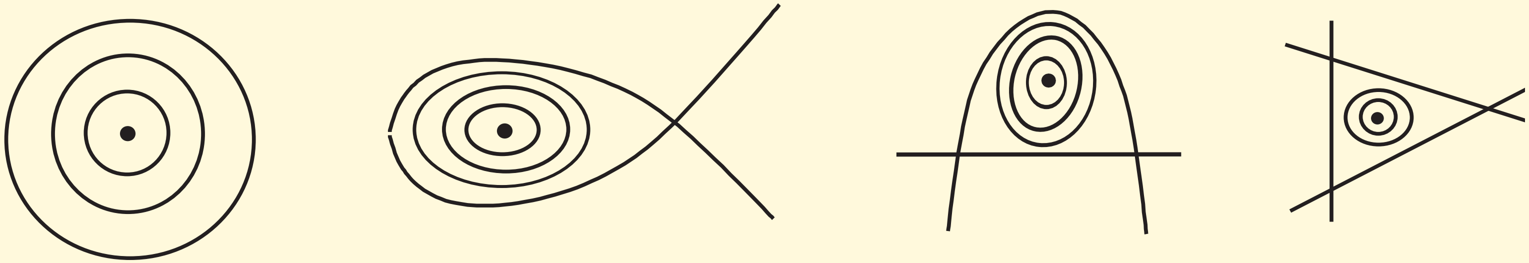
Cyclicities of Period Annulus for Quadratic Integrable Systems Under Quadratic Perturbations

Chengzhi Li

(Peking University and York University)

Southern Ontario Dynamics Day Workshop

Fields Institute, Toronto, May 14, 2010



Periodic Annulus + Perturbation ?

Especially: $X_2 + \varepsilon Y_2$?

This is a special case for the Weak Hilbert's 16th problem, proposed by Arnold in 1977.

This is also a special case for the cyclicity problem, proposed by Dumortier, Roussarie and Rousseau in 1994, and by Rousseau and Huaiping Zhu, and some others.

Classification of Quadratic Integrable systems

By H. Zoladek, JDE 1994:

- Q_3^H： The Hamiltonian class;
- Q_3^R： The reversible class;
- Q_3^{LV}: The Lotka-Volterra class;
- Q_4: The codimension 4 class.

Generic and Degenerate

For example, for the Hamiltonian class:

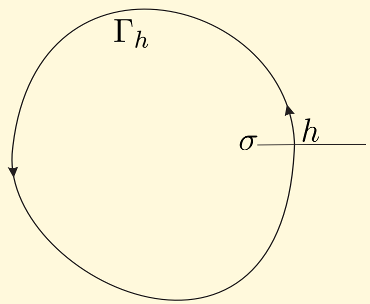
- Generic: $X \in Q_3^H \setminus \{Q_3^R \cup Q_3^{LV} \cup Q_4\}$;
- Degenerate: $X \in Q_3^H \cap \{Q_3^R \cup Q_3^{LV} \cup Q_4\}$.

Similarly for other classes.

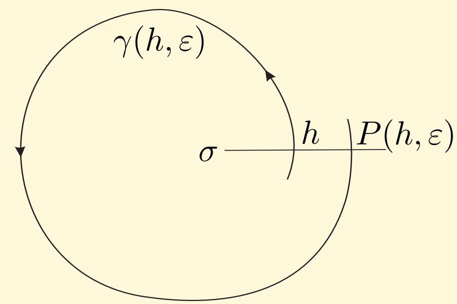
Study of perturbation of Hamiltonian class Q_3^H

$$\frac{dx}{dt} = \frac{\partial H(x,y)}{\partial y} + \epsilon f(x,y), \quad \frac{dy}{dt} = -\frac{\partial H(x,y)}{\partial x} + \epsilon g(x,y).$$

where $\deg H = 3,\ \deg(f,g) = 2$.



(i) $\varepsilon = 0$



(ii) $0 < |\varepsilon| \ll 1$

By using the Poincaré-Pontryagin Theorem we have:
the displacement function

$$d(h,\epsilon) = P(h,\epsilon) - h = \epsilon(I(h) + \epsilon\phi(h,\epsilon)),$$

where

$$I(h) = \oint_{\Gamma_h} f(x,y)dy - g(x,y)dx,$$

is an Abelian integral, and $\phi(h,\epsilon)$ is analytic and uniformly bounded for (h,ε) in a compact region near $(h,0)$.

Cyclicity of Period Annulus

Hence, for perturbed generic quadratic Hamiltonian system,

the cyclicity of period annulus can be defined as

Maximum number of isolated zeros of the $I(h)$ (with their multiplicities) for $h \in (h_1, h_2)$;

which gives

Maximal number of limit cycles bifurcated from a compact region insider the annulus.

The region contains:

– the singular point inside the annulus;

– the homoclinic loop as the boundary (by P. Madisic and R. Roussarie);

– but dos not include the heteroclinic loop as the boundary (by M. Caubergh, F. Dumortier & R. Roussarie about Alien limit cycles).

Study the Cyclicity for generic Q_3^H

A universal unfolding of Q_3^H contains at least 3 parameters, hence the Abelian integrals can be expressed as

$$I(h) = \alpha I_0(h) + \beta I_1(h) + \gamma I_2(h).$$

–A basic tool for the study is the Picard-Fuchs equation, but we must add some $I_3(h)$ in order to find the "closed" differential equation of order 4:

$$G(h)\frac{d\tilde{I}}{dt} = A(h)\tilde{I},$$

where $\tilde{I} = (I_0, I_1, I_2, I_3)^T$.

– Some other methods (in complex or real) are needed to get the final answer:

The cyclicity is two.

The phase portraits of Q_3^H

E. Horozov and I. D. Iliev proved in 1994 that any cubic Hamiltonian, with at least one period annulus contained in its level curves, can be transformed into the following form

$$H(x,y) = \frac{1}{2}(x^2+y^2) - \frac{1}{3}x^3 + a\,xy^2 + \frac{1}{3}b\,y^3,$$

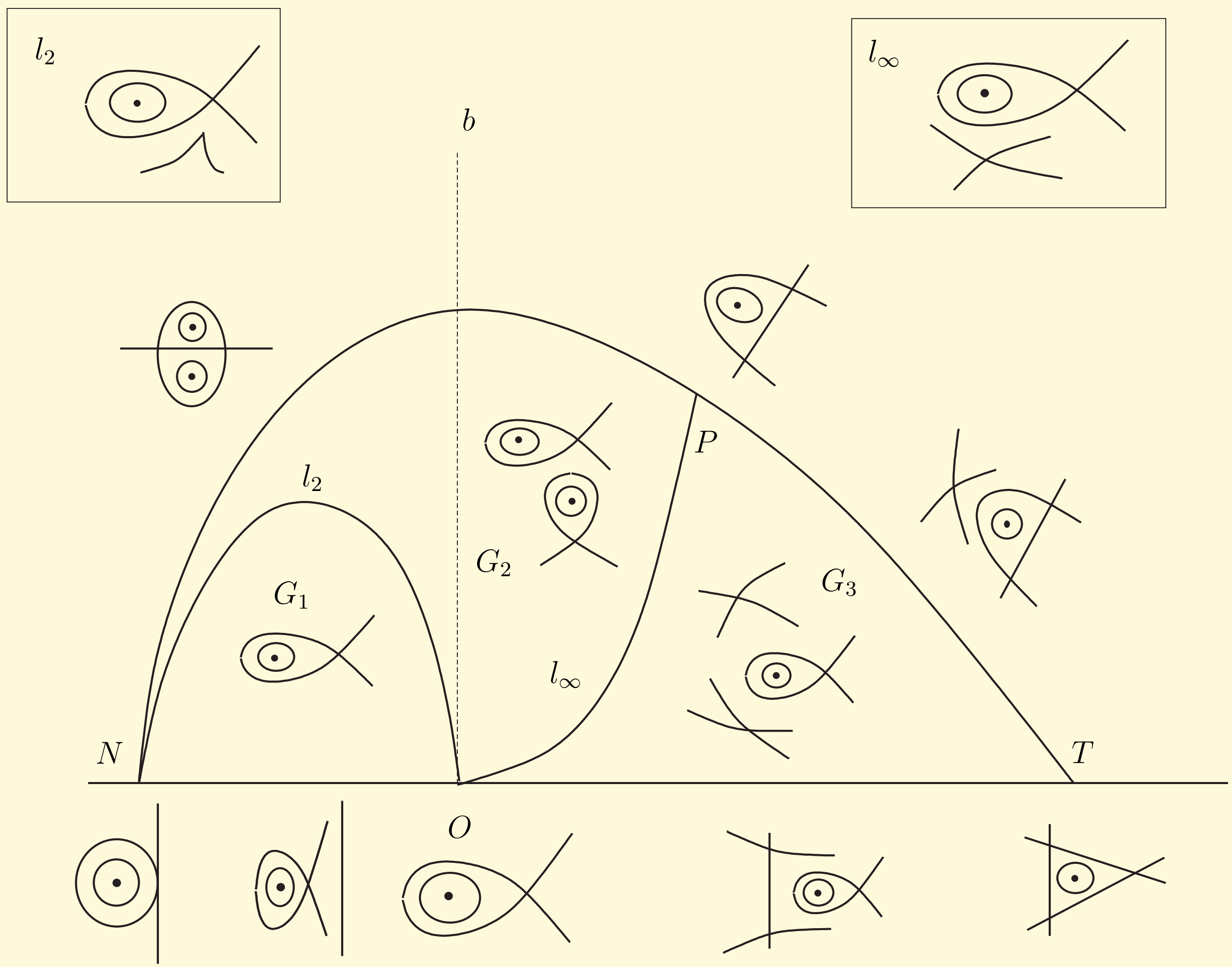
where a, b are parameters lying in the region

$$\bar{G} = \left\{(a,b) : -\frac{1}{2} \le a \le 1,\ 0 \le b \le (1-a)\sqrt{1+2a}\right\}.$$

X_H are generic if $(a,b) \in G = G_1 \cup l_2 \cup G_2 \cup l_\infty \cup G_3$: 5 cases;

and degenerate if $X_H \in \partial\bar{G}$: 8 cases.

The classification of all 13 phase portraits are shown in the next figure:


l_2
l_∞
b
l_2
P
G_1
G_2
G_3
l_∞
N
T
O

Results for generic cases

(1) $(a, b) \in l_\infty$, Z.-F. Zhang and C. Li Adv.in Math.(1987);

(2) $(a, b) \in G_3$, E. Horozov and I. D. Iliev Proc. London Math. Soc. (1994);

(3) $(a, b) \in G_1 \cup G_2$, L. Gavrilov Invent. Math (2001);

(4) $(a, b) \in l_2$, C. Li and Z.-H. Zhang Nonlinearity(2002);

- A unified proof by F. Chen, C. Li, J. Llibre and Z.-H. Zhang JDE(2006).

Basic idea of the unified proof

$$
\begin{aligned}
I(h) &= \alpha I_0(h) + \beta I_1(h) + \gamma I_2(h) \\
&= I_0(h)\left[\alpha + \beta p(h) + \gamma q(h)\right],
\end{aligned}
$$

where $I_0(h) \neq 0$ for $h \in (h_0, h_1]$, $h_0 \sim$ center point, $h_1 \sim$ the loop, and

$$p(h) = \frac{I_1(h)}{I_0(h)},\ \ q(h) = \frac{I_2(h)}{I_0(h)}.$$

In $(p, q)-$plane define a family of curves

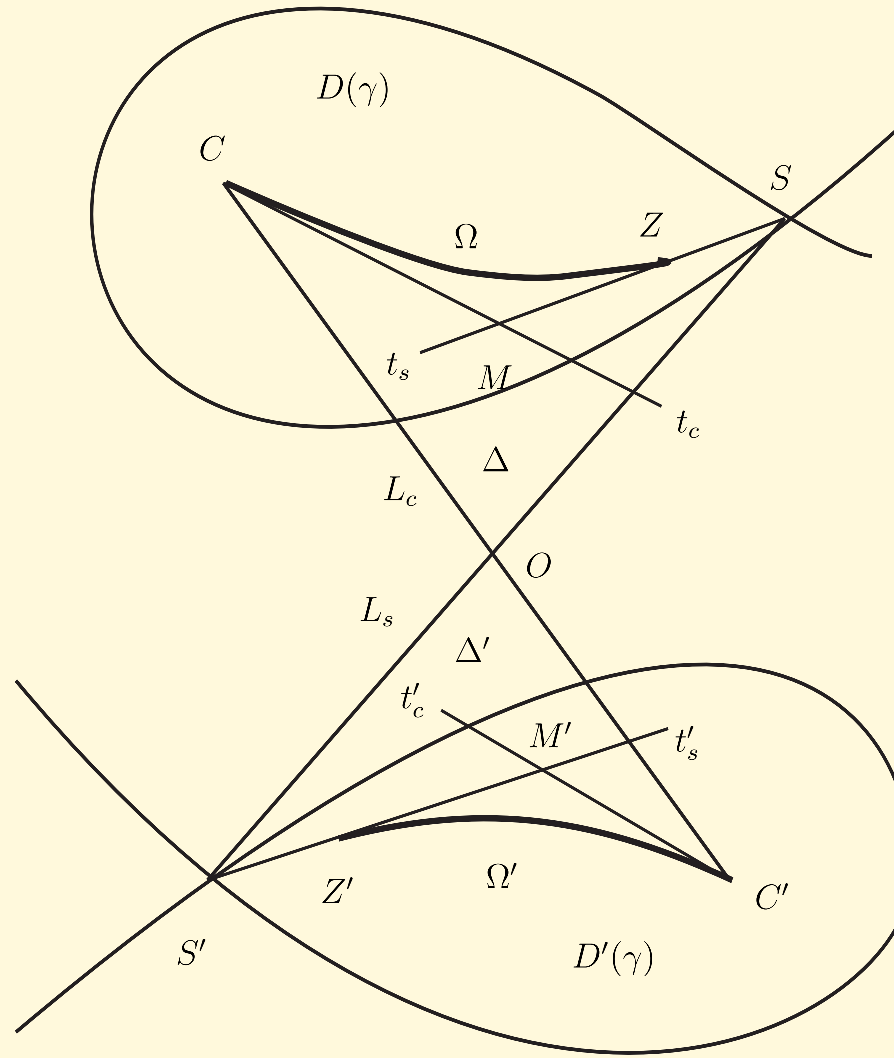
$$\Omega_{a,b} = \Big\{\big(p, q\big)(h) : h_0 \leq h \leq h_1\Big\},$$

and a family of straight lines

$$L_{\alpha\beta\gamma} : \quad \alpha + \beta p + \gamma q = 0.$$

Then

$$\#\{I(h) = 0\} = \#\{\,\Omega_{a,b} \cap\, L_{\alpha\beta\gamma}\}.$$



Implying all configurations of limit cycles:
(0,0), (1,0), (0,1), (2,0), (1,1), (0,2).

The study for degenerate cases

- Good things: The order of Picard-Fuchs equation is 3 or 2; and the Hamiltonian function contains only one parameter.

- Bad things: Instead of $I(h) = M_1(h)$, one has to study $M_2(h)$ or $M_3(h)$:

$$d(h, \epsilon) = P(h, \epsilon) - h = \epsilon M_1(h) + \epsilon^2 M_2(h) + \epsilon^3 M_3(h) + O(\epsilon^4).$$

  $M_2(h)$ and $M_3(h)$ may be pseudo-Abelian integrals.

Answer of the cyclicity of period annulus or annuli for degenerate cases:

- 3 for the Hamiltonian triangle;
- 2 for other cases.

Results for degenerate cases

(1) a saddle loop with a double singularity at infinity, Iliev Adv. Diff. Eq.(1996);

(2) a saddle loop with two more saddles, Chow,Li and Yi Ergod. Th.& Dyn. Sys.(2002);

(3) a triangular heteroclinic loop, Iliev JDE(1998);

(4) a hyperbolic segment loop, Zhao and Zhu Bull.Sci.Math(2001);

(5) a parabolic segment loop, Iliev Adv.Diff.Eq.(1996);

(6) an elliptic segment loop, Chow, Li and Yi Ergod.Th.& Dyn.Sys.(2002);

(7) a non-Morsean point, Zhao etc JDE(2000);

(8) a saddle loop, a pair of complex singularities, Gavrilov and Iliev Ergod.Th.& Dyn.Sys.(2000).

- A unified proof (except the case (3)) by Li and Llibre JDDE(2004).

The number of limit cycles bifurcating from Q_3^H

Only the following cases are open: The limit cycles may appear

- from the cusp point when $(a, b) \in l_2$ (in generic case);
- from infinity (in generic or degenerate cases);

  Partially studied by L. Gavrilov and I.D. Iliev, Can. J. Math, 2002.
- from the non-Morse point when $(a, b) = (-1/2, 0)$ (in degenerate case), can be changed to above case by the Poincar'e transformation;
- from the heteroclinic loop (the boundary of the period annulus in degenerate cases):

  partially studied by C. Li and R. Roussarie, JDE, 2004.

Perturbations of Integrable and non-Hamilton systems

$$\frac{dx}{dt} = P(x,y) + \epsilon f(x,y),$$
$$\frac{dy}{dt} = Q(x,y) + \epsilon g(x,y).$$

We need to use the integrating factor $\mu(x,y) \neq 0$, such that

$$\frac{dx}{dt} = \mu P + \epsilon \mu f = -\frac{\partial H(x,y)}{\partial y} + \epsilon \mu(x,y) f(x,y),$$
$$\frac{dy}{dt} = \mu Q + \epsilon \mu f = \quad \frac{\partial H(x,y)}{\partial x} + \epsilon \mu(x,y) g(x,y).$$

Now we have to study the pseudo-Abelian integral

$$I(h) = \oint_{\Gamma_h} \mu(x,y)(f(x,y)dy - g(x,y)dx),$$

here $H, \mu f, \mu g$ are not polynomials anymore (in general), the study becomes more difficult.

Study the perturbations of Q_4

- For generic Q_4, L. Gavrilov and I. D. Iliev [JMAA, 2009] proved that

$$\text{cyclicity} \leq 8$$

by using the Petrov method (the Argument Principle).

- For $X \in Q_4 \cap Q_3^R$, there are two cases:

$$\dot{z} = -iz + 4z^2 + 2|z|^2 \pm \bar{z}^2.$$

I. D. Ilive proved in both cases

$$\text{cyclicity} \leq 3$$

in "$-$" case: Proc. Royal Sci. Edinburg, 2007; in "$+$" case: Bulletin Sci. Math, 2008.

Study the perturbations of Q_3^{LV}

For the sub-class: with 2 or 3 invariant lines, i. e. the classical Lotka-Volterra class, H. Zoladek proved in 1994: the maximal number of zeros of the first order Melnikov function $M_1(h) = I(h)$ is

- 2, for generic Q_3^{LV};
- 1, for $Q_3^{LV} \cap Q_3^R \setminus Q_3^H$;
- 0, for $Q_3^{LV} \cap Q_3^R \cap Q_3^H$ (Hamiltonian triangle).

Remark: In degenerate cases, this number gives no information about the maximal number of limit cycles bifuracting from the annulus, it is needed to study $M_2(h)$ or $M_3(h)$. In fact, in Hamiltonian triangle case, the cyclicity is 3 (by Ilive, metioned above, it is the maximal number of zeros of $M_3(h)$).

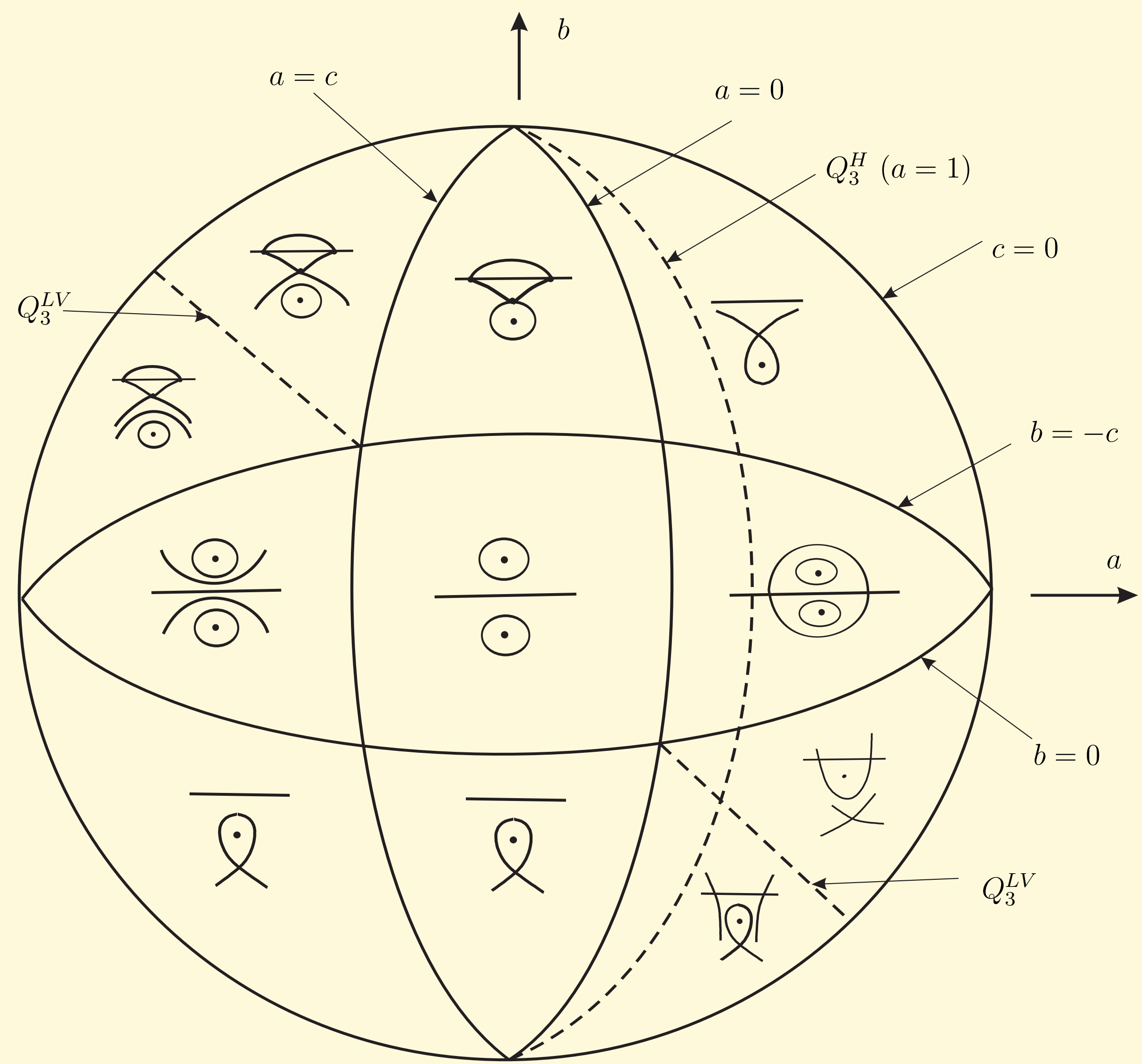
Study the perturbations of Q_3^R

- After perturbations, the reversible class Q_3^R may get more rich bifurcation phenomena.
- The general form of Q_3^R:

$$\frac{dx}{dt} = -y + ax^2 + by^2 \,, \qquad \frac{dy}{dt} = x(1+cy)\,.$$

The map $(x,t) \mapsto (-x,-t)$ does not change the orbits, only changes the direction on the flows, so it is called reversible.

- The topological classification of Q_3^R:



b
$a = c$
$a = 0$
$Q_3^H \ (a = 1)$
$c = 0$
Q_3^{LV}
$b = -c$
a
$b = 0$
Q_3^{LV}

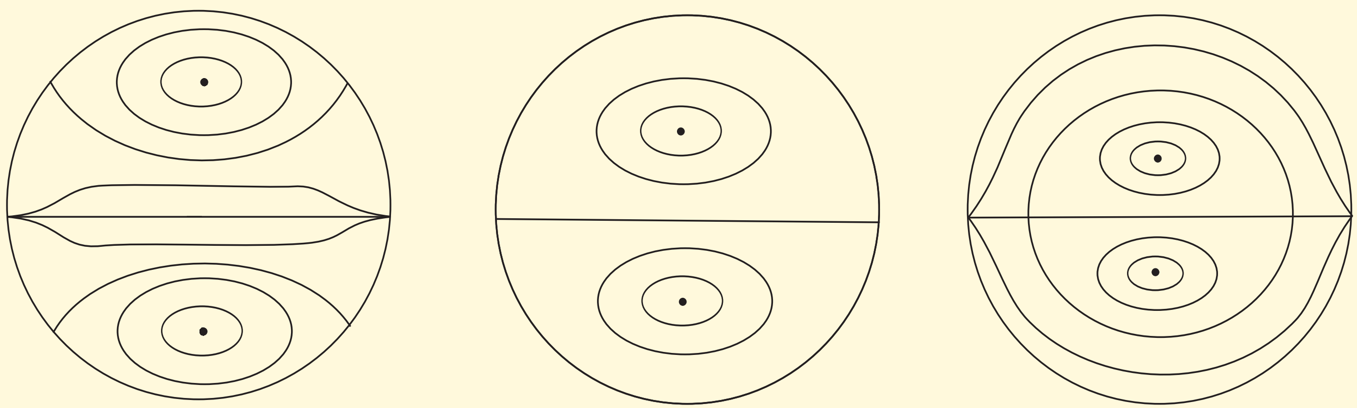
Reversible Quadratic Systems with Two Centers

- $c \neq 0$: taking $c = -2$ (by scaling);
- $0 < b < 2$; $b = 1$corresponds to the symmetry case;
- case 1: $-\infty < a < -2$,

  case 2: $-2 < a < 0$,

  case 3: $0 < a < +\infty$.

  Correspond to 3 kinds of topological phase portraits:



1 2 3

Phase portraits of reversible system with two centers.

The study of Q_3^R with quadratic perturbations

- Case 1, taking $a = -3$
  - $b = 1$, Dumortier, Li & Zhang, JDE, 139(1997)
  - $b \in (0, 2)$, Iliev, Li & Yu, Nonlinearity, 18(2005)

    One center：
  - $b = -1$, Peng, Acta. Math. Sinica (English Series), 18(2002)
  - $b \in (-\infty, 0) \setminus \{-1\}$, Yu & Li, JMAA, 269(2002)
  - $b = 3$ ($X \in Q_3^R \cap Q_3^{LV}$), Li & Llibre, Nonlinearity, 22(2009)
  - $b \in (2, +\infty) \setminus \{3\}$, Iliev, Li & Yu, CPAA, 9(2010)

The study of Q_3^R with quadratic perturbations

- Case 1, taking $a = -3$
  - $b = 1$, Dumortier, Li & Zhang, JDE, 139(1997)
  - $b \in (0, 2)$, Iliev, Li & Yu, Nonlinearity, 18(2005)

    One center：
  - $b = -1$, Peng, Acta. Math. Sinica (English Series), 18(2002)
  - $b \in (-\infty, 0) \setminus \{-1\}$, Yu & Li, JMAA, 269(2002)
  - $b = 3$ ($X \in Q_3^R \cap Q_3^{LV}$), Li & Llibre, Nonlinearity, 22(2009)
  - $b \in (2, +\infty) \setminus \{3\}$, Iliev, Li & Yu, CPAA, 9(2010)
- Case 2
  - $a = -\frac{3}{2}$, $b \in (0, 2)$, Liu, preprint
  - $a = -\frac{1}{2}$, $b \in (0, 2)$, Coll, Li & Prohens, Dis. Contin. Dyn. Sys. 24(2009)

The study of Q_3^R with quadratic perturbations

- Case 1, taking $a=-3$
  - $b=1$, Dumortier, Li & Zhang, JDE, 139(1997)
  - $b\in(0,2)$, Iliev, Li & Yu, Nonlinearity, 18(2005)

    One center：
  - $b=-1$, Peng, Acta. Math. Sinica (English Series), 18(2002)
  - $b\in(-\infty,0)\setminus\{-1\}$, Yu & Li, JMAA, 269(2002)
  - $b=3$ ($X\in Q_3^R\cap Q_3^{LV}$), Li & Llibre, Nonlinearity, 22(2009)
  - $b\in(2,+\infty)\setminus\{3\}$, Iliev, Li & Yu, CPAA, 9(2010)
- Case 2
  - $a=-\frac{3}{2}$, $b\in(0,2)$, Liu, preprint
  - $a=-\frac{1}{2}$, $b\in(0,2)$, Coll, Li & Prohens, Dis. Contin. Dyn. Sys. 24(2009)
- Case 3

  $a=2$, $b\in(0,2)$ and $a=-4$, Chen, Li, Liu & Llibre, Dis. Contin. Dyn. Sys. 16(2006)

A Difficulty: The Order of P-F Equation

The order K of the Picard-Fuchs equation ([CLLL]):

- $K < \infty$ if $a \in \mathbb{Q}$ $(a \neq 0, -1, -2)$; $\quad K = \infty$ if a is irrational.

A Difficulty: The Order of P-F Equation

The order K of the Picard-Fuchs equation ([CLLL]):

- $K < \infty$ if $a \in \mathbb{Q}$ $(a \neq 0, -1, -2)$; $K = \infty$ if a is irrational.
- $K(a, b) = K(-(a+2), 2-b)$, hence only need to consider $a > -1$.

A Difficulty: The Order of P-F Equation

The order K of the Picard-Fuchs equation ([CLLL]):

- $K < \infty$ if $a \in \mathbb{Q}$ ($a \neq 0, -1, -2$); $K = \infty$ if a is irrational.
- $K(a, b) = K(-(a+2), 2-b)$, hence only need to consider $a > -1$.
- If $|a| < 1$ and $a = \pm\frac{m}{n} \in \mathbb{Q}$, $0 < m < n$, $(m, n) = 1$, than $K = 2n$.
- If $a \geq 1$ is an integer, than $K = a + 2$.
- If $a > 1, a \in \mathbb{Q}$ and is not an integer, $a = [a] + \frac{m}{n}$, than $K = ([a] + 2)n$.

A Difficulty: The Order of P-F Equation

The order K of the Picard-Fuchs equation ([CLLL]):

- $K < \infty$ if $a \in \mathbb{Q}$ ($a \neq 0, -1, -2$); $K = \infty$ if a is irrational.
- $K(a, b) = K(-(a+2), 2-b)$, hence only need to consider $a > -1$.
- If $|a| < 1$ and $a = \pm\frac{m}{n} \in \mathbb{Q}$, $0 < m < n$, $(m, n) = 1$, than $K = 2n$.
- If $a \geq 1$ is an integer, than $K = a + 2$.
- If $a > 1, a \in \mathbb{Q}$ and is not an integer, $a = [a] + \frac{m}{n}$, than $K = ([a] + 2)n$.

In particular,

- $K = 3$ if $a = 1$ ($Q_3^R \cap Q_3^H$) or $a = -3$.
- $K = 4$ if $a = 2, -4, -\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}, -\frac{5}{2}$.
- $K \geq 5$, otherwise.

More Difficulties

- By using Picard-Fuchs equation it is possible to get some results locally, not globally.

More Difficulties

- By using Picard-Fuchs equation it is possible to get some results locally, not globally.
- In integrable and non-Hamiltonian case, it is hard to make analytic extension of the integrand function, so it is hard to use the Argument Principle (as Petrov did).

More Difficulties

- By using Picard-Fuchs equation it is possible to get some results locally, not globally.
- In integrable and non-Hamiltonian case, it is hard to make analytic extension of the integrand function, so it is hard to use the Argument Principle (as Petrov did).
- The method of complexification, developed by Arnold, Ilyashenko and Yakovenko, is valid only for the polynomial perturbation of polynomial Hamiltonian. It is invalid for the integrable and non-Hamiltonian case.

More Difficulties

- By using Picard-Fuchs equation it is possible to get some results locally, not globally.
- In integrable and non-Hamiltonian case, it is hard to make analytic extension of the integrand function, so it is hard to use the Argument Principle (as Petrov did).
- The method of complexification, developed by Arnold, Ilyashenko and Yakovenko, is valid only for the polynomial perturbation of polynomial Hamiltonian. It is invalid for the integrable and non-Hamiltonian case.
- Hence, it is necessary to develop some new methods and new techniques.

Example：[DLZ，1997]

- Taking $a = -3, b = 1, c = -2$, then any quadratic perturbation can be changed to the 3-parameters family (universal unfolding)

$$\dot{x} = -y - 3x^2 + y^2 + \delta(\mu_1 x + \mu_2 xy)\,,$$

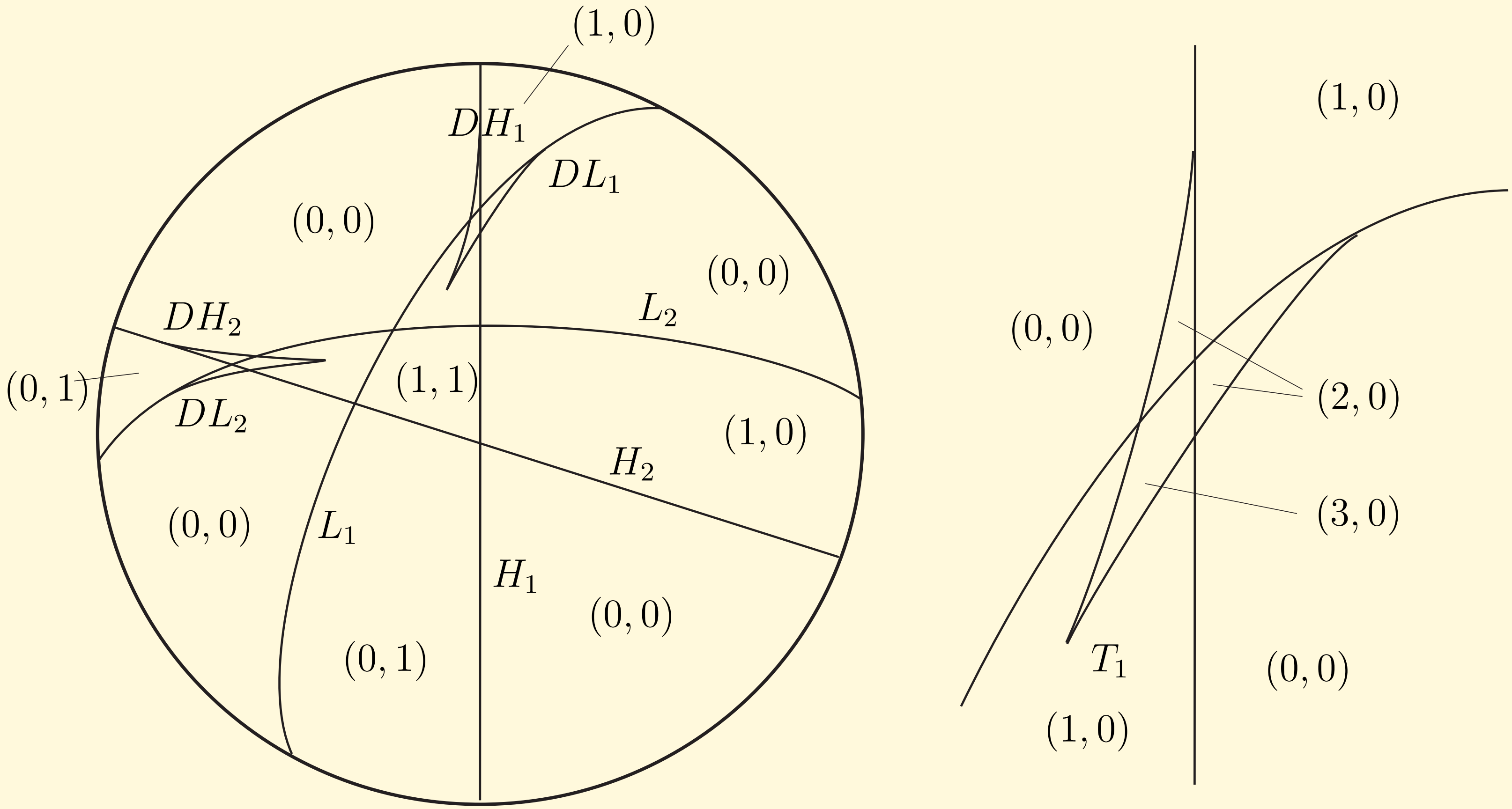
$$\dot{y} = x - 2xy + \delta\mu_3 x^2\,.$$

- Conclusion: the bifurcation diagram and the topological classification of the phase portraits are shown below:

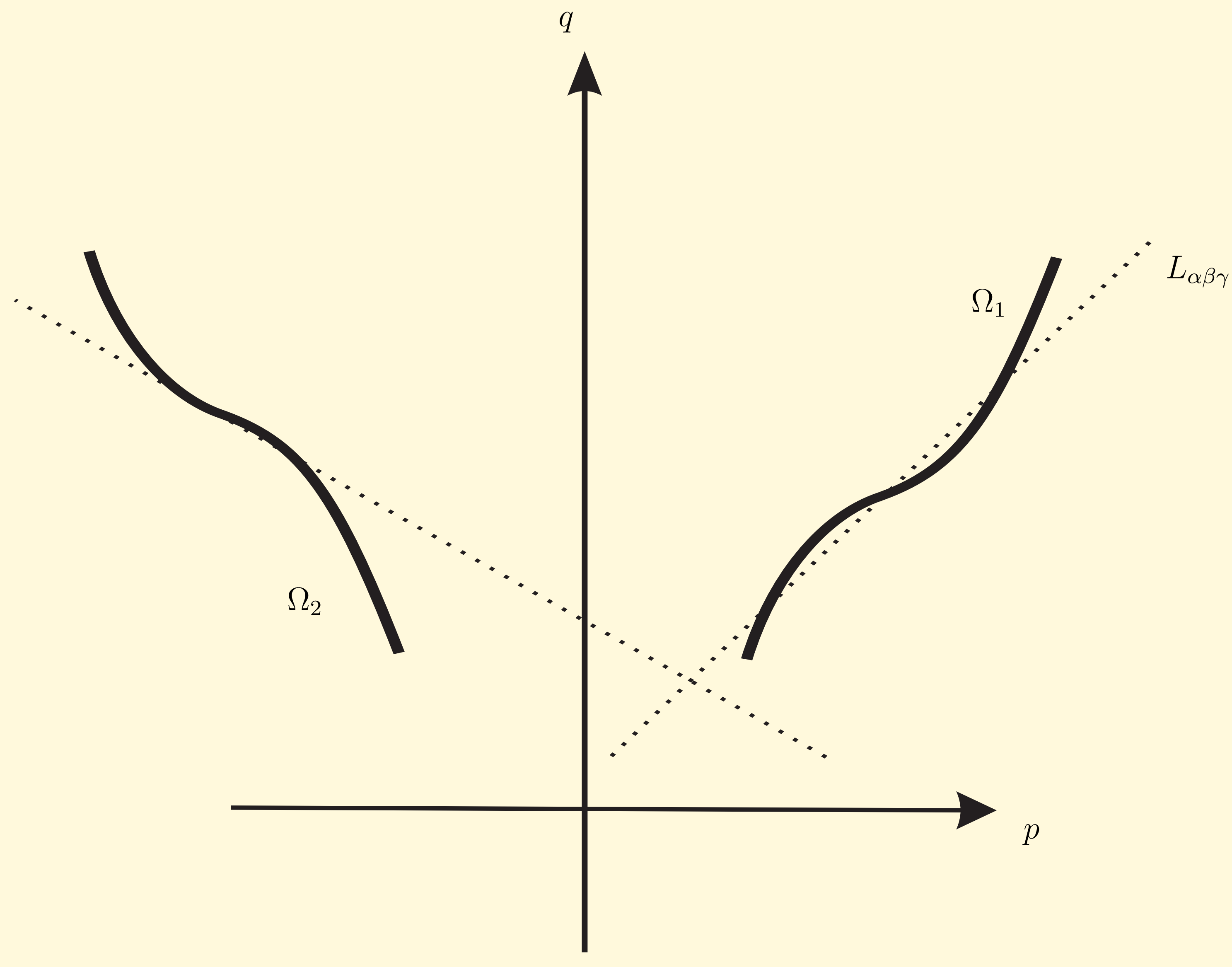
  (since the bifurcation diagram is unchanged under the scaling

$$(\mu_1, \mu_2, \mu_3) \ \mapsto \ (\varepsilon\mu_1, \varepsilon\mu_2, \varepsilon\mu_3)$$

  so we need only consider the intersection of the bifurcation diagram with half sphere, then project the diagram on a plane.)



$(1,0)$
DH_1
DL_1
$(0,0)$
$(0,0)$
L_2
DH_2
$(0,1)$
$(1,1)$
DL_2
$(1,0)$
H_2
$(0,0)$
L_1
H_1
$(0,0)$
$(0,1)$
$(1,0)$
$(0,0)$
$(2,0)$
$(3,0)$
T_1
$(0,0)$
$(1,0)$


q
$L_{\alpha\beta\gamma}$
Ω_1
Ω_2
p

THANK YOU VERY MUCH!