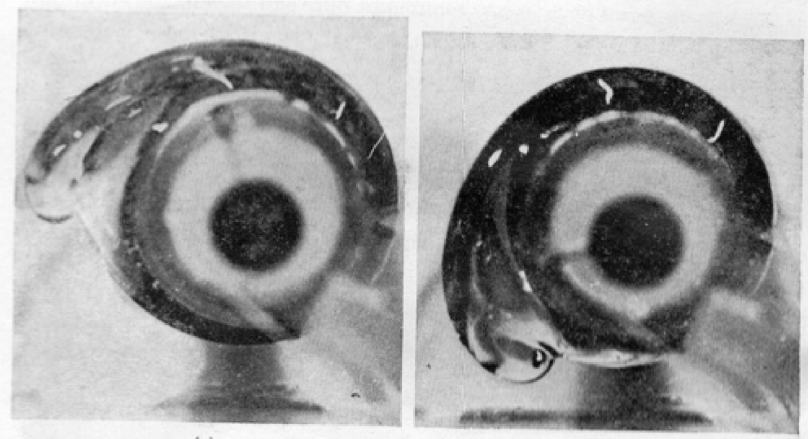
(b) 28.4 rpm



(c) 38.2 rpm



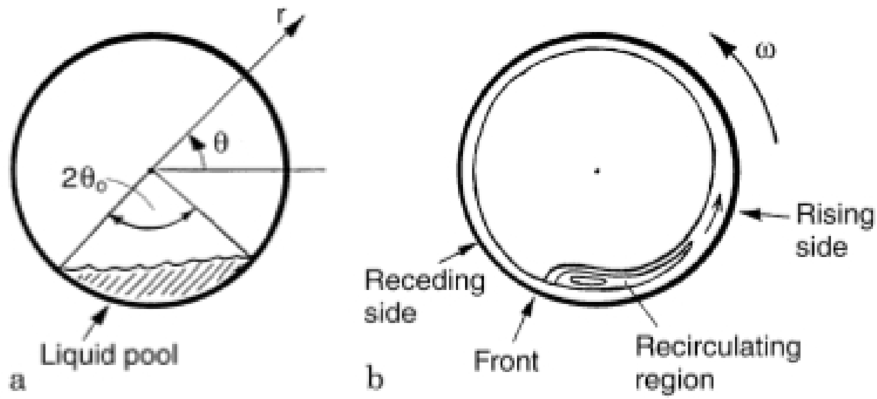
(d) 48.8 rpm

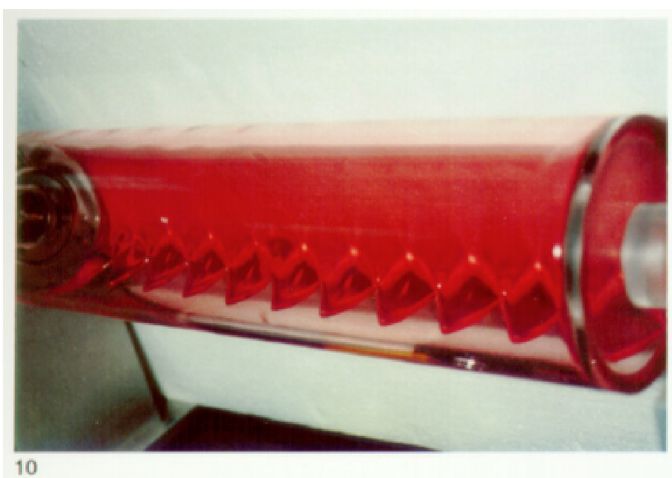


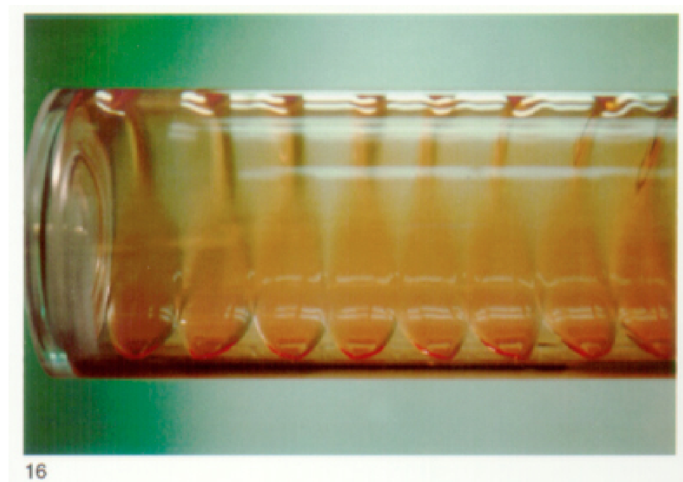
(a) (b)

H.K. Moffatt, Journal de Méchanique 16(1977)5:651–673. Reproduced without author's permission.

Rimming experiments



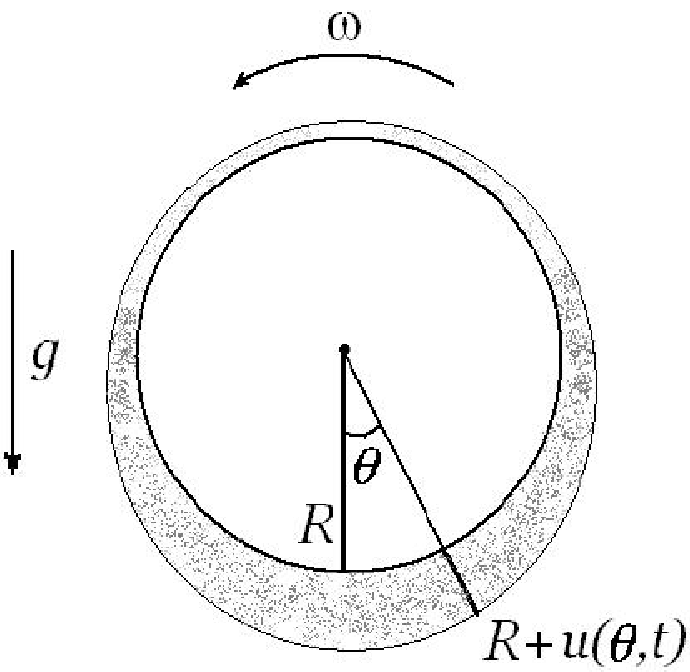




S.T. Thoroddsen and L. Mahadevan Experiments in Fluids 23(1997)1-13. Reproduced without authors' permission.

Model parameters

Consider a thin liquid film on the outer surface of a cylinder:



R is the radius of the cylinder. ω is the rate of rotation. g is the acceleration due to gravity. ν is the kinematic viscosity. ρ is the density. σ is the surface tension.

Lubrication approximation model

Three dimensionless quantities: the Reynolds number $\mathrm{Re} = \frac{R^2\omega}{\nu}$, $\gamma = \frac{g}{R\omega^2}$, **and the Weber number** $\mathrm{We} = \frac{\rho R^3 \omega^2}{\sigma}$.

Modelling assumptions:

- **The fluid flow is modelled by the Navier Stokes equations**
- **There is no slip at the liquid/solid interface**
- **There is surface tension at the liquid/air interface**
- **If** $\bar{u}$ **is the average thickness of the fluid then** $\varepsilon = \bar{u}/R$ **is small**
- $\chi = \frac{\mathrm{Re}}{\mathrm{We}}\varepsilon^3$ **and** $\mu = \gamma\,\mathrm{Re}\,\varepsilon^2$ **have finite, nonzero limits as** $\varepsilon \longrightarrow 0$.

Lubrication approximation model

Assume the flow is constant along the length of the cylinder.

Pukhnachov, Journal of Applied Mechanics and Technical Physics 18(1977)3:344–351:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial \theta}\left(u - \frac{\mu}{3}u^3 \sin(\theta)\right) + \frac{\chi}{3}\frac{\partial}{\partial \theta}\left(u^3\left[\frac{\partial u}{\partial \theta} + \frac{\partial^3 u}{\partial \theta^3}\right]\right) = 0$$

$$\theta \in [-\pi, \pi], \quad \frac{\partial^i u}{\partial \theta^i}(-\pi, t) = \frac{\partial^i u}{\partial \theta^i}(\pi, t) \textbf{ for } t > 0, \ \ i = \overline{0, 3}$$

where $\mu = \gamma \ \mathrm{Re}\ \varepsilon^2 = \frac{gR}{\omega\nu}\varepsilon^2$ **and** $\chi = \frac{\mathrm{Re}}{\mathrm{We}}\varepsilon^3 = \frac{\sigma}{\nu\rho R\omega}\varepsilon^3$ **.**

Moffatt (1973, 1977) found the same evolution equation for the zero surface tension ($\chi = \sigma = 0$) case:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial \theta}\left(u - \frac{\mu}{3}u^3 \sin(\theta)\right) = 0.$$

Steady States: zero surface tension

$$\frac{\partial}{\partial\theta}\left(u - \frac{\mu}{3}u^3\sin(\theta)\right) = 0 \quad \Longrightarrow \quad u - \frac{\mu}{3}u^3\sin(\theta) = q$$

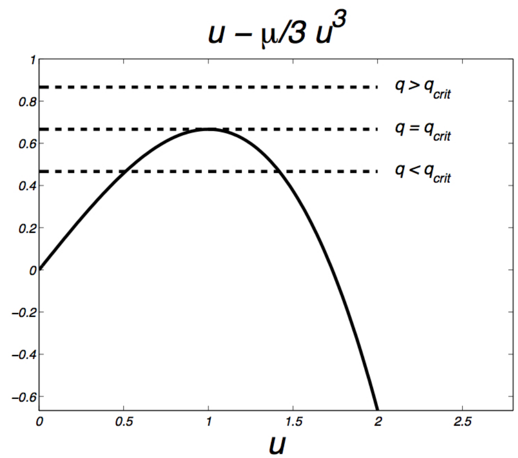
for some fixed q.

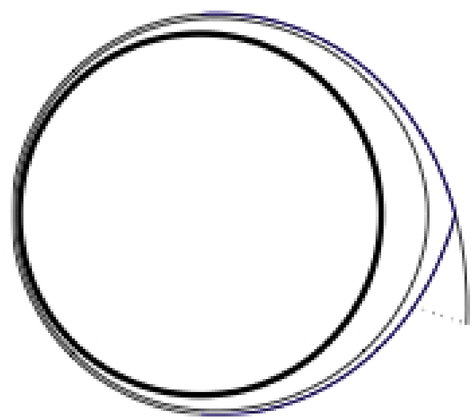
$$\theta = 0, \theta = \pi \quad \Longrightarrow \quad u(0) = u(\pi) = q$$

$$\theta = \pi/2 \quad \Longrightarrow \quad u(\pi/2) \text{ is a root of } u - \frac{\mu}{3}u^3 = q$$

$$\theta = 3\pi/2 \quad \Longrightarrow \quad u(3\pi/2) \text{ is a root of } u + \frac{\mu}{3}u^3 = q$$

At $\theta = \pi/2$ there might be no positive root if q is too big.





If q "small" then smooth solution, at q critical there's a corner, if q is too big the steady state is discontinuous.

Steady states: positive surface tension

If there's no surface tension then $q_{crit} = \frac{2}{3\sqrt{\mu}} = \frac{2}{3\varepsilon}\sqrt{\frac{\omega\nu}{gR}}$ The total amount of "honey" in the steady state is closely related to the value of q and so we see that the larger ω is, the more "honey" you can hold on your knife.

If there is surface tension, Pukhnachov[1] proved that $q_{crit} \leq 2\sqrt{3/\mu} \approx 3.464/\sqrt{\mu}$. We improve on this:

Theorem(ChugPughTara 2009) For positive surface tension, there is no strictly positive 2π periodic steady state with flux $q > \frac{2}{3}\sqrt{\frac{2}{\mu}} \approx 0.943/\sqrt{\mu}$.

The upper bound on q_{crit} doesn't depend on surface tension, but q_{crit} likely will. See Benilov et al.[2] for extensive simulations of steady states.

[1] V.V. Pukhnachov, Mathematics and Continuum Mechanics (2004)191-199.
[2] E.S. Benilov, M.S. Benilov, and N. Kopteva, J Fluid Mechanics 597(2008)91-118

Proof positive surface tension steady state result

The steady state satisfies

$$u - \frac{\mu}{3} u^3 \sin(\theta) + \frac{\chi}{3} u^3 \left(u_{\theta\theta\theta} + u_\theta\right) = q.$$

Rescale the flux q to 1 by introducing $y(\theta) = u(\theta)/q$

$$\mathcal{L} y = \gamma \left(y_{\theta\theta\theta} + y_\theta\right) = \beta \sin(\theta) - \frac{1}{y^2} + \frac{1}{y^3}$$

where $\gamma := \chi q^3/3$ and $\beta := \mu q^2/3$. The righthand side must be orthogonal to the kernel of $\mathcal{L}$ hence

$$\int_{-\pi}^{\pi} \left(\frac{1}{y(\theta)^2} - \frac{1}{y(\theta)^3}\right) \, d\theta = 0, \qquad \int_{-\pi}^{\pi} \left(\frac{1}{y(\theta)^2} - \frac{1}{y(\theta)^3}\right) \sin(\theta) \, d\theta = \beta\pi.$$

Adding these,

$$\int_{-\pi}^{\pi} \left(\frac{1}{y(\theta)^2} - \frac{1}{y(\theta)^3}\right) [1 + \sin(\theta)] \, d\theta = \beta\pi.$$

The term $[\dots]$ is nonnegative, so bound the $(\dots)$ term over $\{y \geq 1\}$...

$$\beta\pi \leq \int_{\{y \geq 1\}} \left(\frac{1}{y(\theta)^2} - \frac{1}{y(\theta)^3}\right) [1 + \sin(\theta)] \, d\theta \leq \int_{\{y \geq 1\}} \frac{4}{27} [1 + \sin(\theta)] \, d\theta \leq \frac{4}{27} 2\pi.$$

So there is no steady state if $\beta > 8/27$ (which then unravels to $q > \frac{2}{3}\sqrt{\frac{2}{\mu}}$).

Second– and fourth–order parabolic PDE

Second–order parabolic equations Solutions obey a comparison principle hence nonnegative initial data yield nonnegative solutions.
Diffusion equation $u_t = Du_{xx}$. Compactly supported initial data yield solutions that instantaneously lose that compact support.
Porous medium equation $u_t = (u^m u_x)_x$ with $m > 0$. Compactly supported initial data yield solutions that continue to have compact support.

Fourth–order parabolic equations Solutions don't obey a comparison principle.
linear equation $u_t = -u_{xxxx}$. Positive initial data can yield solutions that are negative at certain places at certain times. Compactly supported initial data yield solutions that instantaneously lose that compact support.
Thin film equation $u_t = -(u^n u_{xxx})_x$ with $n > 0$. Given nonnegative, compactly supported initial data, one can construct solutions that are nonnegative and have compact support.

Nonnegative solutions

The classic thin film equation:

$$u_t = -(u^n u_{xxx})_x, \quad u_x(\pm a) = u_{xxx}(\pm a) = 0, \;\; \Omega = (-a, a)$$
$$P = \overline{Q}_T - (\{u = 0\} \cup \{t = 0\}).$$

If a solution is positive then it is smooth. But exact solutions (travelling wave, source-type, etc) that go to zero at a point have a discontinuous u_{xx} at that point. And so if one is interested in solutions that have contact lines (or develop contact lines) one must work with weak solutions.

The seminal work in this area is by Bernis and Friedman: Journal of Differential Equations 83(1990)1:179-206. *Cited 212 times and counting. . .*

Weak solutions

Two popular types of weak solutions for $u_t = -(f(u)u_{xxx})_x$:

- **weak generalized solution**

$$\iint_{Q_T} u\phi_t + \iint_P f(u)u_{xxx}\phi_x = 0,$$

$$u \in C_{x,t}^{1/2,1/8}, \qquad f(u)u_{xxx} \in L^2(P).$$

(See Bernis & Friedman JDE 83(1990)1:179-206.)

- **strong generalized solution**

$$\iint_{Q_T} u\phi_t - \iint_{Q_T} f(u)u_{xx}\phi_{xx} - \iint_{Q_T} f'(u)u_x u_{xx}\phi_x = 0,$$

$$u \in L^2(0,T; H_0^2(\Omega)).$$

(See Beretta, Bertsch, & Dal Passo, ARMA 129(1995)2:175–200 and Bertozzi & Pugh, CPAM 49(1996)2:85-123.)

Constructing a nonnegative weak solution for $0 < n < 4$

1) Approximate the PDE **For $u_t = -(u^n u_{xxx})_x$ consider the approximate problem**

$$u_t = -\left(\frac{u^{n+4}}{u^4 + \epsilon u^n} u_{xxx}\right)_x$$

where $\epsilon > 0$. *What?!? Why not just look at $u_t = -((u^n + \epsilon)u_{xxx})_x$? That'd be so much easier and nondegenerate to boot...*

2) Approximate the nonnegative initial data **Take your nonnegative initial data u_0 and "lift" it, giving the initial data**

$$u_{0\epsilon} = u_0 + \epsilon^{2/5}$$

to the approximate PDE. *Why muck with the initial data? Why 2/5?*

3) Study the solution of the approximate problem **The approximate problem has a unique, smooth, strictly positive solution $u_\epsilon(x,t)$ for all time.** *Whoah. How did that work when a nice equation like $u_t = -u_{xxxx}$ wouldn't necessarily give a strictly positive solution?*

Continuing the construction

4) Take $\epsilon \to 0$ In the limit, the positive functions $u_\epsilon(x,t)$ will have a nonnegative limit $u(x,t)$.

5) Show the limiting function is a weak solution The smooth solutions u_ϵ conserve mass:

$$\int u_{0\epsilon}(x)\,dx = \int u_\epsilon(x,t)\,dx \qquad \forall t.$$

They dissipate energy:

$$\int u_{\epsilon x}^2(x,t)\,dx \le \int u_{0\epsilon x}^2(x)\,dx \qquad \forall t.$$

They dissipate the "entropy"

$$\int \frac{1}{u_\epsilon^{n-2}(x,t)}\,dx \le \int \frac{1}{u_{0\epsilon}^{n-2}(x)}\,dx \qquad \forall t.$$

This control of the energy and entropy allows one to argue that initially positive solution of the approximate problem remains positive and hence exists for all time.

Finishing the construction

One shows that in the limit, the function $u(x,t)$ inherits some of this energy and entropy dissipation and, as a result, is a weak generalized solution

Up to this point, all the above is due to Bernis and Friedman, 1990. In 1993, Leo Kadanoff found another dissipated entropy: the "α entropy"

$$\int \frac{u_\epsilon^\alpha(x,t)}{u_\epsilon^{n-2}(x,t)}\,dx \le \int \frac{u_{0\epsilon}^\alpha(x)}{u_{0\epsilon}^{n-2}(x)}\,dx \qquad \forall t.$$

where $-1/2 < \alpha < 1$. It was used[3] to prove that the weak generalized solution is a strong generalized solution.

For Pukhnachov's model, we don't have dissipated quantities but we do have an energy and an entropy we can control. These are used to prove

[3] Baretta+Bertsch+Dal Passo 1995, Bertozzi+Pugh 1996

<u>Theorem</u>(ChugPughTara 2009) Consider nonnegative initial data $u_0 \in H^1$ **which has finite entropy. Then given a time** $T < \infty$ **there is a nonnegative strong generalized solution** $u \in L^2(0, T; H^2_{per}(\Omega))$ **for:**

$$u_t + \Big(|u|^3(a_0\, u_{\theta\theta\theta} + a_1 u_\theta + a_2 w'(\theta))\Big)_\theta + a_3 u_\theta = 0$$

where a_1, a_2, a_3 **are arbitrary constants, constant** $a_0 > 0$**, and** $w(\theta)$ **is periodic.**

Pukhnachov's model

$$u_t + \Big[|u|^3 \left(u_{\theta\theta\theta} + \alpha^2\, u_\theta - \sin\theta\right) + \omega u\Big]_\theta = 0, \;\; \theta \in \Omega = (-\pi, \pi)$$

is a special case of the equation above.

Thank you !

THANK YOU FOR YOUR PATIENCE