Integer Matrices with Constrained Eigenvalues

Cyclotomic matrices and graphs

Graeme Taylor

Edinburgh

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A question

Which symmetric integer matrices have all eigenvalues in $[-2, 2]$?

Mahler Measure

Let $P(z) = a_0 z^d + \cdots + a_d = a_0 \prod_{i=1}^{d}(z - \alpha_i)$ be a non-constant polynomial.

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- Clearly, $M(P) \geq 1$ for all P.
- If $M(P) = 1$, then all roots of P lie in the closed unit disc.
- $\forall \lambda \geq 1$, $\exists P$ s.t. $M(P) = \lambda$.

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- So $M(P) = 1 \Leftrightarrow P$ cyclotomic.
- What about noncyclotomic polynomials?

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- If not, then there exists some $\lambda > 1$ such that $M(P) > 1 \Rightarrow M(P) > \lambda$, forcing a 'gap' between cyclotomic and non-cyclotomic polynomials.

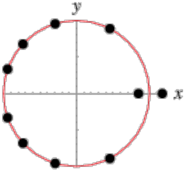
Lehmer's Conjecture

The smallest known Mahler measure greater than 1 for a monic polynomial from $\mathbb{Z}[z]$ is

$$\lambda_0 = 1.176280818$$

which is the larger real root of the *Lehmer polynomial*

$$z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$



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- Likely candidates for small Mahler measure are polynomials that are ‘almost cyclotomic’- as few roots outside the unit circle as possible.
- Difficulty: There’s no obvious way to obtain such an ‘almost cyclotomic’ integer polynomial from a cyclotomic one.

Associated Polynomials

- If A is an $n \times n$ integer symmetric matrix, then its *associated polynomial* is $R_A(z) := z^n \chi_A(z + 1/z)$

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- If A is an $n \times n$ integer symmetric matrix, then its *associated polynomial* is $R_A(z) := z^n \chi_A(z + 1/z)$
- If A has all eigenvalues in $[-2, 2]$, then R_A is a cyclotomic polynomial- We describe A as a *cyclotomic matrix*.

From cyclotomic to noncyclotomic

Theorem (Cauchy Interlacing Theorem)

Let A be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.
Let B be obtained from A by deleting row i and column i from A.
Then the eigenvalues $\mu_1 \leq \cdots \leq \mu_{n-1}$ of B interlace with those of A: that is,

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n$$

From cyclotomic to noncyclotomic

We can run this process in reverse. Let B be a cyclotomic matrix, so its eigenvalues satisfy

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So

$$\lambda_2, \ldots, \lambda_{n-1} \in [\mu_1, \mu_{n-1}] \subseteq [-2, 2]$$

At worst,

$$\lambda_1, \lambda_n \notin [-2, 2]$$

Cyclotomic Matrices: Entries

Lemma

The only cyclotomic 1×1 *matrices are*

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Corollary

By interlacing, the entries of an integer cyclotomic matrix must be elements of $\{0, 1, -1, 2, -2\}$.

Cyclotomic Matrices: Indecomposability

If M decomposes as a block-diagonal matrix, then its eigenvalues are those of the blocks; thus a cyclotomic matrix decomposes into one or more indecomp. cyclotomic matrices, and it suffices to classify these.

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Lemma

Apart from the matrices

$$(2), (-2), \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

any indecomp. cyclotomic matrix has all entries from $\{0, 1, -1\}$.

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Theorem (McKee, Smyth)

Any non-maximal indecomp. cyclotomic matrix is contained in a maximal one.

Cyclotomic Matrices: Equivalence

Let $O_n(\mathbb{Z})$ be the orthogonal group of $n \times n$ signed permutation matrices, generated by permutation matrices and matrices of the form

$$diag(1, 1, \ldots, 1, -1, 1, \ldots, 1)$$

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- A matrix M' is then described as *equivalent* to M if it is strongly equivalent to either M or $-M$.

The question, refined

Our original question thus reduces to classifying all *maximal, indecomposable, cyclotomic, symmetric* $\{-1, 0, 1\}$*-matrices, up to equivalence*.

Charged Signed Graphs

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- $M_{ii} = 0$ gives a neutral vertex i, denoted $\bullet$.
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- $M_{ij} = 1$, $i \neq j$ gives a positive edge between vertices i and j, denoted —— .
- $M_{ij} = -1$, $i \neq j$ gives a negative edge between vertices i and j, denoted $\cdots\cdots$.

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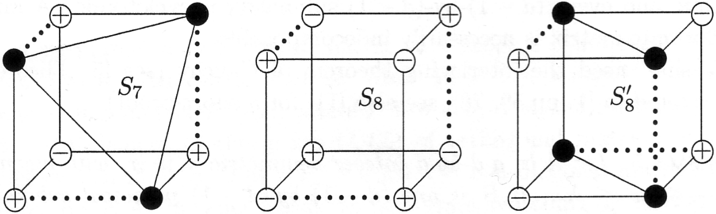
- Conjugation of M by kth diagonal matrix $\Leftrightarrow$ Switching of signs of all edges incident at vertex k of G.

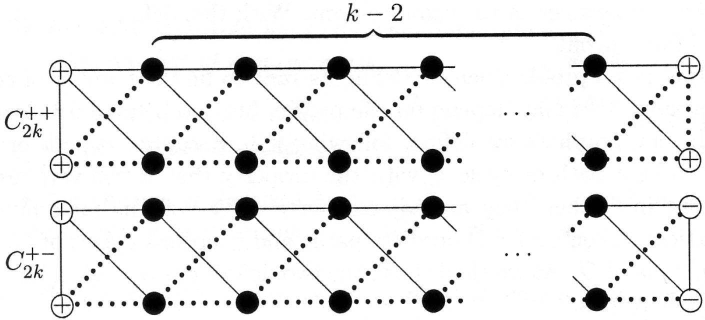
Classification

Charged Sporadics S_7, S_8, S_8':



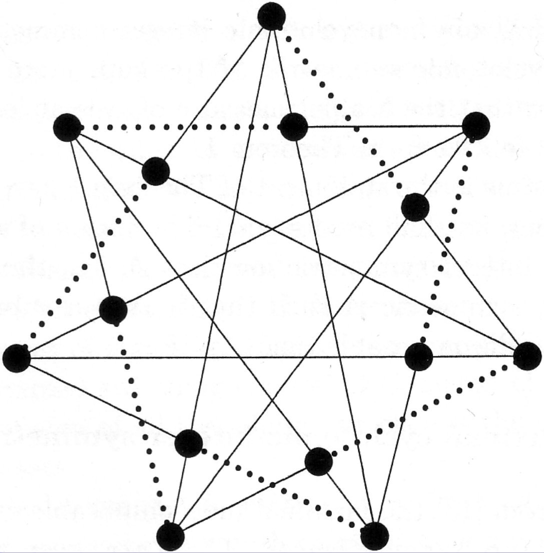
Classification

Infinite family $C_{2k}^{+\pm}$, $k \geq 2$:



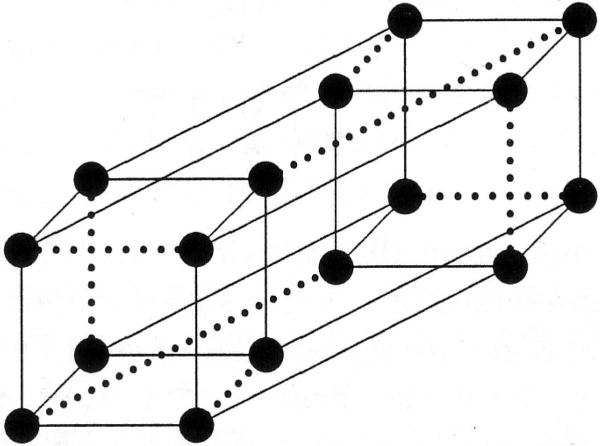
Classification

Uncharged Sporadic S_{14}:



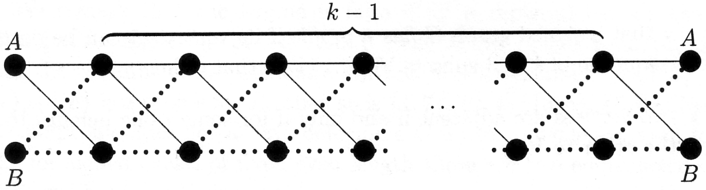
Classification

Uncharged Sporadic S_{16}:



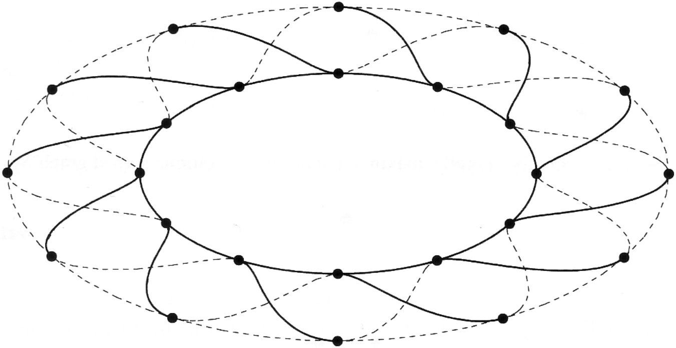
Classification

Infinite family T_{2k}, $k \geq 3$:



Classification

Example: T_{24}



A special case of Lehmer's Problem

Theorem (McKee,Smyth)

If A is a noncyclotomic integer symmetric matrix then

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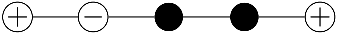

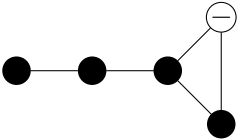

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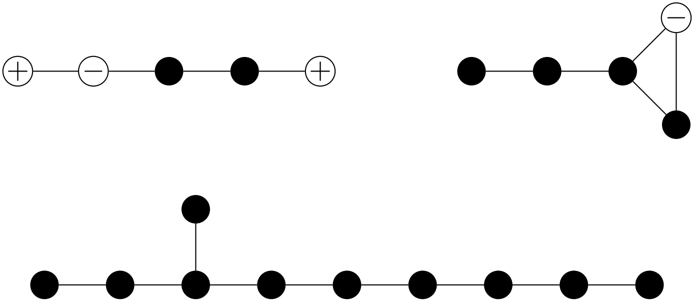

Cyclotomic Matrices over $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d < 0$ squarefree

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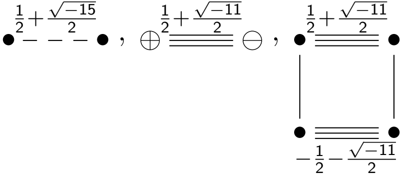
- Interlacing Theorem still holds for Hermitian matrices.
- $M_{i,i} \in \{0, \pm 1, \pm 2\}$ as before.
- Off-diagonal entries satisfy $M_{i,j} M_{j,i} = N(M_{i,j}) \leq 4$.

Cyclotomic Matrices over $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d \leq -11$ squarefree

- For $d \leq -17$, $\{x \in R \mid N(x) \leq 4\} \subset \mathbb{Z}$.

Cyclotomic Matrices over $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d \leq -11$ squarefree

- For $d \leq -17$, $\{x \in R \mid N(x) \leq 4\} \subset \mathbb{Z}$.
- For $d = -15, -11$, only finitely many cyclotomic matrices with entries from $R \backslash \mathbb{Z}$:



Cyclotomic Matrices over $R = \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $-7 \le d \le -1$

Many possible entries!

d	$N(x) = 1$	$N(x) = 2$	$N(x) = 3$	$N(x) = 4$
-1 :	$\pm 1, \pm i$	$\pm 1 \pm i$		$\pm 2, \pm 2i$
-2 :	± 1	$\pm\sqrt{-2}$	$\pm 1 \pm \sqrt{-2}$	± 2
-3 :	$\pm 1, \pm\frac{1}{2} \pm \frac{\sqrt{-3}}{2}$		$\pm\frac{3}{2} \pm \frac{\sqrt{-3}}{2}, \pm\sqrt{-3}$	$\pm 2, \pm 1 \pm \sqrt{-3}$
-7 :	± 1	$\pm\frac{1}{2} \pm \frac{\sqrt{-7}}{2}$		$\pm 2, \pm\frac{3}{2} \pm \frac{\sqrt{-7}}{2}$

4-Cyclotomic Matrices

Observation

For $R = \mathbb{Z}, \mathcal{O}_{\mathbb{Q}(\sqrt{-15})}, \mathcal{O}_{\mathbb{Q}(\sqrt{-11})}$:

$$M \in Mat(R) \text{ maximal cyclotomic} \Leftrightarrow M^2 = 4I$$

4-Cyclotomic Matrices

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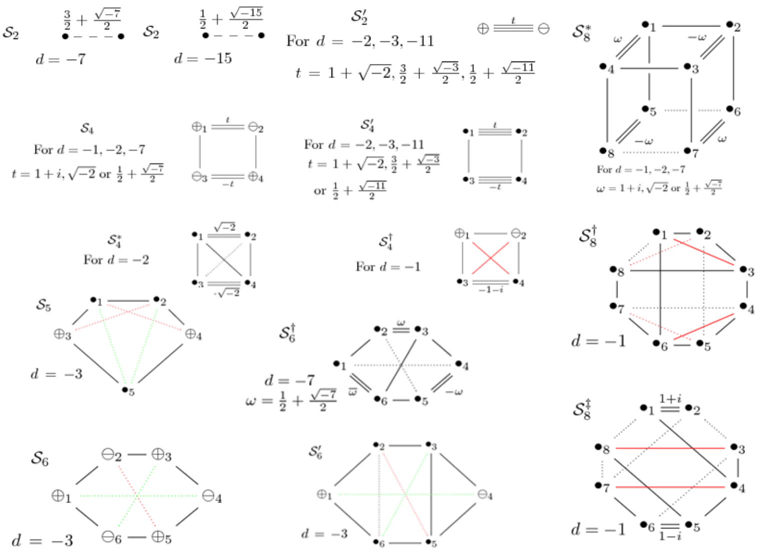
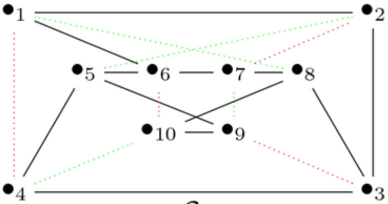
Determining 4-cyclotomic matrices for $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $d \in \{-1, -2, -3, -7\}$ is computationally feasible!

Classification

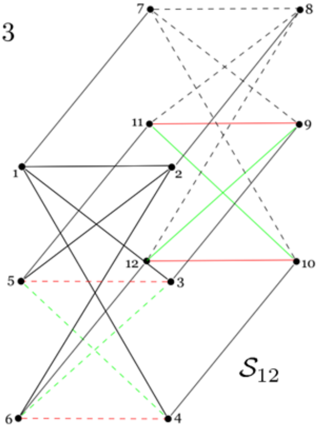


Classification

$d = -3$

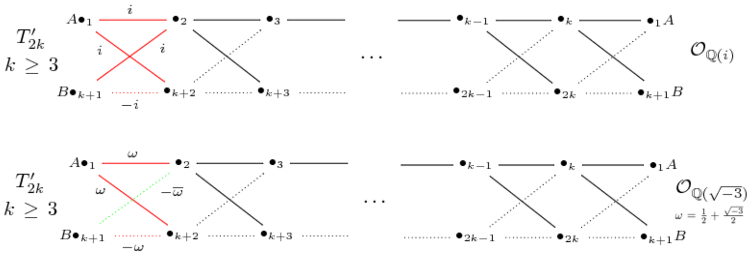


$\mathcal{S}_{10}$

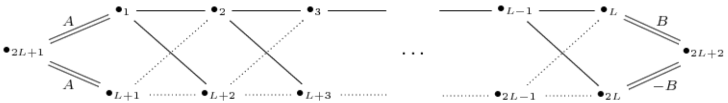


$\mathcal{S}_{12}$

Classification



Classification



$T_{2k}^4 \quad k \geq 2 \quad \mathcal{O}_{\mathbb{Q}(i)} \quad A = B = 1 + i$

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$T_{2k}^4 \quad k \geq 2 \quad \mathcal{O}_{\mathbb{Q}(\sqrt{-7})} \quad A = B = \frac{1}{2} + \frac{\sqrt{-7}}{2}$

${T_{2k}^4}' \quad k \geq 2 \quad \mathcal{O}_{\mathbb{Q}(\sqrt{-7})} \quad A = \omega = \frac{1}{2} + \frac{\sqrt{-7}}{2} \quad B = \overline{\omega}$

Classification

C_{2k}^{2+} $k \geq 3$ $\mathcal{O}_{\mathbb{Q}(i)}$ $A = 1 + i$

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- $\Rightarrow$ holds for $d \leq -11$, or $R = \mathbb{Z}$.
- Caution: Not true for adjacency matrices of graphs!

Maximal Cyclotomic Graphs

Theorem (Smith)

The connected cyclotomic graphs are precisely the induced subgraphs of the graphs $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ and those of the $(n+1)$-vertex graphs $\tilde{A}_n (n \geq 2)$, $\tilde{D}_n, (n \geq 4)$: