Ramanujan and the ζ-function

M. Ram Murty
Queen’s University

$$\alpha^{-n}\left\{\frac{1}{2}\zeta(2n+1)+\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\alpha k}-1}\right\}$$

$$=(-\beta)^{-n}\left\{\frac{1}{2}\zeta(2n+1)+\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\beta k}-1}\right\}$$

$$-2^{2n}\sum_{k=0}^{n+1}(-1)^{k}\frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!}\alpha^{n+1-k}\beta^{k}$$

Euler (1707-1783)

- In 1735, Euler discovered experimentally that $1 + \frac{1}{4} + 1/9 + \ldots = \pi^2/6$.
- He gave a “rigorous” proof of this much later, in 1742.

Sketch of Euler's proof

- The polynomial $(1-x/r_1)(1-x/r_2)\ldots(1-x/r_n)$ has roots equal to $r_1, r_2, \ldots, r_n$.
- When we expand the polynomial, the coefficient of x is $-(1/r_1 + 1/r_2 + \ldots + 1/r_n)$.
- A polynomial is determined by its roots. Knowing its roots is the same as knowing the polynomial.
- Suppose that $\sin \pi x$ "behaves" like a polynomial.

“Factoring” sin $\pi x = \pi x - (\pi x)^3 / 3!$ …

- sin πx has roots at $x=0, \pm 1, \pm 2, \ldots$
- Put $f(x) = (\sin \pi x)/\pi x$. By l’Hopital’s rule, $f(0)=1$.
- Now $f(x)$ has roots at $x=\pm 1, \pm 2, \ldots$
- Perhaps $f(x)= (1-x)(1+x)(1-x/2)(1+x/2)\ldots$
- That is, $f(x) = (1-x^2)(1-x^2/4)(1-x^2/9)\ldots$
- The coefficient of x^2 on the right hand side is $-(1 + 1/4 + 1/9 + 1/16 + \ldots)$

- Comparing coefficients gives
- $\Sigma_{n\geq 1}\ 1/n^2 = \pi^2/6$.
- Can this proof be justified?

Euler's proof can be made rigorous using Hadamard's theory of factorization of entire functions, a theory developed much later in 1892 in his doctoral thesis.

Can Euler's result be generalized?
For example, can we evaluate
$\Sigma_{n\geq 1}\ 1/n^3$ or $\Sigma_{n\geq 1}\ 1/n^4$?

Euler had difficulty with the first question but managed to show that $\Sigma_{n\geq 1}\ 1/n^4 = \pi^4/90$ and more generally that $\Sigma_{n\geq 1}\ 1/n^{2k} = \pi^{2k}$(rational number).

Modifying Euler's proof

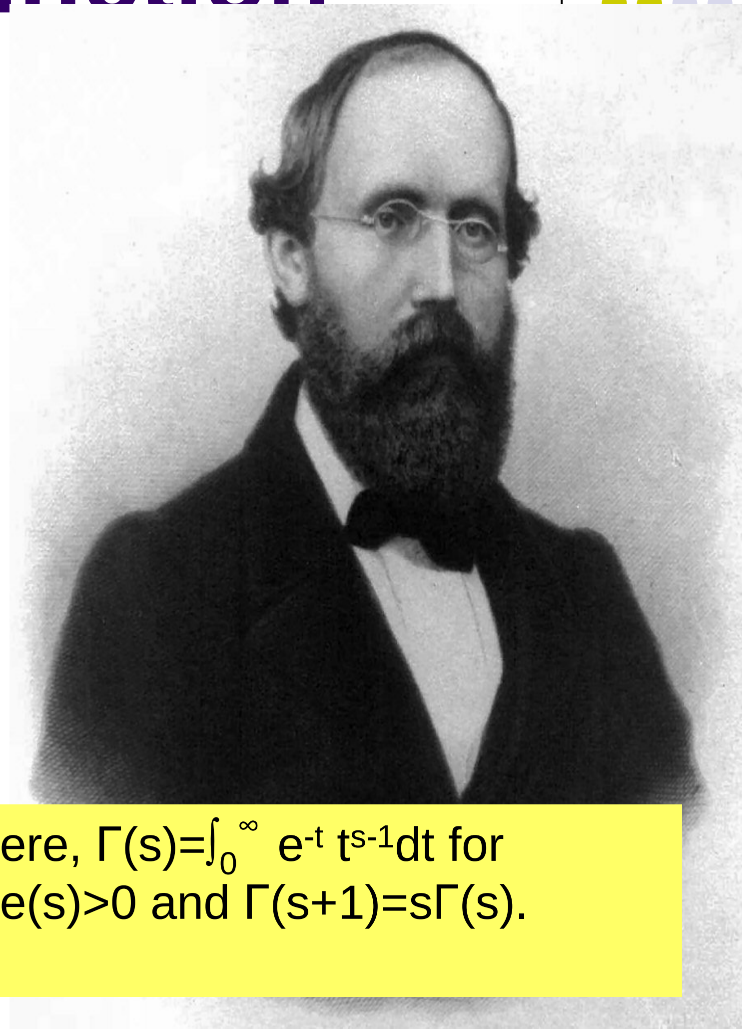
- Recall $f(x) = (\sin \pi x)/ \pi x = (1-x^2)(1-x^2/4)\ldots$
- Thus $f(ix)=(\sin \pi ix)/\pi ix = (1+x^2)(1+x^2/4)\ldots$ Here $i=\sqrt{-1}$.
- Multiplying these two together gives us:
- $(1-x^4)(1-x^4/2^4)(1-x^4/3^4)\ldots$
- But the Taylor expansion of $f(x)f(ix)$ is
- $(1- \pi^2x^2/3! +\pi^4x^4/5! - \ldots)(1+\pi^2x^2/3!+\pi^4x^4/5!+\ldots)$
- Computing the coefficient of x^4 gives:
- $\Sigma_{n\geq 1}\ 1/n^4 = \pi^4/90.$

Algebraic and transcendental numbers

- A complex number z is called an algebraic number if it is the root of a monic polynomial with rational coefficients.
- A complex number which is not algebraic is called transcendental.
- Algebraic numbers form a countable set and so the set of transcendental numbers is uncountable.
- Numbers such as $\sqrt{3}$, $2^{1/7}$, $(1+\sqrt{5})/2$ are algebraic.
- Numbers like π and e are transcendental.
- The set of all algebraic numbers forms a field under addition and multiplication.
- In this talk, we will be interested if $\zeta(k)$ is an algebraic number or a transcendental number for natural numbers $k \geq 2$.

The Riemann zeta function

- $\zeta(s)=\Sigma_{n\geq 1} 1/n^s$ converges for $\mathrm{Re}(s)>1$.
- Riemann showed how to analytically continue this function to the entire complex plane and established a functional equation.
- Put $\psi(s):=\pi^{-s/2}\Gamma(s/2)\zeta(s)$
- Then $\psi(s)=\psi(1-s)$.



Here, $\Gamma(s)=\int_0^\infty e^{-t}\, t^{s-1}dt$ for $\mathrm{Re}(s)>0$ and $\Gamma(s+1)=s\Gamma(s)$.

G.F.B. Riemann (1826-1866)

The Prime Number Theorem

- In the same paper, Riemann indicated but did not prove how his ζ-function can be used to prove theorems about prime numbers.
- More precisely, he indicated a program for proving the prime number theorem, originally conjectured by Gauss, that the number of primes $P(x)$ up to x is asymptotic to $x/\log x$ as x tends to infinity.

Hadamard and de la Vallée Poussin

- In 1896, Hadamard and de la Vallée Poussin (independently) proved the prime number theorem.

Zeros of the Riemann ζ-function

- In their proof of the prime number theorem, Hadamard and de la Vallée Poussin proved that $\zeta(1+it) \neq 0$ for all real $t \neq 0$.
- Riemann conjectured that if $\zeta(s_0)=0$ and $0<\mathrm{Re}(s_0)<1$, then $\mathrm{Re}(s_0)=1/2$.
- This is called the Riemann hypothesis and to this day,it is still an open problem.
- In 2004, Gourdon and Demichel checked that the first 10^{13} zeros are on the line!

Other mysteries of the ζ-function

- If the Riemann hypothesis is true, how are the zeros distributed on the line Re(s)=1/2?
- What about its growth rate on the critical line?
- It is conjectured that $|\zeta(1/2 + it)|=O(t^{\varepsilon})$ for any $\varepsilon>0$. This is called the Lindelof Hypothesis.
- It is a consequence of the Riemann hypothesis.
- What about special values, like $\zeta(2)$ or $\zeta(3)$?

Euler's theorem

- $\zeta(2k) \in \pi^{2k}\mathbb{Q}$.
- Actually, Euler proved a more precise theorem.
- Define the Bernoulli numbers as the Taylor coefficients of $x/(e^x-1) = \Sigma_{n\geq 1} B_n x^n/n!$.
- The B_n's are rational numbers.
- Euler proved that $2\zeta(2k)=(-1)^{k-1}B_{2k}(2\pi)^{2k}/(2k)!$.
- Since π is transcendental, we see that $\zeta(2k)$ is transcendental for every $k\geq 1$.
- What about $\zeta(3)$?

Apery's Theorem

- In 1979, Roger Apery showed that ζ(3) is irrational.
- His starting point was the following remarkable formula:

$$\zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

Conjectures concerning $\zeta(3)$, $\zeta(5)$, …

- One expects $\zeta(3)$, $\zeta(5)$, … to be transcendental.
- In fact, one expects that π, $\zeta(3)$, $\zeta(5)$, … to be algebraically independent.
- In 2000, Rivoal proved that infinitely many of the numbers $\zeta(2k+1)$, $k \geq 1$, are irrational.
- In 2001, Rivoal and Zudilin showed that at least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational.
- The $\mathbb{Q}$-vector space spanned by $\zeta(3)$, $\zeta(5)$,… is of infinite dimension. (Rivoal, Ball and Zudilin, 2003)

Ramanujan's formula

- Ramanujan discovered the following formula for $\zeta(3)$:
- $\zeta(3) + 2\Sigma_{n\geq 1}\, n^{-3}(e^{2\pi n}-1)^{-1} = 7\pi^3/180.$
- At least one of the two terms on the left hand side is transcendental!

S. Ramanujan (1887-1920)

A page from Ramanujan’s notebook

21 i. If $\alpha\beta = \pi^2$ and n any integer,

$$(4\alpha)^{1-n}\left\{\tfrac{1}{2}S_{2n-1} + \frac{1}{1^{2n-1}(e^{2\alpha}-1)} + \frac{1}{2^{2n-1}(e^{4\alpha}-1)} + \&c\right\}$$

$$-(-4\beta)^{1-n}\left\{\tfrac{1}{2}S_{2n-1} + \frac{1}{1^{2n-1}(e^{2\beta}-1)} + \frac{1}{2^{2n-1}(e^{4\beta}-1)} + \&c\right\}$$

$$= \frac{B_{2n}}{\underline{|2n}}\left\{(-\alpha)^n + \beta^n\right\} + \pi^2 \frac{B_2}{\underline{|2}}\frac{B_{2n-2}}{\underline{|2n-2}}\left\{(-\alpha)^{n-2} + \beta^{n-2}\right\}$$

$$- \pi^4 \frac{B_4}{\underline{|4}}\frac{B_{2n-4}}{\underline{|2n-4}}\left\{(-\alpha)^{n-4} + \beta^{n-4}\right\} + \&c$$ the last term

being $-\pi^n \frac{B_n}{\underline{|n}}\cdot\frac{B_n}{\underline{|n}}(-1)^{\frac{n}{2}}$ or $\pi^{n-1}\frac{B_{n-1}}{\underline{|n-1}}\frac{B_{n+1}}{\underline{|n+1}}(-1)^{\frac{n+1}{2}}\{(-\alpha)+\beta\}$

according as n is even or odd.

ii. If $\alpha\beta = \pi^2$ and n any integer, then

$$\alpha^{1-n}\left\{\frac{1}{1^{2n-1}(e^{\frac{\alpha}{2}}+e^{-\frac{\alpha}{2}})} - \frac{1}{3^{2n-1}(e^{\frac{3\alpha}{2}}+e^{-\frac{3\alpha}{2}})} + \&c\right\}\cdot\frac{2^{2n+1}}{\pi}$$

$$+(-\beta)^{1-n}\left\{\frac{1}{1^{2n-1}(e^{\frac{\beta}{2}}+e^{-\frac{\beta}{2}})} - \frac{1}{3^{2n-1}(e^{\frac{3\beta}{2}}+e^{-\frac{3\beta}{2}})} + \&c\right\}\frac{2^{2n+1}}{\pi}$$

$$= \frac{E_1 E_{2n-1}}{\underline{|2n-2}}\left\{(-\alpha)^{n-1} + \beta^{n-1}\right\} - \frac{E_3 E_{2n-3}}{\underline{|2}\,\underline{|2n-4}}\left\{(-\alpha)^{n-3} + \beta^{n-3}\right\}$$

$$+ \frac{E_5 E_{2n-5}}{\underline{|4}\,\underline{|2n-6}}\left\{(-\alpha)^{n-5} + \beta^{n-5}\right\} - \&c$$ the last term being

$(-1)^{\frac{n-1}{2}}\left(\frac{E_n}{\underline{|n-1}}\right)^2$ or $(-1)^{\frac{n}{2}}\frac{E_{n-1}}{\underline{|n-2}}\cdot\frac{E_{n+1}}{\underline{|n}}(\alpha-\beta)$ according as n is

odd or even.

iii. If $\alpha\beta = \pi^2$ and n any integer, $\frac{\sqrt{\alpha}}{\alpha^n}\left\{\tfrac{1}{2}\left(\frac{1}{1^{2n}} - \frac{1}{3^{2n}} + \frac{1}{5^{2n}} \ldots\right)\right.$

$$\left. + \frac{1}{1^{2n}(e^{\alpha}-1)} - \frac{1}{3^{2n}(e^{3\alpha}-1)} + \frac{1}{5^{2n}(e^{5\alpha}-1)} - \&c\right\} =$$

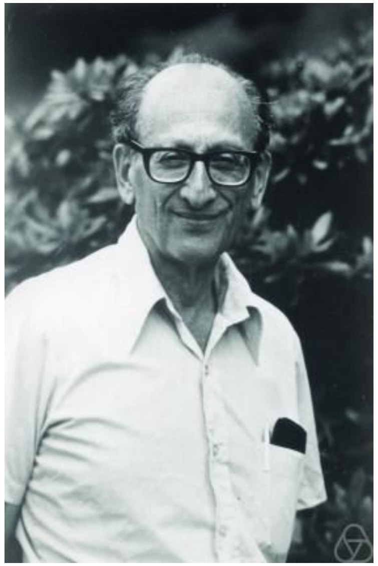
A more general formula of Ramanujan

$$\alpha^{-n}\left\{\tfrac{1}{2}\zeta(2n+1)+\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\alpha k}-1}\right\}$$

$$=(-\beta)^{-n}\left\{\tfrac{1}{2}\zeta(2n+1)+\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\beta k}-1}\right\}$$

$$-2^{2n}\sum_{k=0}^{n+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!}\alpha^{n+1-k}\beta^k$$

Grosswald's generalization



Emil Grosswald
(1912-1989)

- Define

$$R_{2k+1}(z) = \sum_{j=0}^{k+1} \frac{B_{2j} B_{2k+2-2j}}{(2j)!(2k+2-2j)!} z^{2k+2-2j},$$

$$\sigma_k(n) = \sum_{d|n} d^k$$

and set

$$F_k(z) = \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^k} e^{2\pi i n z}$$

for $\Im(z) > 0$. Then

$$F_{2k+1}(z) - z^{2k} F_{2k+1}(-1/z)$$
$$= \frac{1}{2}\zeta(2k+1)(z^{2k}-1) + \frac{(2\pi i)^{2k+1}}{2z} R_{2k+1}(z).$$

$F_k(z)$ for $z=i\alpha$

- $F_k(i\alpha) = \Sigma_{n\geq 1} \sigma_k(n)n^{-k}e^{-2\pi n\alpha} = \Sigma_{n\geq 1} n^{-k}(e^{2\pi n\alpha} -1)^{-1}$.
- Thus, putting $z=i\alpha$ in Grosswald's formula recovers Ramanujan's formula.
- The values of $F_k(i\alpha)$ are examples of Eichler integrals.
- However, Grosswald's formula opens up new possible expressions for $\zeta(2k+1)$.
- Indeed, if z_0 is a zero of $R_{2k+1}(z)$ and z_0 is not a (2k)-th root of unity, then we get an expression of $\zeta(2k+1)$ as a difference of two Eichler integrals.

Zeros of Ramanujan polynomials

- Joint work with Rob Wang (NSERC Summer Research student)

$$R_1(z) = \frac{z^2+1}{2 \cdot 3!} \text{ (this is the trivial case)}$$

$$\text{Roots: } \pm i$$

$$R_3(z) = \frac{-z^4 + 5z^2 - 1}{6!}$$

$$\text{Roots: } \pm\sqrt{\frac{5 \pm \sqrt{21}}{2}} \; (\pm 2.1889, \pm 0.4569)$$

$$R_5(z) = \frac{2z^6 - 7z^4 - 7z^2 + 2}{12 \cdot 7!}$$

$$\text{Roots: } \pm i, \pm\sqrt{\frac{9 \pm \sqrt{65}}{4}} \; (\pm 2.0653, \pm 0.4842)$$

$$R_7(z) = \frac{-3z^8 + 10z^6 + 7z^4 + 10z^2 - 3}{10!}$$

$$\text{Roots: } \pm\sqrt{\frac{13 \pm \sqrt{133}}{6}}, \pm\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

$$(\pm 2.0221, \pm 0.4945, \pm 0.5 \pm 0.8660i)$$

More numerical data

$$R_9(z) = \frac{10z^{10} - 33z^8 - 22z^6 - 22z^4 - 33z^2 + 10}{12!}$$

$$\text{Roots: } \pm i, \pm\sqrt{\frac{43}{40} + \frac{3\sqrt{201}}{40} \pm \frac{1}{2}\sqrt{\frac{1029}{200} + \frac{129\sqrt{201}}{200}}}$$

$$\pm\sqrt{\frac{43}{40} - \frac{3\sqrt{201}}{40} \pm \frac{i}{2}\sqrt{\frac{-1029}{200} + \frac{129\sqrt{201}}{200}}}$$

$$(\pm 2.0071, \pm 0.4982, \pm 0.7112 \pm 0.7030i)$$

And a few more cases (all roots other than $\pm i$ are approximations):

$$R_{11}(z) = \frac{-1382z^{12} + 4550z^{10} + 3003z^8 + 2860z^6 + 3003z^4 + 4550z^2 - 1382}{2 \cdot 15!}$$

Roots: $\pm 2.0022, \pm 0.4995, \pm 0.3081 \pm 0.9513i, \pm 0.8146 \pm 0.5800i$

$$R_{13}(z) = \frac{210z^{14} - 691z^{12} - 455z^{10} - 429z^8 - 429z^6 - 455z^4 - 691z^2 + 210}{12 \cdot 15!}$$

Roots: $\pm i, \pm 2.0006, \pm 0.4998, \pm 0.5 \pm 0.8660i, \pm 0.8715 \pm 0.4904i$

$$R_{15}(z) = \frac{-10851z^{16} + 35700z^{14} + 23494z^{12} + 22100z^{10} + 21879z^8}{5 \cdot 18!}$$
$$\frac{+22100z^6 + 23494z^4 + 35700z^2 - 10851}{5 \cdot 18!}$$

Roots: $\pm 2.0002, \pm 0.5000, \pm 0.2219 \pm 0.9751i, \pm 0.9058 \pm 0.4238i, \pm 0.6247 \pm 0.7809i$

$$R_{17}(z) = \frac{438670z^{18} - 1443183z^{16} - 949620z^{14} - 892772z^{12} - 881790z^{10}}{21!}$$
$$\frac{-881790z^8 - 892772z^6 - 949620z^4 - 1443183z^2 + 438670}{21!}$$

Roots: $\pm i, \pm 2.0001, \pm 0.5000, \pm 0.3822 \pm 0.9241i, \pm 0.9279 \pm 0.3729i, \pm 0.7091 \pm 0.7051i$

Notice that, for each $k \geq 1$, $R_{2k+1}(z)$ seems to possess exactly 4 real roots. Furthermore, the largest of the real roots is always less than or equal to 2.2 (and it seems to be approaching 2 as k increases. On the other hand, the complex roots seem to lie exactly on the unit circle.

Theorem on zeros of Ramanujan polynomials

- Theorem (R. Murty & R. Wang) All the zeros of the Ramanujan polynomials lie in the disk $|z| < 2.2$. The real zeros of $R_{2k+1}(z)$ are four in number and approach 2, -2, ½, - ½ , as k tends to infinity. When k is even, $R_{2k+1}(z)$ has $z=\pm i$ as zeros. When $3|k$, $R_{2k+1}(z)$ has $z=\rho, \rho^2$ as zeros (here, ρ is a primitive cube root of unity).
- Conjecture: All the non-real zeros of $R_{2k+1}(z)$ lie on the unit circle.

Some transcendence results

$$E_k(z) = \gamma_k + \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}, \qquad \gamma_k = (-1)^{(k/2-1)} \frac{B_k}{4k}.$$

Then for any odd $k > 1$, we have

$$F_k(z) = \frac{(2\pi i)^k}{(k-1)!} \int_{i\infty}^{z} [E_{k+1}(\tau) - \gamma_{k+1}](\tau - z)^{k-1} d\tau,$$

Report on some joint work with S. Gun and P. Rath

Theorem 1. *Let k be non-negative. With at most $2k+3$ exceptions, the number $F_{2k+1}(\alpha) - \alpha^{2k} F_{2k+1}(-1/\alpha)$ is transcendental for every algebraic $\alpha \in \mathbb{H}$. In other words, there are at most $2k+3$ algebraic numbers $\alpha \in \mathbb{H}$ such that $F_{2k+1}(\alpha)$ and $F_{2k+1}(-1/\alpha)$ are both algebraic.*

A general theorem of Grosswald (1973)

for $s \in \mathbb{C}$, let

$$\Delta(s) = \prod_{\nu=1}^{M} \Gamma(\alpha_\nu s + \beta_\nu), \quad \alpha_\nu > 0, \ \beta_\nu \in \mathbb{C}$$

Suppose a Dirichlet series

$$\phi(s) = \sum_{n=1}^{\infty} a(n) e^{-\lambda_k s}$$

is convergent for $\Im(s) = \sigma \geq \sigma_0 > 0$ and define for $s \in \mathbb{C}$

$$\Phi(s) = \phi(s)\Delta(s)P(s),$$

where $P(s)$ is a rational function. Suppose that

$$\Phi(s) = (-1)^\delta \Phi(r - s)$$

with $\delta = 0$ or $\delta = 1$. For $z \in \mathbb{H}$, let

$$F(z) = \frac{1}{2\pi i} \int_{(\sigma_2)} \Phi(s)\,(z/i)^{-s}\ ds,$$

where $\sigma_2 = \sigma_0 + \epsilon$ and $\int_{(\sigma_2)} = \lim_{T\to\infty} \int_{\sigma-iT}^{\sigma+iT}$. Also define for $u \in \mathbb{C}$ and $\sigma_2 = \sigma_0 + \epsilon, \ \sigma_1 = r - \sigma_2,$

$$S(u) = \sum_{\sigma_1 \leq \sigma \leq \sigma_2} \operatorname{Res}\left\{\Phi(s)\ u^s\right\}.$$

Generalized period polynomials

Then Grosswald ([15], page 116) proved that

$$F(-1/z) - (-1)^{\delta}(z/i)^{r}F(z) = S(z/i)$$

which in particular shows that

$$2iF'(i) + rF(i) = -\sum_{\sigma_1 \leq \sigma \leq \sigma_2} \operatorname{Res}\{s\Phi(s)\} \quad \text{if } \delta = 0, \tag{4}$$

$$2F(i) = \sum_{\sigma_1 \leq \sigma \leq \sigma_2} \operatorname{Res}\{\Phi(s)\} \quad \text{if } \delta = 1. \tag{5}$$

If we apply this lemma to the series $\phi(s) = \zeta(s)\zeta(s+2k+1)$ which satisfies the functional equation

$$\Phi(s) = (-1)^{k}\Phi(-2k-s),$$

with

$$\Phi(s) = (2\pi)^{-s}\Gamma(s)\phi(s),$$

we deduce Ramanujan's formula, or more precisely, (3) of which (1) is a special case.

The modular case

As noted earlier, the modular analogue of Ramanujan's formula had been worked out by many authors, the most notable being Weil [31]. Razar [23] and Weil [31] derive the following result. Let

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda},$$

$$g(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z/\lambda},$$

$$f^*(z) = \frac{a_0 z^{k-1}}{(k-1)!} + \left(\frac{2\pi i}{\lambda}\right)^{-(k-1)} \sum_{n=1}^{\infty} \frac{a_n}{n^{k-1}} e^{2\pi i n z/\lambda},$$

$$g^*(z) = \frac{b_0 z^{k-1}}{(k-1)!} + \left(\frac{2\pi i}{\lambda}\right)^{-(k-1)} \sum_{n=1}^{\infty} \frac{b_n}{n^{k-1}} e^{2\pi i n z/\lambda},$$

and

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

If k is a positive integer and γ a complex number such that

$$f(z) = \gamma z^{-k} g(-1/z),$$

then

$$(6) \qquad f^*(z) - \gamma z^{k-2} g^*(-1/z) = \sum_{j=0}^{k-2} \frac{\phi(k-1-j)}{j!} \left(\frac{2\pi i}{\lambda}\right)^{-(k-1-j)} z^j.$$

Some results on transcendence

- Joint work with S. Gun and P. Rath

Let f be a normalized Hecke eigenform of level N. Following the theory of Eichler, Shimura and Manin, we know that there are two "periods", ω_+ and ω_- such that $L(f,j) \in \omega_\pm \pi^j \overline{\mathbb{Q}}$. It is perhaps true that the numbers π, ω_+, ω_- are algebraically independent.

Theorem 2. *Let f be a normalized Hecke eigenform of weight k and level N. Suppose that π, ω_+, ω_- are as above and algebraically independent. Then, there are at most k algebraic values of z in the upper half plane such that*

$$f^*(z) - \gamma z^{k-2} f^*(-1/z)$$

is algebraic.

The case k=2

- In this case, the right hand side is constant and equal to φ(1)(λ/2πi).
- The left hand side is

$$f^*(z) - \gamma z^{k-2} g^*(-1/z)$$

where

$$f^*(z) = \frac{a_0 z^{k-1}}{(k-1)!} + \left(\frac{2\pi i}{\lambda}\right)^{-(k-1)} \sum_{n=1}^{\infty} \frac{a_n}{n^{k-1}} e^{2\pi i n z/\lambda},$$

$$g^*(z) = \frac{b_0 z^{k-1}}{(k-1)!} + \left(\frac{2\pi i}{\lambda}\right)^{-(k-1)} \sum_{n=1}^{\infty} \frac{b_n}{n^{k-1}} e^{2\pi i n z/\lambda},$$

An application of a theorem of Bertrand

Proposition 6.1. *Let $f(z)$ be a normalized Hecke eigenform of weight 2 on $\Gamma_0(N)$. Let τ be a rational number or an element of the upper half-plane such that the modular invariant $j(\tau)$ is algebraic. Then, any determination of the integral*

$$2\pi i \int_{i\infty}^{\tau} f(z)dz$$

is either zero or transcendental.

An application of this theorem to our setting leads to the following corollary.

Let f be a normalized Hecke eigenform of weight 2 on $\Gamma_0(N)$. Then $L(f,1)/\pi$ is either zero or transcendental.

A more general theorem

- Theorem (S. Gun, R. Murty & P. Rath) Let f be a normalized Hecke eigenform of weight k on $\Gamma_0(N)$. For σ in $\Gamma_0(N)$, the function $f^*(z) - (f|\sigma)^*(z)$ can take on at most k+3 algebraic values as z ranges over algebraic numbers in the upper half-plane.

Summary

- The Ramanujan-Grosswald formula gives the value of $\zeta(2k+1)$ as an algebraic linear combination of π^{2k+1} and the sum of two Eichler integrals evaluated at algebraic points in the upper half-plane.
- If our conjecture concerning the location of zeros of the Ramanujan polynomials is correct, and if in addition, these zeros are not 2k-th roots of unity, as numerical evidence seems to suggest, then one can deduce that $\zeta(2k+1)$ is the sum of two Eichler integrals evaluated at certain algebraic numbers lying on the unit circle.