Szpiro's Conjecture

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Workshop on Discovery and Experimentation in Number Theory

Let $E/\mathbb{Q}$ be an elliptic curve. Let

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, a_i \in \mathbb{Q}$$

be a Weierstrass equation for $E/\mathbb{Q}$.
By simple changes of variable we can find a simpler Weierstrass equation

$$y^2 = x^3 + Ax + B.$$

The discriminant of this Weierstrass equation is given by

$$\Delta(E) = -16(4A^3 - 27B^2).$$

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Discriminant is not an invariant of E, but rather an invariant of the model chosen for E:

Let $X = u^2 x$ and $Y = u^3 y$, then

$$E' : Y^2 = X^3 + Au^{-4}X + Bu^{-6},$$

and we get

$$\Delta' = u^{-12}\Delta.$$

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Definition

If the Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

is such that $a_i \in \mathbb{Z}$ and for any other Weierstrass equation of E with $a_i' \in \mathbb{Z}$ we have

$$|\Delta'| \geq |\Delta|,$$

we say that it is a minimal Weierstrass equation.

The discriminant of this minimal model is an invariant of $E/\mathbb{Q}$. We call this discriminant the minimal discriminant of E, and we denote it by Δ_E.

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We now know that $E/\mathbb{Q}$ is modular.

That means, there is an integer N and a rational surjective morphism

$$X_0(N) \to E.$$

The conductor of E is the smallest N such that such a map exist.
For $E/\mathbb{Q}$, let Δ_E be the minimal discriminant of E. Then

$$N_E = \prod_{p \mid \Delta_E} p^{\nu_p}$$

where

$$\nu_p = \begin{cases} 1 & \text{if } p \text{ is a prime of multiplicative reduction,} \\ 2 & \text{if } p > 3 \text{ and is a prime of additive reduction,} \\ \leq 8 & \text{in general.} \end{cases}$$

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Conjecture

For any $\varepsilon > 0$ there exists a positive number C_ε such that for any elliptic curve $E/\mathbb{Q}$ we have

$$|\Delta_E| \leq C_\varepsilon (N_E)^{6+\varepsilon},$$

where Δ_E is the minimal discriminant of E, and N_E is the conductor of E.

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The above conjecture generalizes to any number field, and in general global fields in obvious way.

- Szpiro's conjecture is true for function field case with $\varepsilon = 0$.
- Stewart and Yu's result for ABC conjecture shows that

$$|\Delta_E| < K^{(N_E)^{1/3+\varepsilon}},$$

  for some K.
- There are few families where we can prove Szpiro uncoditionally:
  - Szpiro's conjecture is true for elliptic curves with prime conductor.
  - Szpiro's conjecture is true for semistable elliptic curves with perfect power discriminant $\Delta_E = D^l$.

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- The ABC conjecture for number fields implies Szpiro's conjecture.
- Vojta's conjecture implies Szpiro's Conjecture
- It is true for function field case with $\varepsilon = 0$.
- For families of elliptic curves E_t, one gets that

$$\liminf \frac{\log |\Delta_{E_t}|}{\log N_{E_t}} \leq 6.$$

- It is such a simple conjecture with such wonderful consequences (a lot of them proven). It will be a shame if it's not true.

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- Szpiro's conjecture for $\mathbb{Q}$ implies the ABC conjecture (with slightly different constants) for $\mathbb{Q}$.
- One can construct elliptic curves E such that
$$\frac{\log|\Delta_E|}{\log N_E}$$
is fairly large (around $8.8\ldots$.)
- To make things worst, many of the above examples happen for elliptic curves with large conductor.
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Searching through Cremona's table of elliptic curves, we find the following elliptic curves.

N	Szpiro Ratio
1290h1	8.90370022
9690m2	8.801596
858k2	8.75731615
1218h4	8.16352289
910e1	8.11481288
174a1	7.88245679
1938b1	7.67049417
522m2	7.55196781
1110c4	7.53394369
1938b2	7.44459994

Well, there aren't any good examples in this table, since these are searched by bounding the discriminant.

Elliptic curves with 7 torsion points are parametrized by

$$E_7 : y^2 + (s^2 - st - t^2)xy - s^2t^3y = x^3 - s^2t(s-t)x^2,$$

and it has discriminant

$$\Delta_7 = s^7t^7(s-t)^7(s^3 - 8s^2t + 5st^2 + t^3).$$

Quotiening by the 7 torsion pint we get E_7' with discriminant $\Delta_7' = st(s-t)(s^3 - 8s^2t + 5st^2 + t^3)^7$.

Choosing s and t really powerful numbers, one expects to get elliptic curves with high Szpiro ratio.

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7-Torsion Search

s	t	N	σ
11	2	858	8.75731614557112
312500	49221	50476302089404230	7.92756011861372
312500	49221	50476302089404230	7.53984409257145
486	487	202824786	7.44460382021831
425984	369285	175354399584002730	7.36245698593439
304	21	120369522	7.32780268502485
707281	10935	9246380170145557110	7.31803486329741
⋮	⋮	⋮	⋮
1712421	868096	1363130725497912232710	7.11870486158338
147392	127223	13911787389465294	7.10618688719188

Good Szpiro Ratio and non-trivial Torsion

N	$\lvert T \rvert$	σ
2526810	4	8.811944
9690	2	8.801596
858	7	8.75731615
167490523410	4	8.688968
610537970	3	8.596580
29070	2	8.502119
391491534	5	8.48609917
91910	2	8.485421
33641790	3	8.245590

Here is a typical application of Szpiro's Conjecture to Diophantine equations.

Proposition

Assume for some $\varepsilon > 0$ and some constant C_ε, Szpiro's conjecture is true. Then there are only finitely many solutions to

$$A^a + B^b = C^c,$$

with A, B, C coprime integers, and $1/a + 1/b + 1/c < 1$.

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Proof.

For any such triple (a, b, c), Darmon and Granville have showed that the above equation has only finitely many solutions.

So assume that $\max(a, b, c)$ is really really large. For simplicity, assume that $\min(a, b, c) > 6 + \varepsilon$. Construct the Frey elliptic curve

$$y^2 = x(x - A^a)(x + B^b),$$

which has minimal discriminant and conductor

$$\Delta_E = 2^r A^a B^b C^c, N_E = \prod_{p \mid ABC} p < ABC$$

Then Szpiro's conjecture says

$$|2^r A^a B^b C^c| < C_\varepsilon (ABC)^{6+\varepsilon}.$$

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Local Szpiro's Conjecture

Note that Szpiro was used to prove that not all powers of primes in the discriminant can be large.

Conjecture

Let $E/\mathbb{Q}$ be a semistable elliptic curve with conductor N_E and minimal discriminant Δ_E. Then there exists a prime number $p|N_E$ such that

$$v_p(\Delta_E) \leq 6.$$

The above conjecture can prove many of the results that Szpiro's conjecture proves.

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We have the following (very) partial result in this direction:

Theorem

For any integer M, there exist a constant $C(M)$ such that for any $p > C(M)$ and any elliptic curve $E/\mathbb{Q}$ of conductor Mp and a non-trivial rational isogeny we have

$$v_p(\Delta_E) \leq 6.$$

Here is a problem that local Szpiro conjecture implies, which I do not know a proof of.

Conjecture

Let $l > 6$ be a prime. Let $E/\mathbb{Q}$ be a semistable elliptic curve with minimal discriminant $\Delta_E = p^r M^l$ for some integer M coprime to p. Then $r \leq 6$.

This is closely related to the following Diophantine problem:

Conjecture

Let $F/\mathbb{Q}$ be an elliptic curve with a prime conductor p. Let $X_F(l)$ be the twist of $X(l)$ by $F[l]$-torsion structure. Then for any $E \in X_F(l)(\mathbb{Q})$ we have $v_p(j(E)) \geq -6$.

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