Mosquitos driven oscillations in vector-borne diseases

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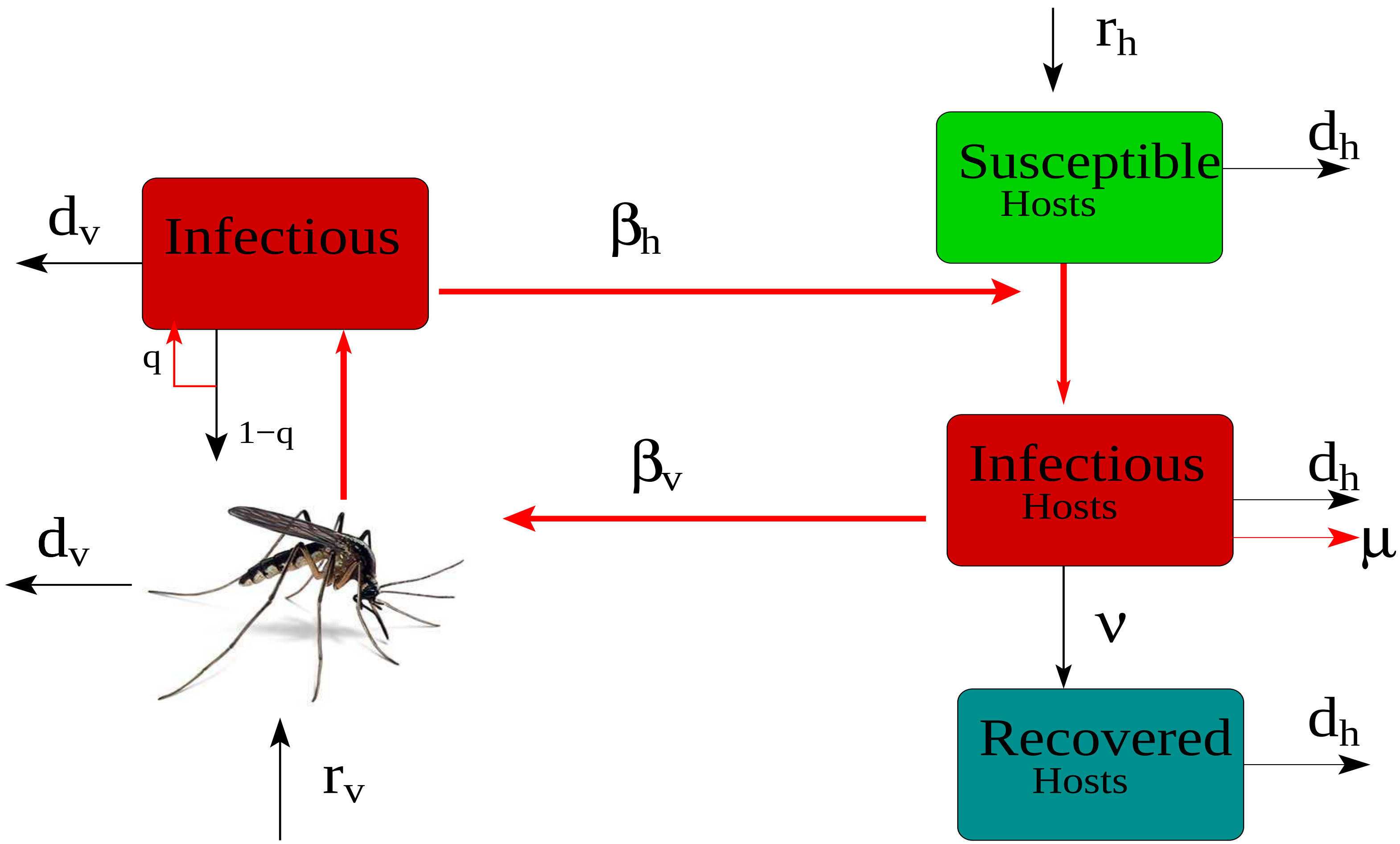
Outline

- ◇ A general model for vector borne diseases.
- ◇ Analytical results about bifurcation and oscillations.
- ◇ Case study for West Nile virus
- ◇ Numerical simulations

1. Mosquito-born Diseases:

1.1 The flow diagram



1.2 Main parameters

- For vectors:
  - d_v is the natural death rate.
  - r_v is the production rate.
  - $q \in [0, 1]$ is the vertical transmission rate.
- For Hosts:
  - μ is the disease induced death rate
  - ν is the recovery rate
- Interaction parameters:
  - β_v is the contact transmission rate from hosts to vectors.
  - β_h is the contact transmission rate from vectors to hosts.

1.3 The Model

$$\left\{\begin{array}{rcl}
\dfrac{\mathrm{d}V_s(t)}{\mathrm{d}t} & = & r_v(V(t-\tau_v))\Big(V_s(t-\tau_v)+(1-q)V_i(t-\tau_v)\Big)e^{-d_v\tau_v} \\
 & & -d_vV_s(t) - \beta_v(V(t),H(t))V_s(t)H_i(t), \\
\dfrac{\mathrm{d}V_i(t)}{\mathrm{d}t} & = & qr_v(V(t-\tau_v))V_i(t-\tau_v)e^{-d_v\tau_v} - d_vV_i(t) \\
 & & +\beta_v(V(t),H(t))V_s(t)H_i(t), \\
\dfrac{\mathrm{d}H_s(t)}{\mathrm{d}t} & = & r_h(H(t))H(t) - \beta_h(V(t),H(t))V_i(t)H_s(t) - d_hH_s(t), \\
\dfrac{\mathrm{d}H_i(t)}{\mathrm{d}t} & = & \beta_h(V(t),H(t))V_i(t)H_s(t) - (\mu+\nu+d_h)H_i(t), \\
\dfrac{\mathrm{d}H_r(t)}{\mathrm{d}t} & = & \nu H_i(t) - d_hH_r(t),
\end{array}\right.$$

1.2 Model for mosquitoes

- ◇ The total vectors $V(t)$ satisfies

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = r_v(V(t-\tau_v))V(t-\tau_v)e^{-d_v\tau_v} - d_vV(t) \tag{1}$$

- ◇ The disease has no impact on the total mosquitoes.
- ◇ Take $r_m(V(t-\tau_v)) = r_m e^{-\alpha V(t-\tau_v)}$.
- ◇ There exists $\tau_2 > \tau_1$ so that

$\diamond$ If we let $V = V_s + V_i$, then

$$
\left\{
\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= r_v(V(t-\tau_v))V(t-\tau_v)e^{-d_v\tau_v} - d_vV(t) \\
\frac{\mathrm{d}V_i(t)}{\mathrm{d}t} &= qr_v(V(t-\tau_v))V_i(t-\tau_v)e^{-d_v\tau_v} - d_vV_i(t) \\
&\quad +\beta_v(V(t),H(t))(V(t)-V_i(t))H_i(t), \\
\frac{\mathrm{d}H_s(t)}{\mathrm{d}t} &= r_h(H(t))H(t) - \beta_h(V(t),H(t))V_i(t)H_s(t) - d_hH_s(t), \\
\frac{\mathrm{d}H_i(t)}{\mathrm{d}t} &= \beta_h(V(t),H(t))V_i(t)H_s(t) - (\mu+\nu+d_h)H_i(t), \\
\frac{\mathrm{d}H_r(t)}{\mathrm{d}t} &= \nu H_i(t) - d_hH_r(t),
\end{aligned}
\right.
\tag{2}
$$

2. Analysis of the model:

2.1 Linearization at equilibria

- ◇ Assume that (2) has at least one positive equilibrium denoted as $E^* = (V^*, V_i^*, H_s^*, H_i^*, H_r^*)$.
- ◇ The characteristic equation evaluating at E^* is

$$(-d_v - \lambda + (r_v' V + r_v)e^{-(\lambda+d_v)\tau_v})\mathcal{F}(\lambda) = 0$$

- ◇ The characteristic equation is a product of the characteristic of total vectors and $\mathcal{F}(\lambda)$, which is given in the following page.
- ◇ This means that the full system may undergo a Hopf bifurcation if the model for total vectors has a Hopf bifurcation.

◇

$$
\begin{aligned}
\mathcal{F}(\lambda) = \quad & \beta_h H_s(d_h+\lambda)\left(\mu \tfrac{\partial \beta_v}{\partial H} V_s H_i + \beta_v V_s (r_h + r'_h H - d_h - \lambda)\right) \\
& + \Bigg((r_h + r'_h H - d_h - \lambda)(\lambda + (\mu + \nu + d_h) + \beta_h V_i)(d_h + \lambda) \\
& + (r_h + r'_h H - d_h - \lambda)\nu \beta_h V_i - \mu(d_h+\lambda)(\beta_h V_i + \tfrac{\partial \beta_h}{\partial H} V_i H_s) \Bigg) \\
& \cdot \left(-d_v - \lambda - \beta_v H_i + q r_v e^{-(\lambda + d_v)\tau_v} \right)
\end{aligned}
$$

◇ It is also possible that the full model undergoes a Hopf bifurcation due to $\mathcal{F}(\lambda) = 0$ has a pair of pure imaginary roots cross the imaginary axis.

◇ Ongoing work: ???Hopf-Hopf bifurcation or double Hopf???

3. A case study for West Nile virus:

3.1 The model

$$
\left\{
\begin{array}{rcl}
\dfrac{dN_m(t)}{dt} & = & r_m N_m(t-\tau)e^{-d_j\tau}e^{-\alpha N_m(t-\tau)} - d_m N_m(t) \\
\dfrac{dM_i(t)}{dt} & = & q r_m M_i(t-\tau)e^{-d_j\tau}e^{-\alpha N_M(t-\tau)} - d_m M_i(t) \\
 & & +\beta_m \frac{(N_m - M_i(t))\kappa B_i(t)}{N_b(t)}, \\
\dfrac{\mathrm{d}B_s(t)}{\mathrm{d}t} & = & r_b - \frac{\kappa\beta_b M_i(t) B_s(t)}{N_b(t)} - d_b B_s(t), \\
\dfrac{\mathrm{d}B_i(t)}{\mathrm{d}t} & = & \frac{\kappa\beta_b M_i(t) B_s(t)}{N_b(t)} - (\mu + \nu + d_b)B_i(t), \\
\dfrac{\mathrm{d}B_r(t)}{\mathrm{d}t} & = & \nu B_i(t) - d_b B_r(t),
\end{array}
\right.
\tag{3}
$$

$\diamond$ For total mosquitoes, there exists $N_m^* = \frac{1}{\alpha}\ln\left(\frac{r_m}{d_m e^{d_j\tau}}\right) > 0$ if $\tau < \tau_2 = \frac{1}{d_j}\ln\left(\frac{r_m}{d_m}\right)$.

3.2 Hopf bifurcation

- The characteristic equation at N_m^*

$$\lambda + d_m - d_m \left(1 - \ln\left(\frac{r_m}{d_m e^{d_j \tau}}\right)\right) e^{-\lambda\tau} = 0. \tag{4}$$

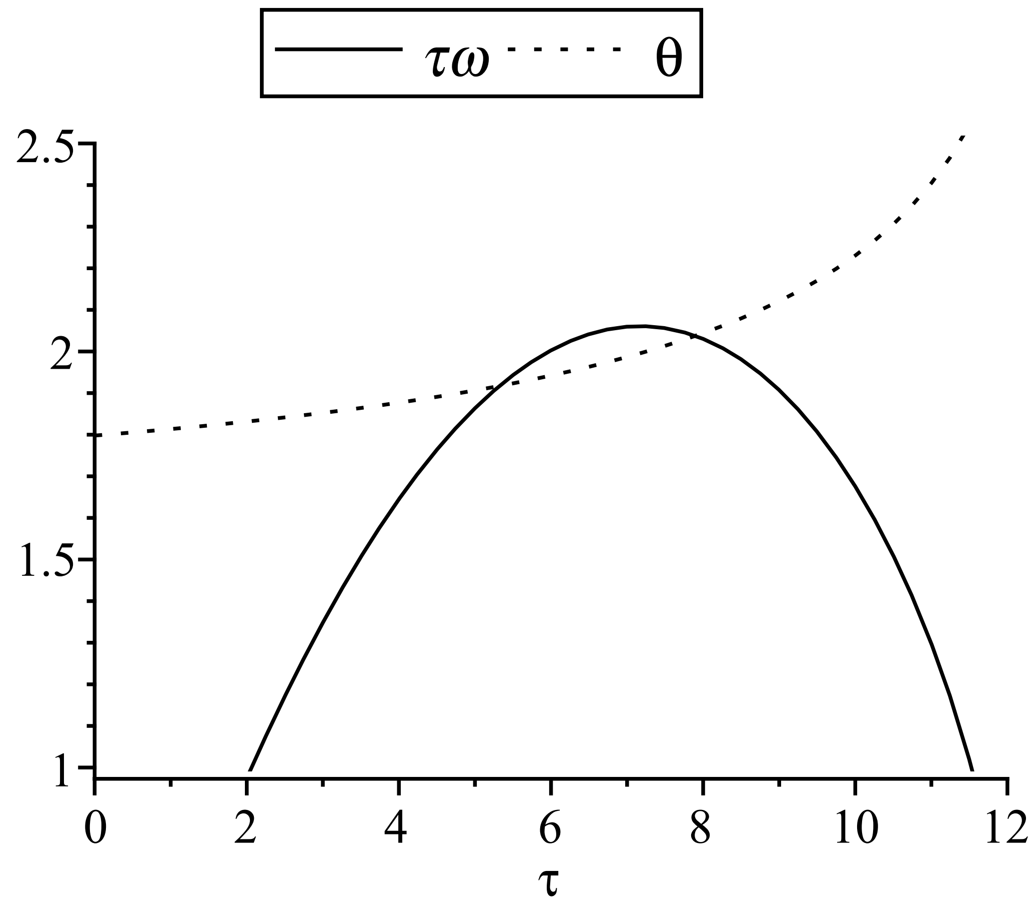
- Assume that $0 \leqslant \tau < \tau_1 (= \frac{1}{d_j}\left(\ln\left(\frac{r_m}{d_m}\right) - 2\right))$ and define

$$\omega(\tau) = d_m \sqrt{\ln\left(\frac{r_m}{d_m e^{d_j \tau}}\right)\left(\ln\left(\frac{r_m}{d_m e^{d_j \tau}}\right) - 2\right)},$$

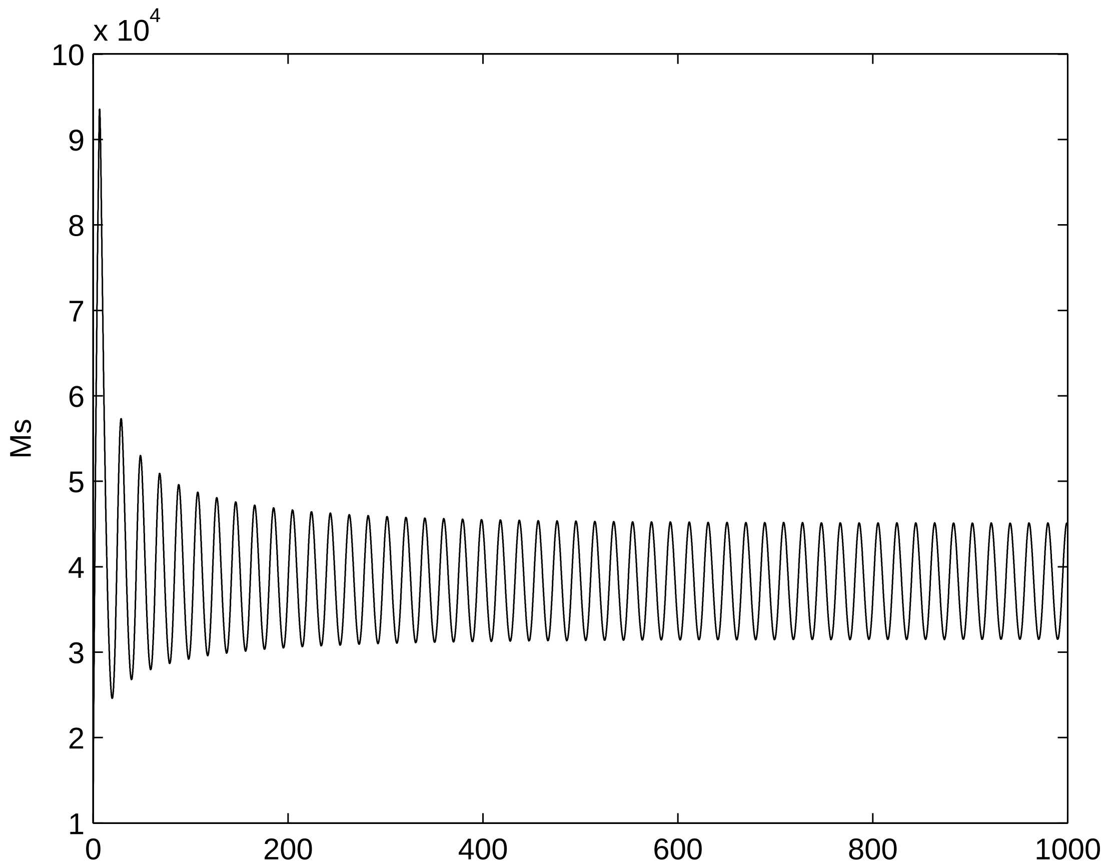
$$\theta(\tau) = \arccos\left(\frac{1}{1 - \ln\left(\frac{r_m}{d_m e^{d_j \tau}}\right)}\right),$$

- Functions $\theta(\tau) + 2n\pi$ and $\tau\omega(\tau)$ has an intersection on $[0, \tau_1]$.
- They are not tangent to each other at the intersection, there is a Hopf bifurcation (see Beretta and Kuang 2002)

3.3 Determine the critical delay values

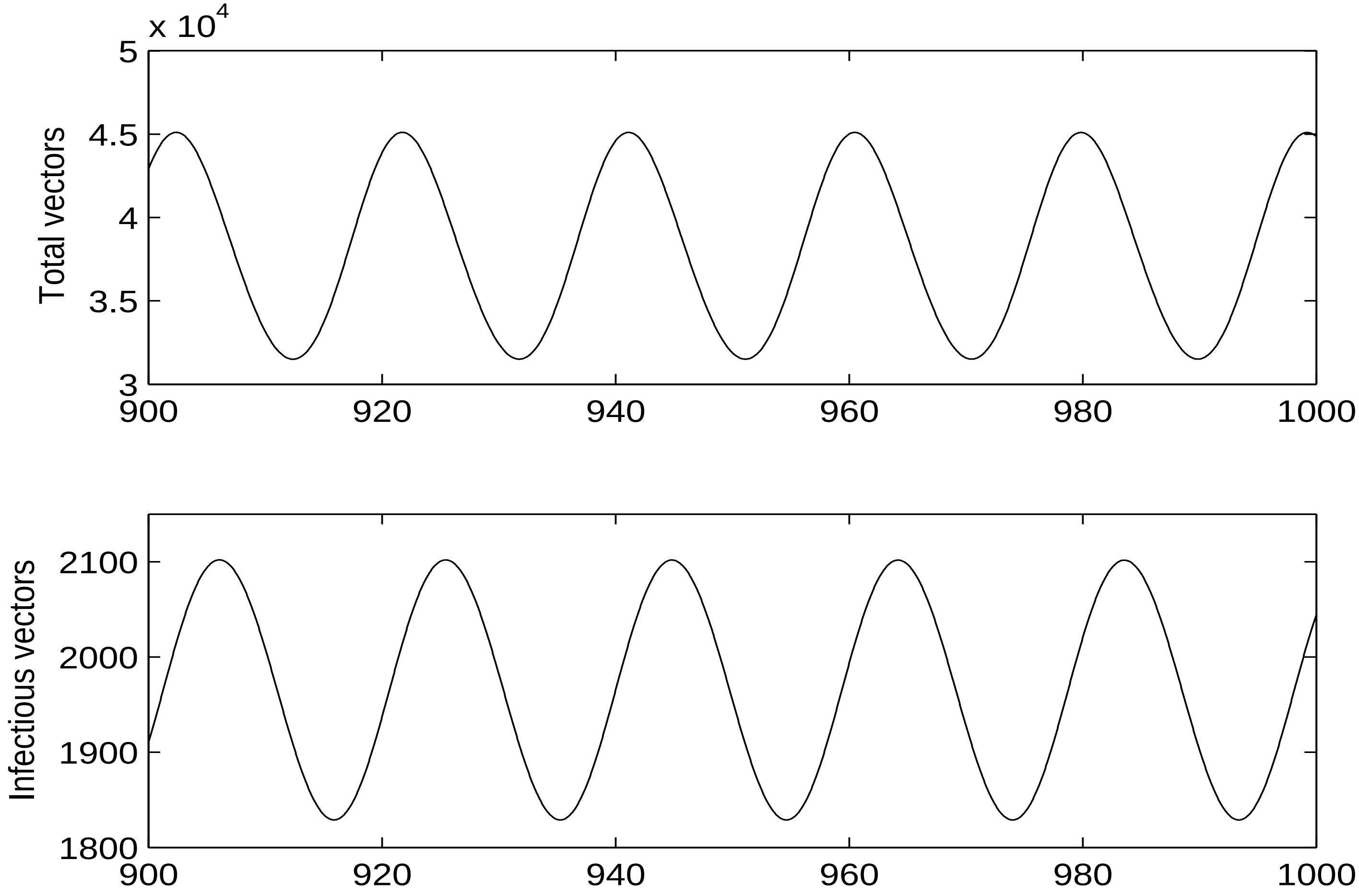


Oscillations in total mosquitoes



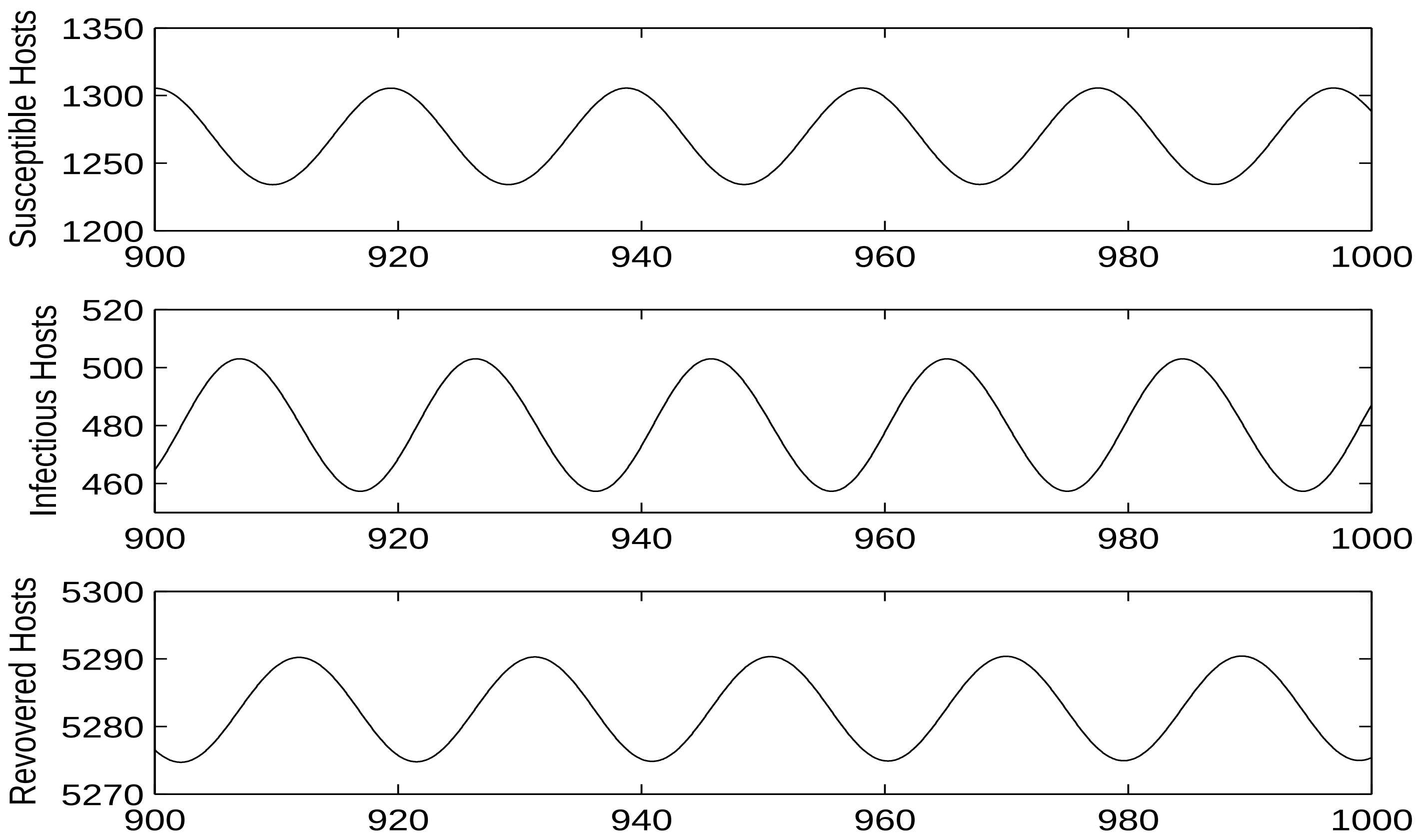
Oscillations due to vectors' forcing

◇ N_m, M_i:



Oscillation in birds

◇ B_s, B_i, and B_r:



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