Global Stability of the Endemic Equilibrium of a Staged-progression Model with Immigration

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Fields Institute, May 14, 2010

Outline

- Introduction
- Earlier results
- Model formulation
- Model analysis
- Conclusion

Disease characters

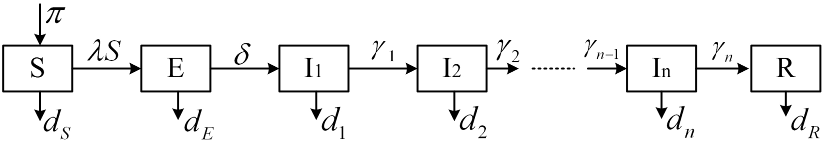
- For infectious disease with
  - long latent period (TB, Malaria etc)
  - long infectious period (HIV, HBV etc)
  - both long latent period and infectious period

  dividing the long latent or infectious period into multiple successive compartments (SP model)

- Use Gamma distribution for disease progression

- Use more general distribution for disease progression

Staged-progression (SP) model

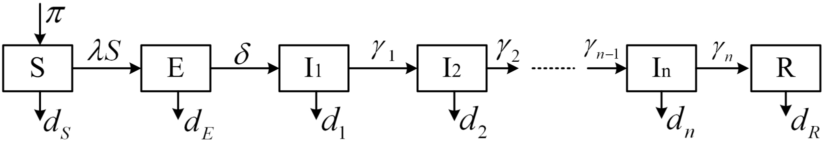


- S : Susceptible individuals
- E : Exposed but not yet infectious individuals
- I_1 : Infectious individuals in the 1st stage
- I_i : Infectious individuals in the i-th stage ($i = 2, \cdots, n$)
- R : Removed/recovered individuals who are not infectious

Total active population: $N = S + E + \sum_{i=1}^{n} I_i$

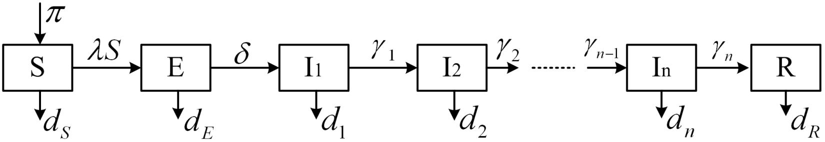
Force of infection: $\lambda S = \sum_{i=1}^{n} \beta_i I_i S$ (bilinear) or $\lambda S = \sum_{i=1}^{n} \frac{\beta_i I_i S}{N}$ (standard)

Staged-progression (SP) model



- π : recruitment to susceptible class
- λS: force of infection (bilinear or standard)
- δ : transfer rate to the 1-st infectious class ($1/\delta$ is latent period)
- γ_i : transfer rate to the (i+1)-th stage ($i = 1, \cdots, n-1$)
- $d_{\#}$: removal rate due to natural death and/or disease induced death

Staged-progression (SP) model

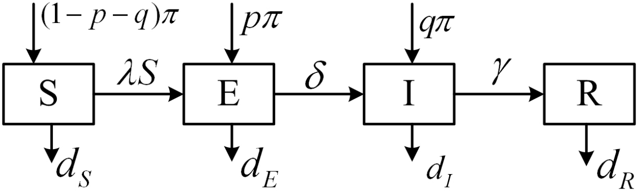


When λS is bilinear incidence, global dynamics were established (Guo & Li MBE, 2006).

Theorem

When $R_0 \leq 1$, the disease-free equilibrium is globally asymptotically stable (GAS); when $R_0 > 1$, the DFE is unstable and the endemic equilibrium is GAS.

SEIR model with constant immigration



$$\begin{cases} S' = (1-p-q)\pi - \lambda S - d_S S, \\ E' = p\pi + \lambda S - (d_E + \delta)E, \\ I' = q\pi + \delta E - (d_I + \gamma)I, \\ R' = \gamma I - d_R R. \end{cases}$$

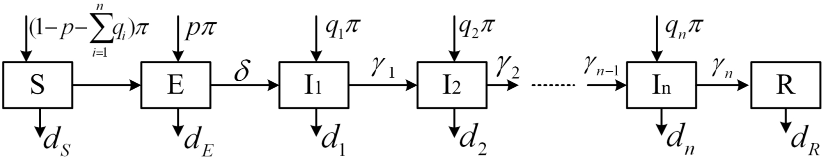
Dynamical behaviors:

- No disease-free equilibrium
- No basic reproduction number
- The unique endemic equilibrium is GAS

Earlier results for immigration model

- F. Brauer and P. van den Driessshe, Math. Biosci. (2001).
  - SIR model with immigration to infective and bilinear incidence
  - GAS: (reduce to 2D, Poincare-Bendixson Theorem)
- C. McCluskey and P. van den Driessche, JDDE. (2004).
  - SEI model with immigration to latent and infective and standard incidence
  - GAS: (Li and Muldowney's Geometric approach by compound matrix)
- G. Li, W. Wang and Z. Jin, Chaos, Solit. Fract. (2006).
  - SEIR model with immigration to latent and standard incidence
  - GAS: (Li and Muldowney's Geometric approach by compound matrix)
- H. Guo and J. Wu, MBE. (2010).
  - SEEI model with immigration to latent classes and bilinear incidence
  - GAS: Lyapunov function (Volterra-Goh type)

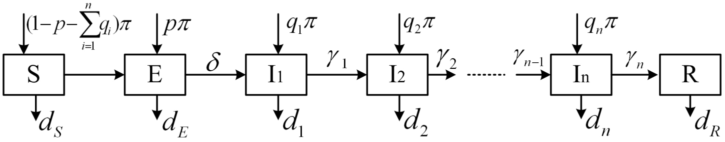
Staged-progression model with constant immigration



- p : the fraction of constant immigration to latent class
- q_i: the fraction of constant immigration to i-th infectious class

If $p = q_i = 0$, the system reduces to standard SP model.

System of equations



$$
\begin{cases}
S' = (1 - p - \sum_{i=1}^{n} q_i)\pi - \sum_{i=1}^{n} \beta_i S I_i - d_S S, \\
E' = p\pi + \sum_{i=1}^{n} \beta_i S I_i - (d_E + \delta)E, \\
I_1' = q_1\pi + \delta E - (d_1 + \gamma_1)I_1, \\
I_i' = q_i\pi + \gamma_{i-1}I_{i-1} - (d_i + \gamma_i)I_i, \qquad i = 2, \cdots, n. \\
R' = \gamma_n I_n - d_R R.
\end{cases}
$$

Reduced system

$$
\begin{cases}
S' = (1 - p - \sum_{i=1}^{n} q_i)\pi - \sum_{i=1}^{n} \beta_i S I_i - d_S S, \\
E' = p\pi + \sum_{i=1}^{n} \beta_i S I_i - (d_E + \delta)E, \\
I_1' = q_1\pi + \delta E - (d_1 + \gamma_1) I_1, \\
I_i' = q_i\pi + \gamma_{i-1} I_{i-1} - (d_i + \gamma_i) I_i, & i = 2, \cdots, n.
\end{cases}
$$

Endemic equilibrium $P^* = (S^*, E^*, I_1^*, \cdots, I_n^*)$, i.e., $S^* > 0, E^* > 0, I_i^* > 0$.

GAS of endemic equilibrium

Theorem

The reduced system has a unique endemic equilibrium P^* which is GAS if $p^2 + q_i^2 \neq 0$.

Proof: Consider a global Lyapunov function

$$V(x) = \left(S - S^* - S^* \ln \frac{S}{S^*}\right) + \left(E - E^* - E^* \ln \frac{E}{E^*}\right) + \sum_{i=1}^{n} x_i \left(I_i - I_i^* - I_i^* \ln \frac{I_i}{I_i^*}\right)$$

where $x_i > 0$. P^* is the endemic equilibrium of the system.

By mean inequality

$$a_1 + a_2 + \cdots + a_n \geq n \sqrt[n]{a_1 \cdot a_2 \cdots a_n}, \quad \text{for} \quad a_i \geq 0, \quad i = 1, \cdots, n.$$

We can prove

$$V' = \left(1 - \frac{S^*}{S}\right) S' + \left(1 - \frac{E^*}{E}\right) E' + \sum_{i=1}^{n} x_i \left(1 - \frac{I_i^*}{I_i}\right) I_i' \leq 0.$$

Conclusions

- high-dimensional system with immigration preserves same behaviors
- immigration factor drives endemic equilibrium more 'endemic'
- global Lyapunov functions are really useful in proving GAS of EE