Approximating the Stability Region of a Neural Network with a General Distribution of Delays.

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Motivation

Typical delay differential equation model with discrete delay:

$$\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t-\tau)).$$

Delay $\tau > 0$ arises due to gestation, maturation, propagation of information from one part of system to the other.

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$g(u)$ is the *kernel* of the distribution. Can be thought of as a probability distribution. Satisfies

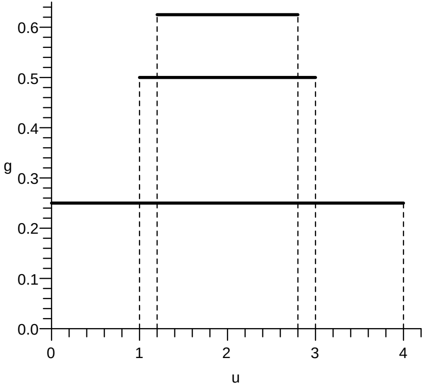
$$\int_0^\infty g(u)\,du = 1.$$

References: Cushing (1977), MacDonald (1978)

Motivation

Uniform distribution with mean τ

$$g(u) = \begin{cases} \frac{1}{\tau\rho}, & \text{for } \tau(1-\frac{\rho}{2}) \le u \le \tau(1+\frac{\rho}{2}) \\ 0, & \text{elsewhere.} \end{cases}$$

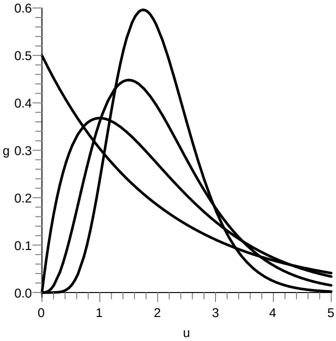


$$\tau = 2 \ \rho = 0.8, 1, 2$$

Motivation

Gamma distribution with mean $\tau = \frac{p}{a}$.

$$g(u) = \frac{u^{p-1} a^p e^{-au}}{\Gamma(p)},$$



$$\tau = 2 \; p = 1, 2, 4, 8$$

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References: Wolkowicz et al. (1997,1999); Bernard et al. (2001); Adimy et al. (2005); Arino et al. (2006); Ruan (2006); Gopalsamy et al. (1994, 1992, 2008); Chen (2002); Faria et al. (2008)

Overview

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- Stability Analysis - Distribution Independent Results

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- Summary/Conclusions

Model - Artificial Neural Network

Artificial neural network with identical neurons

$$Cv_k'(t) = -\frac{v_k(t)}{R} + \sum_{j=1}^{n} a_{kj} f(v_j(t)), \quad k = 1, \ldots, n.$$

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- C, R are capacitance and resistance of each neuron
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- C, R are capacitance and resistance of each neuron
- a_{kj} are synaptic weights
- $f(u)$ is the activation function. Assumed properties:
  - monotonically increasing and differentiable on $(-\infty, \infty)$
  - $f(0) = 0$, $0 < f'(x) \leq f'(0) = \beta$ for any $x \in \mathbb{R}$
  - $\lim_{x \to \pm\infty} f(x) = \pm 1$

References: Cohen-Grossberg (1983); Hopfield (1984)

Model - Neural Network with Discrete Delays

Dividing through by C and taking into account propagation time and signal processing time:

$$v_k'(t) = -\alpha v_k(t) + \sum_{j=1}^{n} w_{kj} f(v_j(t-\tau)), \quad k = 1, \ldots, n.$$

where

- $\alpha = \frac{1}{CR}$ is the intrinsic decay rate of the neuron
- $w_{jk} = \frac{a_{kj}}{C}$, $\mathbf{W} = [w_{jk}]$ is the connection matrix
- $\tau > 0$ is the time delay

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References: Grossberg (1967, 1968); Marcus and Westervelt (1989); See literature reviews in: Horikawa and Kitajima (2009); Singh (2009); Yuan et al. (2008)

Model - Neural Network with Distribution of Delays

Allowing for delay to vary from one instance to the next:

$$v_k'(t) = -\alpha v_k(t) + \sum_{j=1}^{n} w_{kj} \int_0^\infty f(v_j(t-u))g(u)\,du, \quad k = 1, \ldots, n.$$

where $g(u)$ is a the kernel of the distribution with

$$\int_0^\infty g(u)\,du = 1.$$

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Rescale so that the mean delay $\tau = \int_0^\infty u g(u)\, du$ occurs explicitly:

$$v_k'(t) = -\alpha\tau v_k(t) + \tau \sum_{j=1}^{n} w_{kj} \int_0^\infty f(v_j(t-u))\hat{g}(u)\, du, \quad k = 1, \ldots, n.$$

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Model admits the trivial solution. Linearization:

$$x_k'(t) = -\alpha\tau x_k(t) + \tau \sum_{j=1}^{n} w_{kj} \int_0^\infty x_j(t-u)\hat{g}(u)\, du, \quad k = 1, \ldots, n.$$

Linearization

Vector form of linearization:

$$\dot{\mathbf{x}}(s) = -\alpha\tau\mathbf{x}(s) + \beta\tau\mathbf{W}\int_0^\infty \mathbf{x}(s-v)\hat{g}(v)\,dv,$$

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$\mathbf{W}$ symmetric implies z_k real, otherwise z_k complex.

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Let $\mathbf{y} = e^{\lambda s}\mathbf{c}$ to find characteristic equation.
Reference: Bélair et al. (1996)

Stability Analysis

Characteristic equation:

$$\Delta(\lambda) = \prod_{k=1}^{n} \Delta_k(\lambda) = \prod_{k=1}^{n} \left(\lambda + \alpha\tau - \beta\tau z_k \int_0^\infty e^{-\lambda v} \hat{g}(v)\, dv \right) = 0.$$

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Stability depends on zeros of $\Delta_k(\lambda)$ which depends on eigenvalues, z_k, of connection matrix, and parameters α, β, τ.

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The trivial equilibrium point will be asymptotically stable if all roots of each $\Delta_k(\lambda)$ have negative real part.

The trivial equilibrium point will be unstable if one $\Delta_k(\lambda)$ has a root with positive real part.

Theorem 1

If **W** is symmetric and $\int_0^\infty \hat{g}(v) e^{-\lambda v}\, dv$ is analytic in $\mathrm{Re}(\lambda) \geq 0$, then the trivial equilibrium point is locally asymptotically stable if, for each $k = 1, \ldots, n$, either

$$(1)\ |z_k| < \frac{\alpha}{\beta},$$

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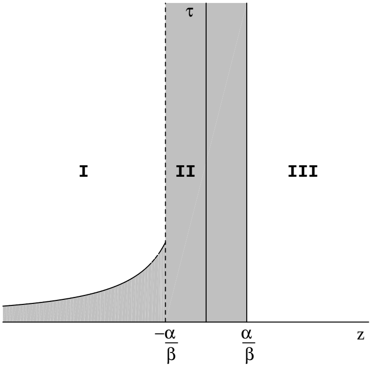

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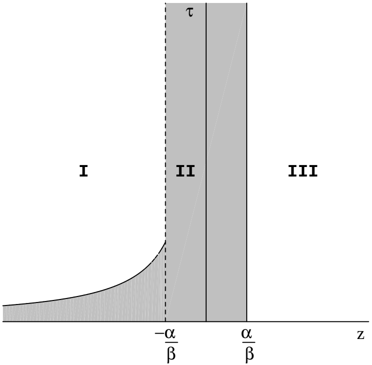

Recall

$$\Delta_k(\lambda) = \lambda + \alpha\tau - \beta\tau z_k \int_0^\infty e^{-\lambda v}\hat{g}(v)\,dv$$

Let $z_k = a_k + ib_k$. The $\lambda = i\omega$ is a zero of $\Delta_k(\lambda)$ if

$$\begin{aligned} \alpha &= \beta a_k C(\omega) + \beta b_k S(\omega), \\ -\omega &= \beta\tau a_k S(\omega) - \beta\tau b_k C(\omega). \end{aligned}$$

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Note: For model with a discrete delay $C(\omega) = \cos(\omega)$, $S(\omega) = \sin(\omega)$.

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The trivial equilibrium point is locally asymptotically stable for any distribution, g, if $|z_k| < \alpha/\beta$, $k = 1, 2, \ldots, n$.

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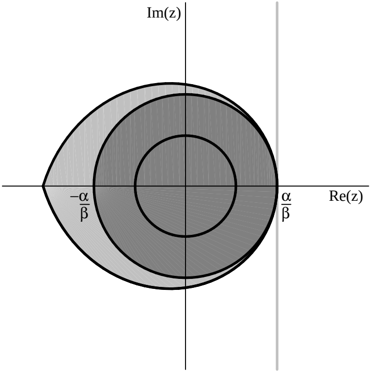

Theorem 4

Let α, β and τ be fixed. The trivial equilibrium point is locally asymptotically stable if for each $k = 1, 2, \ldots, n$ the point (a_k, b_k) lies inside the curve $(R(\omega), I(\omega))$, $\omega \in [-\bar{\omega}, \bar{\omega}]$ where

$$R(\omega) = \frac{\tau\alpha C(\omega) - \omega S(\omega)}{\beta\tau(C^2(\omega) + S^2(\omega))}$$
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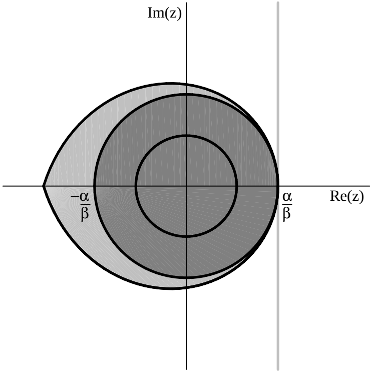

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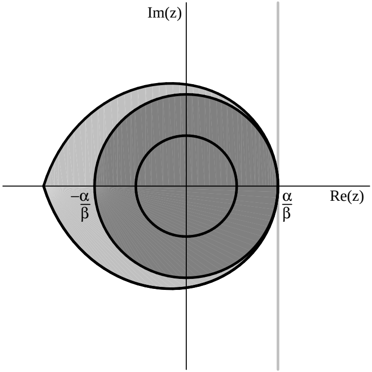
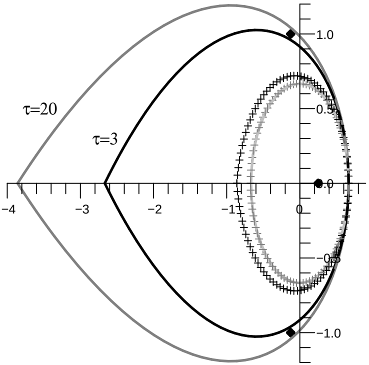


We will call the region defined by $(R(\omega), I(\omega))$ the **stability region**.

Theorem 5

In the limit $\tau \to \infty$, the stability region corresponding to a discrete delay lies inside or is the same as the stability region corresponding to any distribution of delays.



Stability Analysis - Approximations

Stability region bounded by the curve $(R(\omega), I(\omega)),\ \omega \in [-\bar{\omega}, \bar{\omega}]$ where

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Approximate stability boundary by approximating $C(\omega)$ and $S(\omega)$.

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$$\frac{d^n}{dt^n}\phi(t)\bigg|_{t=0} = i^n m_n \qquad \text{and} \qquad \frac{d^n}{dt^n}\ln\phi(t)\bigg|_{t=0} = i^n \kappa_n.$$

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Note: $m_0 = \phi(0) = 1$ and $\kappa_0 = \ln\phi(0) = 0$.

Since we have normalized $\hat{g}$ by its mean, $\kappa_1 = m_1 = 1$.

The moments and cumulants are related, e.g.:

$$\kappa_2 = m_2 - m_1^2,$$
$$\kappa_3 = m_3 - 3m_1 m_2 + 2m_1^3,$$

Stability Analysis - Approximations

Expanding in $\phi(t)$ a Taylor series around $t = 0$ and substituting $t = -\omega$:

$$\phi(i\omega) = \int_0^\infty e^{-i\omega v}\hat{g}(v)\,dv = \sum_{n=0}^{\infty}(-1)^n i^n m_n \frac{\omega^n}{n!} = \exp\left\{\sum_{n=0}^{\infty}(-1)^n i^n \kappa_n \frac{\omega^n}{n!}\right\}.$$

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But $\int_0^\infty e^{-i\omega v} \hat{g}(v)\, dv = C(\omega) - iS(\omega)$, i.e.,

$$C(\omega) = \mathrm{Re}\left(\int_0^\infty e^{-i\omega v} \hat{g}(v)\, dv \right) \quad \text{and} \quad S(\omega) = -\mathrm{Im}\left(\int_0^\infty e^{-i\omega v} \hat{g}(v)\, dv \right).$$

Stability Analysis - Approximations

Thus we obtain expansions in terms of the moments and cumulants:

$$
\begin{aligned}
C(\omega) &= \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n}}{(2n)!} m_{2n} \\
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S(\omega) &= \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n+1}}{(2n+1)!} m_{2n+1} \\
&= \exp\left\{ \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n}}{(2n)!} \kappa_{2n} \right\} \sin\left\{ \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n+1}}{(2n+1)!} \kappa_{2n+1} \right\}
\end{aligned}
$$

Stability Analysis - Approximations

Thus we obtain expansions in terms of the moments and cumulants:

$$\begin{aligned} C(\omega) &= \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n}}{(2n)!} m_{2n} \\ &= \exp\left\{ \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n}}{(2n)!} \kappa_{2n} \right\} \cos\left\{ \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n+1}}{(2n+1)!} \kappa_{2n+1} \right\} \end{aligned}$$

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Approximations may be made by truncating these series.

Stability Analysis - Approximations

Using moments:

(M,N)	$C(\omega)$	$S(\omega)$
$(0,0)$	1	ω
$(1,0)$	$1-\frac{m_2}{2}\omega^2$	ω
$(1,1)$	$1-\frac{m_2}{2}\omega^2$	$\omega-\frac{m_3}{6}\omega^3$

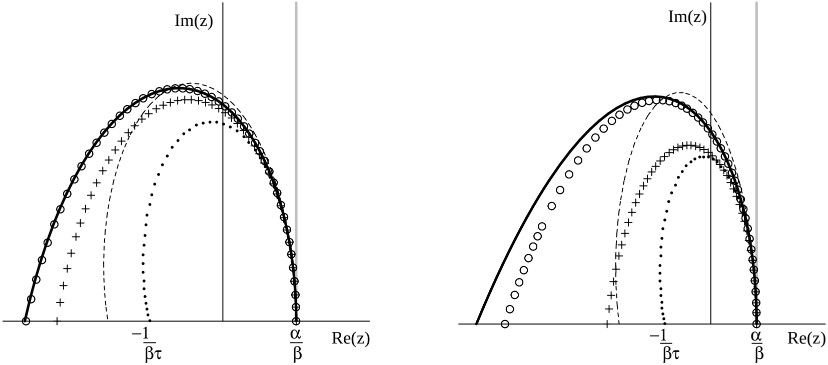
Using cumulants

(M,N)	$C(\omega)$	$S(\omega)$
$(0,0)$	$\cos(\omega)$	$\sin(\omega)$
$(1,0)$	$\exp\left(-\kappa_2\frac{\omega^2}{2}\right)\cos(\omega)$	$\exp\left(-\kappa_2\frac{\omega^2}{2}\right)\sin(\omega)$

Note: $(0,0)$ cumulant approximation recovers the results for discrete delay

Stability Analysis - Approximations

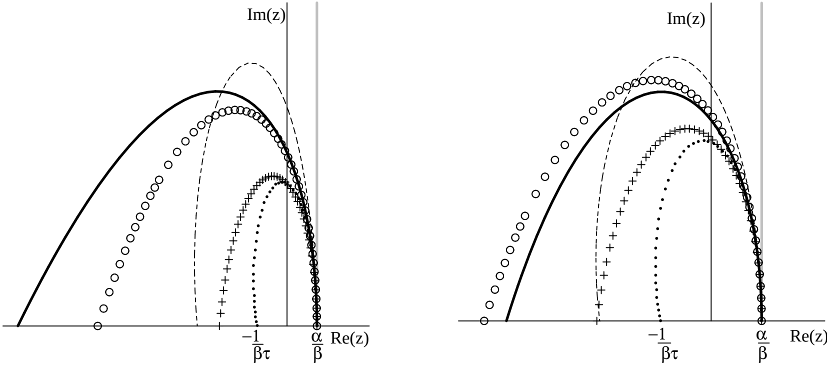
Uniform distribution with $\tau = 1/2$ and $\rho = 1, 2$



True boundary: solid black curve.
Moment approximations: $(1, 0)$ dotted, $(1, 1)$ dashed.
Cumulant approximations: $(0, 0)$ crosses, $(1, 0)$ circles.

Stability Analysis - Approximations

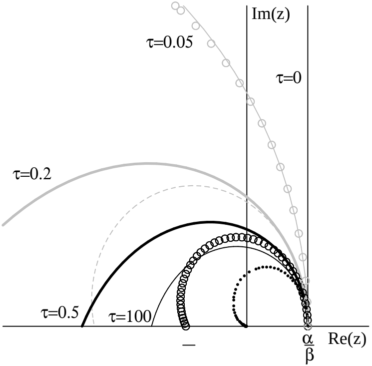
Gamma distribution with $\tau = 1/2$ and $p = 2, 3$



True boundary: solid black curve.
Moment approximations: $(1, 0)$ dotted, $(1, 1)$ dashed.
Cumulant approximations: $(0, 0)$ crosses, $(1, 0)$ circles.

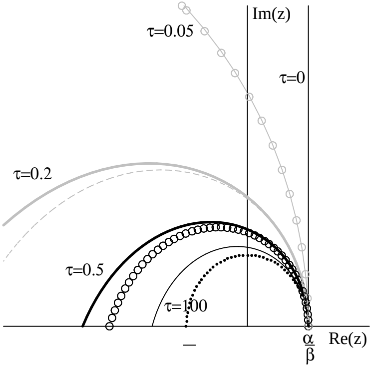
Uniform distribution with $\rho = 1$ and varying τ

True boundaries: solid curves



Moments

$(1, 0)$ Approximations



Cumulants

$(0, 0)$ Approximations

Conclusions

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Future Work:

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Future Work:

- Apply approximation technique to study criticality of Hopf bifurcation (in progress).

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References