Addition laws on elliptic curves

D. J. Bernstein
University of Illinois at Chicago

Joint work with:
Tanja Lange
Technische Universiteit Eindhoven

2007.01.10, 09:00 (yikes!),
Leiden University, part of
"Mathematics: Algorithms and
Proofs" week at Lorentz Center:

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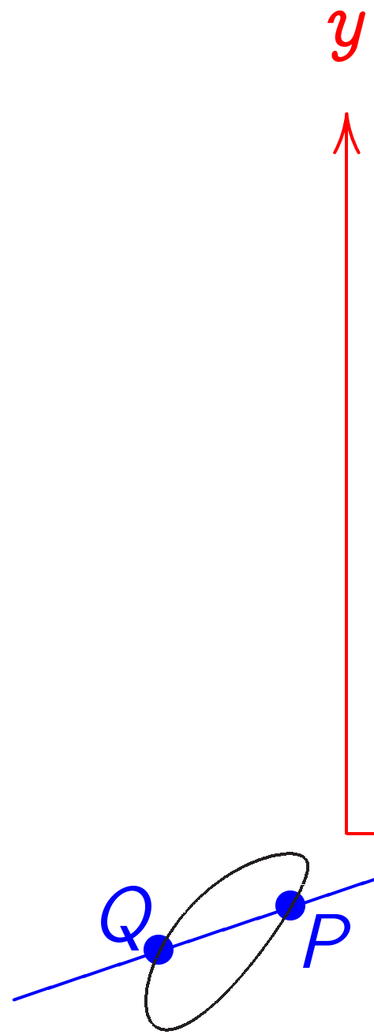

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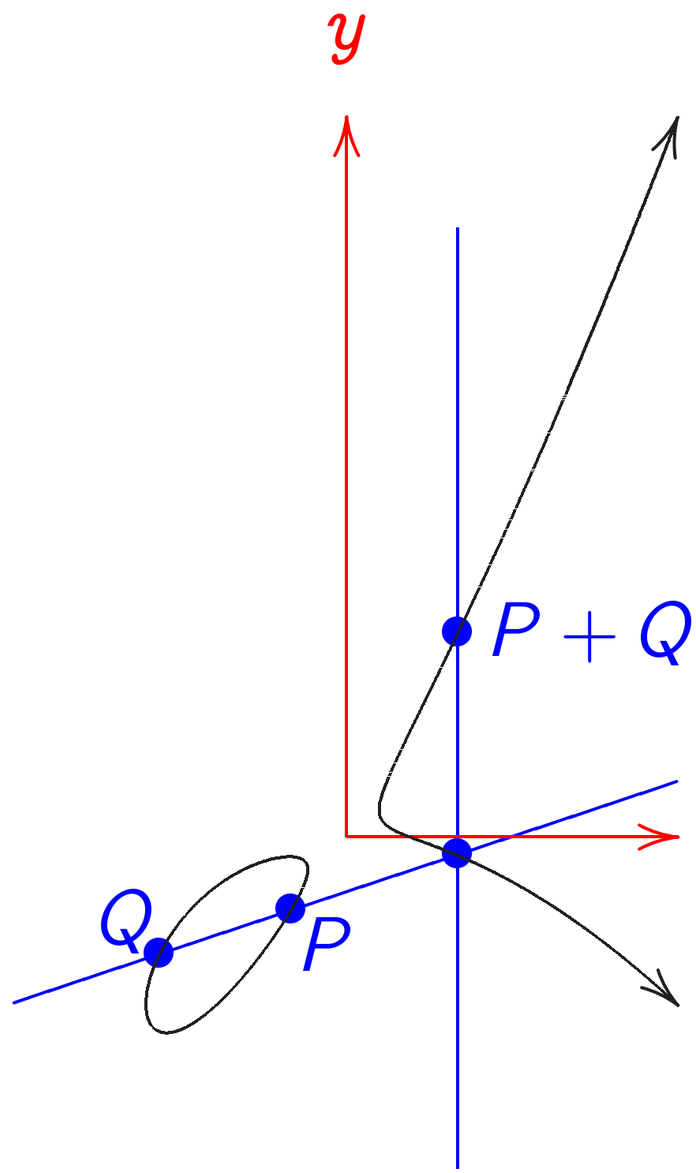

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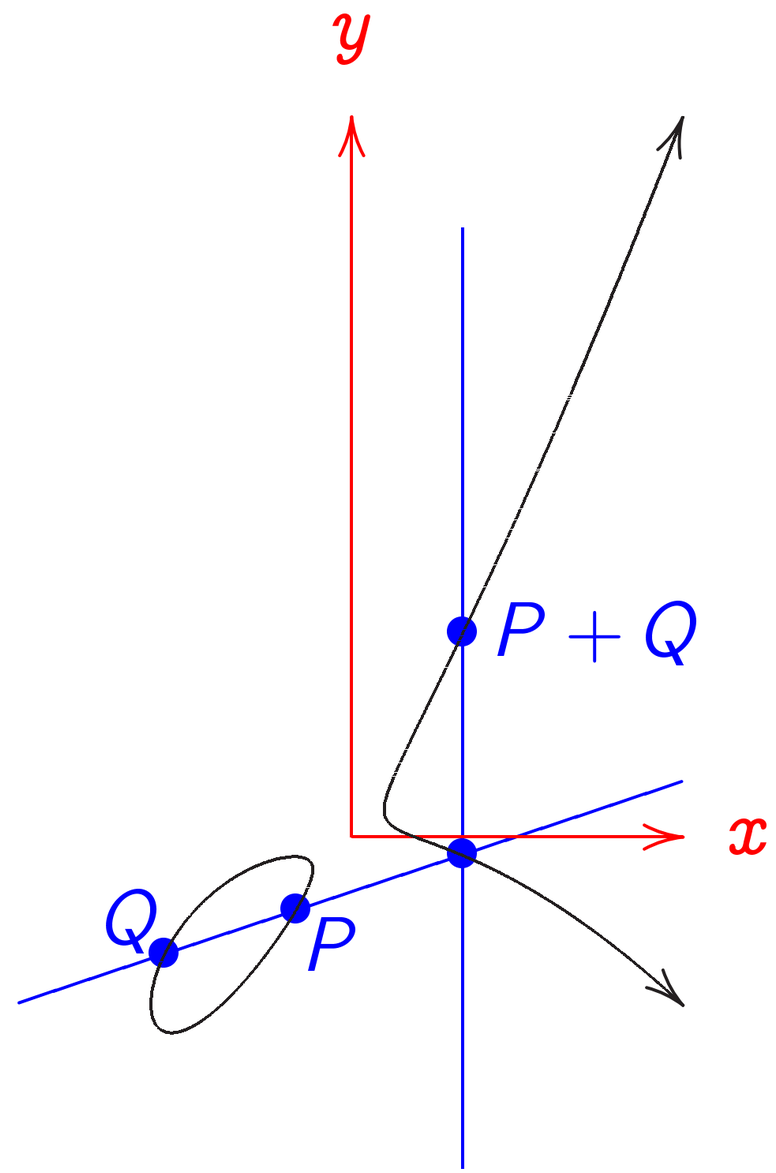

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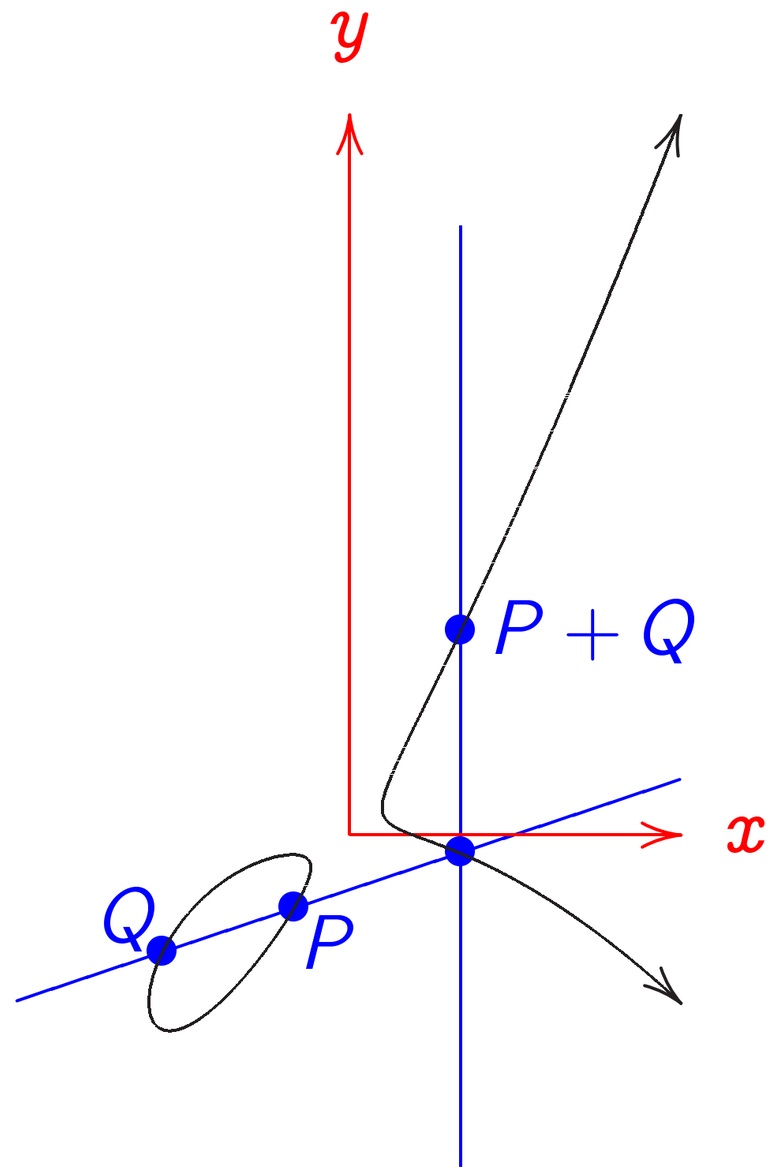

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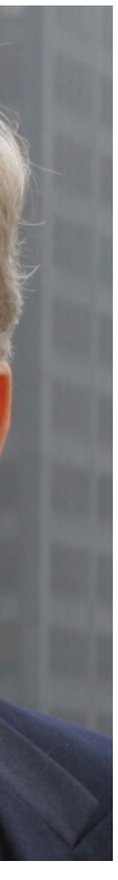

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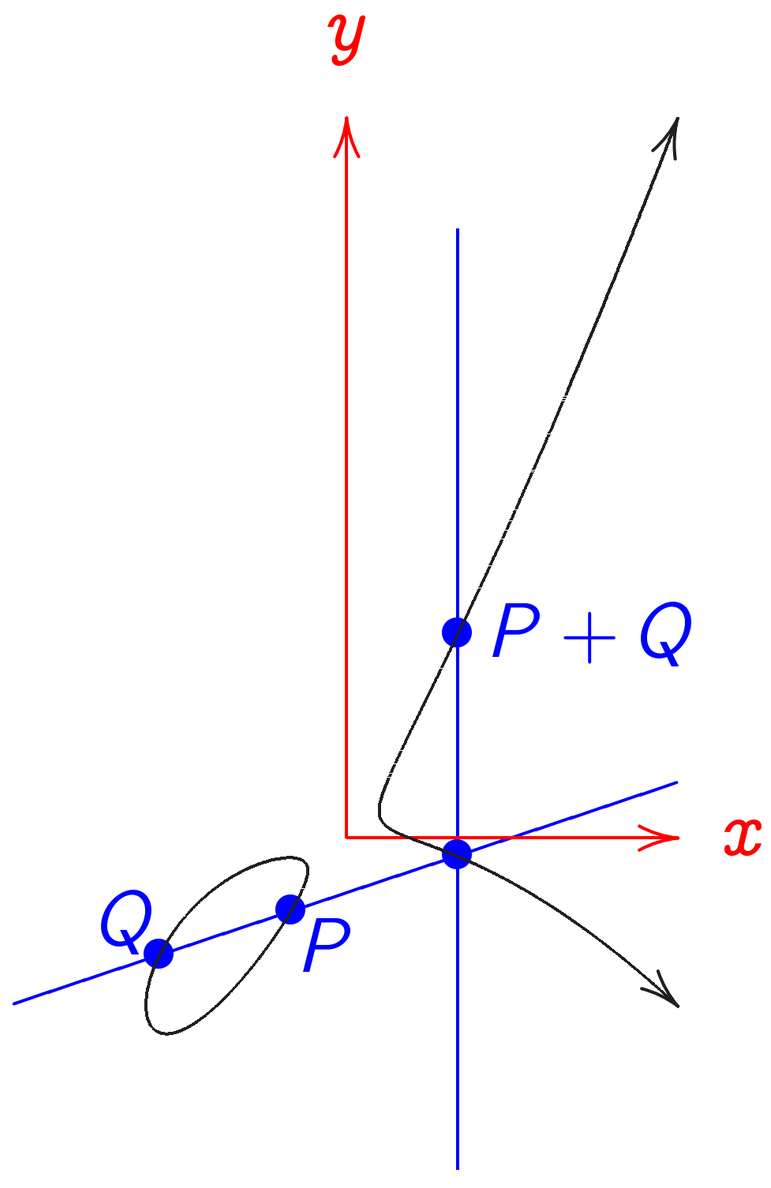

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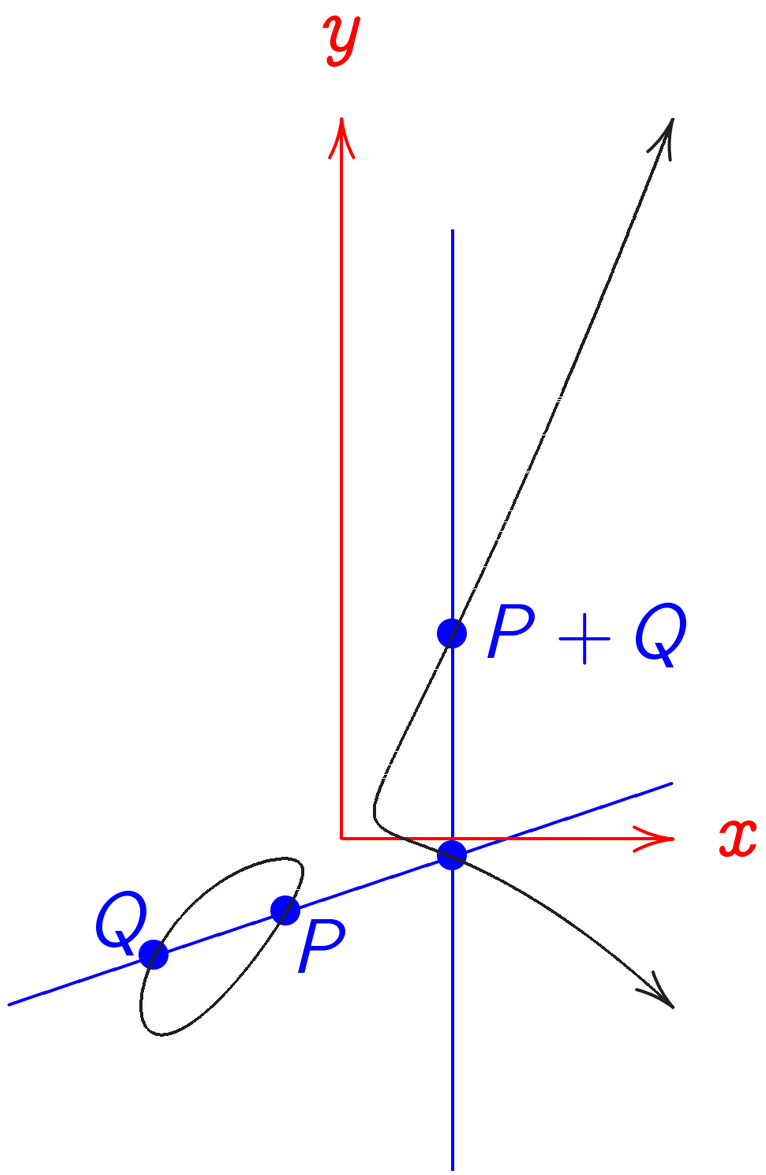

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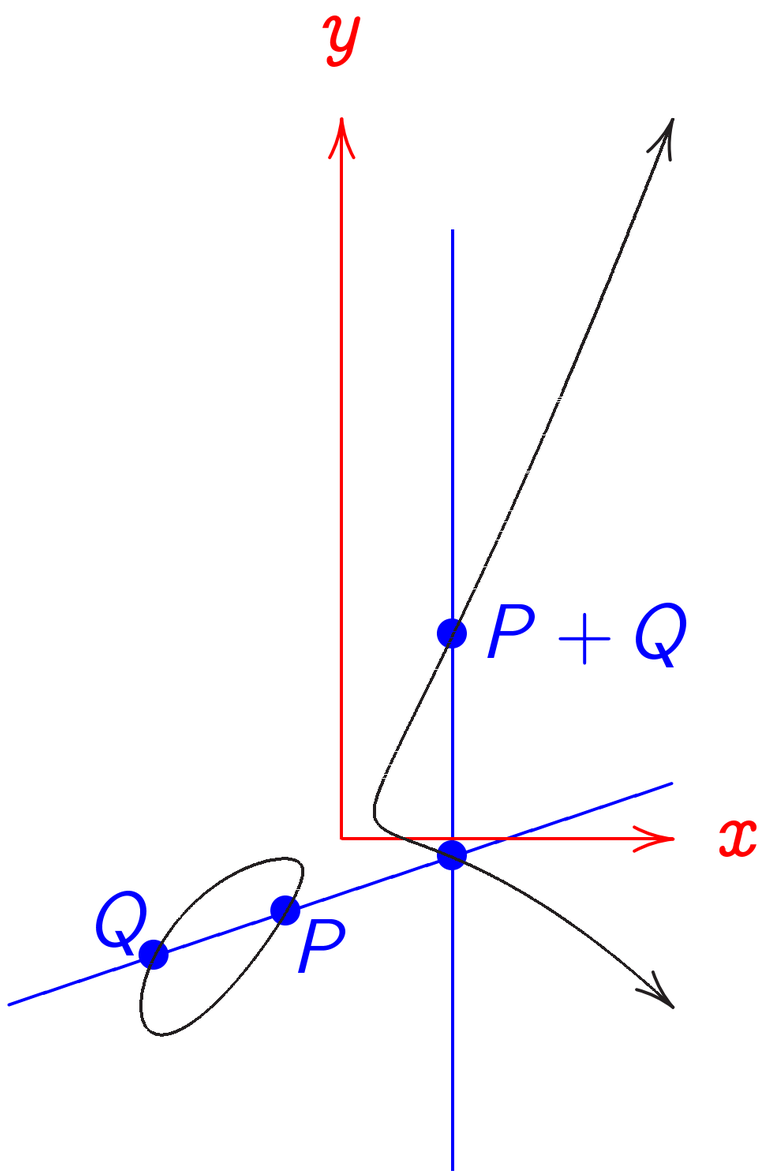

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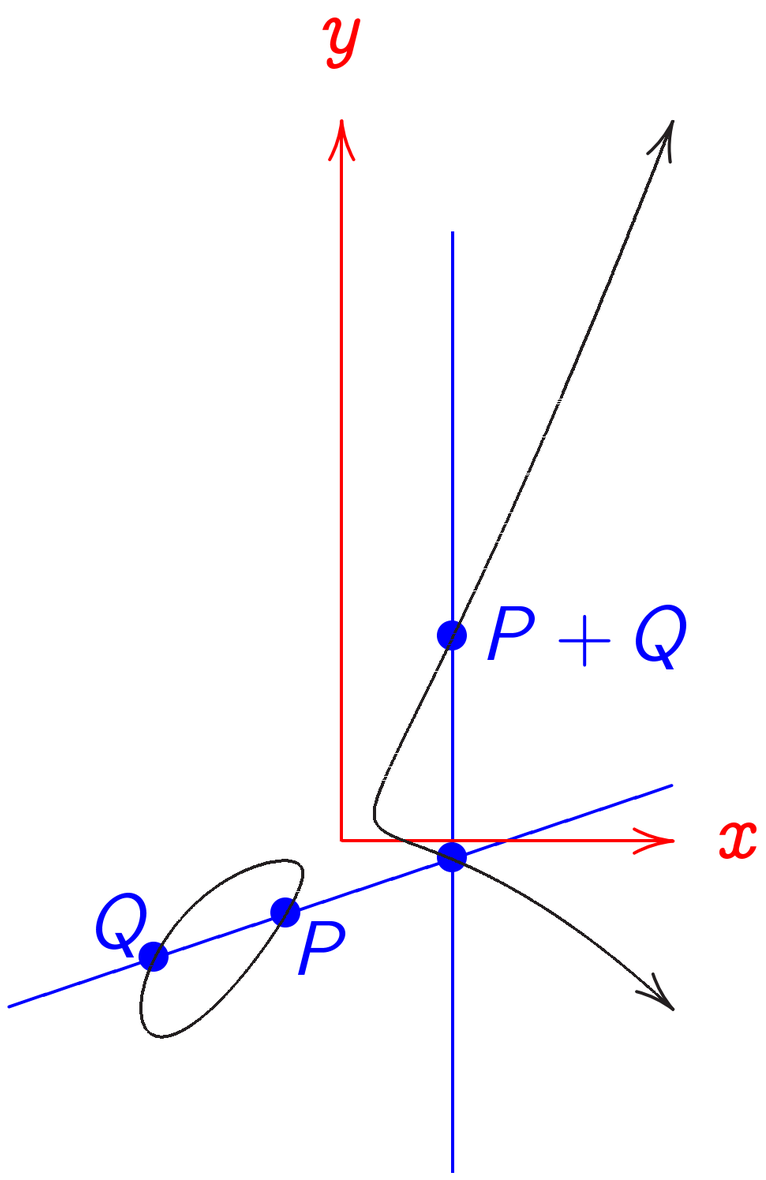

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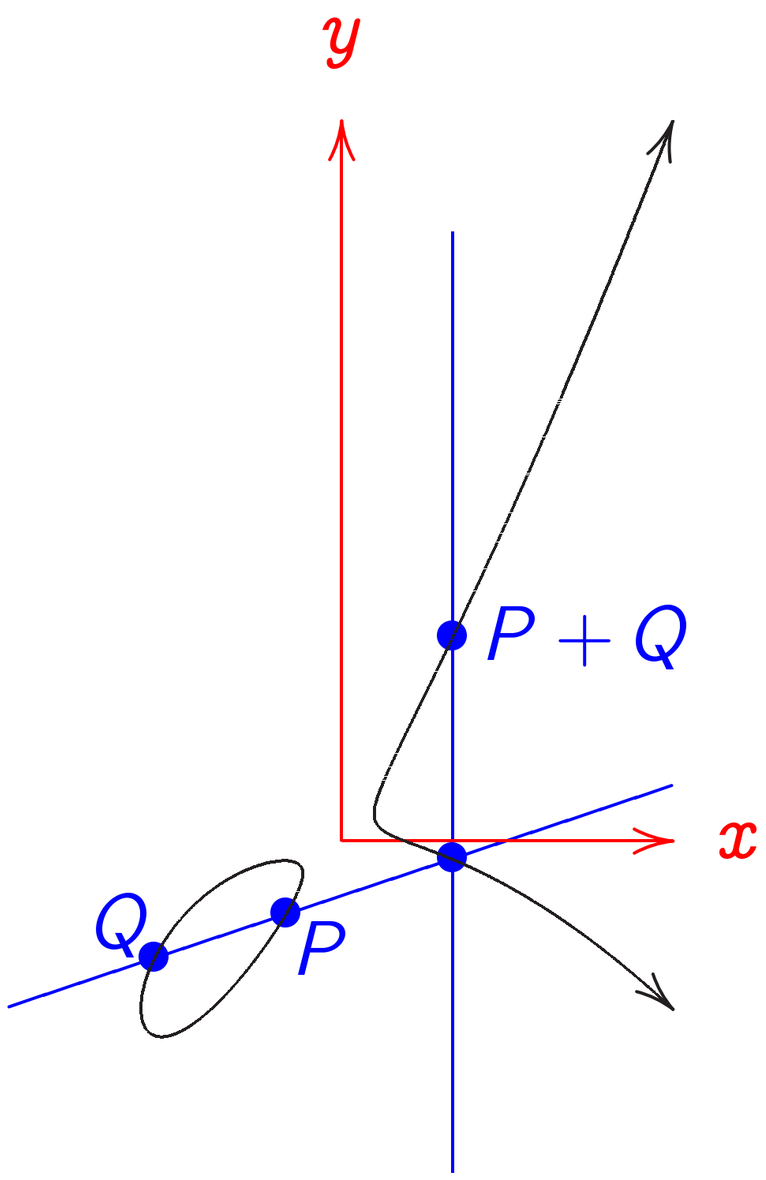

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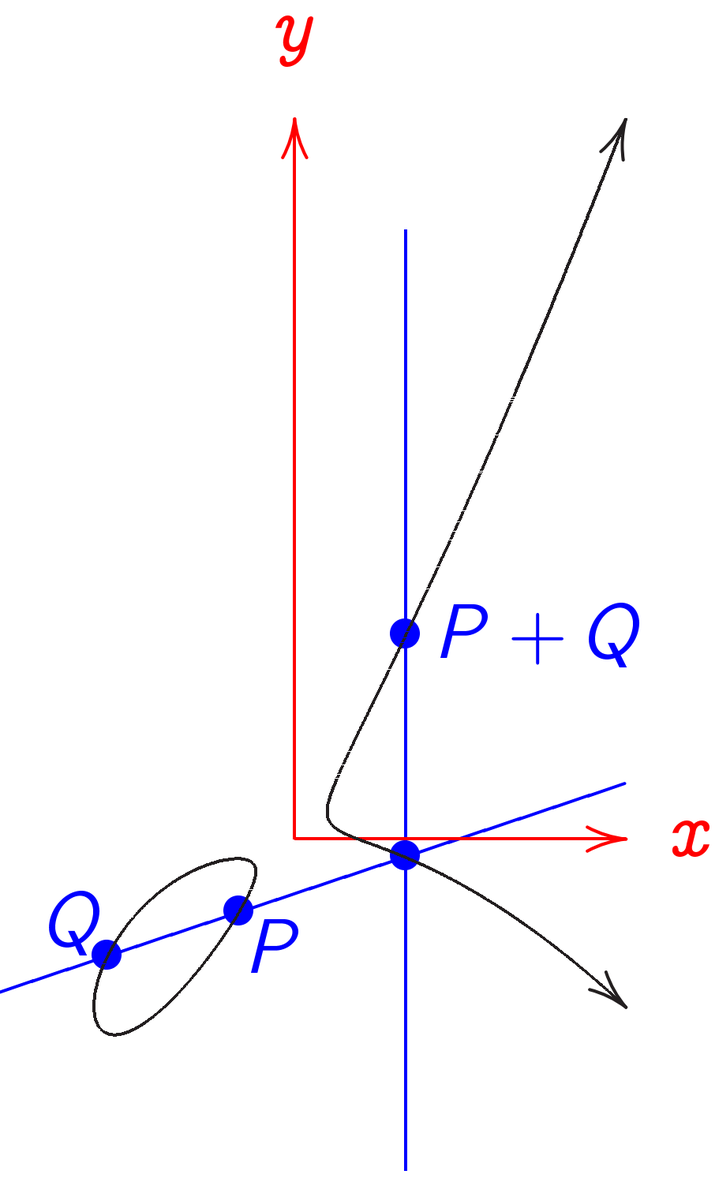

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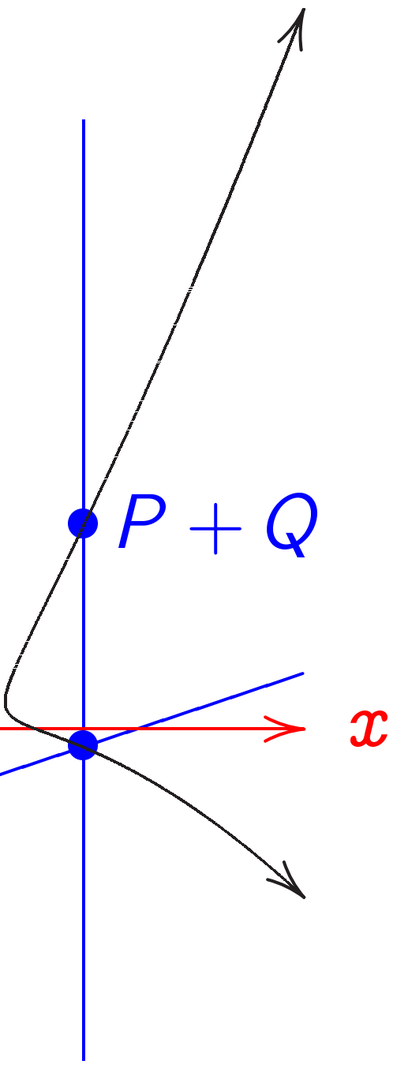

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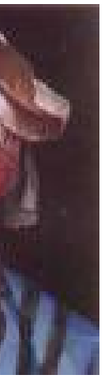
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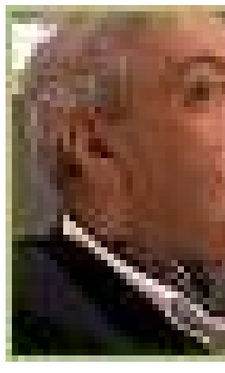

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Pick a prime p; e.g. 47.
Pick curve param $d \in \mathbf{F}_p$.

Enumerate all affine points
$(x, y) \in \mathbf{F}_p \times \mathbf{F}_p$ satisfying
$x^2 + y^2 = 1 + dx^2y^2$.

Use generalized addition la
to make an addition table
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Check associativity etc.

Convenient to take $c = 1$
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$(c, d) \equiv (1, dc^4)$.

$x^2 + y^2 = 1 + dx^2y^2$ has
neutral element $(0, 1)$, addition
$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ with

$$x_3 = \frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2},$$

$$y_3 = \frac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1y_2}.$$

Hmmm, does this really work?

Easiest way to check
the generalized addition law:
pull out the computer!

Pick a prime p; e.g. 47.
Pick curve param $d \in \mathbf{F}_p$.

Enumerate all affine points
$(x, y) \in \mathbf{F}_p \times \mathbf{F}_p$ satisfying
$x^2 + y^2 = 1 + dx^2y^2$.

Use generalized addition law
to make an addition table
for all pairs of points.
Check associativity etc.

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Addition law works generic
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Unified addition law
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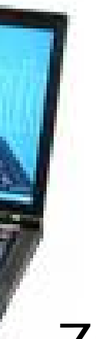

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$= d^2x_1^2y_1^2x_2^2y_2^2 + dx_1^2y_1^2 + 2dx_1^2y_1^2x_2y_2$
$= 1 + dx_1^2y_1^2 \pm 2x_1y_1$
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2008 B.–Birkner–L.–Peters:
"twisted Edwards curves"
$ax^2 + y^2 = 1 + dx^2y^2$
cover all Montgomery curves.

Almost as fast as $a = 1$;
brings Edwards speed
to larger class of curves.

2008 B.–B.–Joye–L.–P.:
every elliptic curve over $\mathbf{F}_p$
where 4 divides group order
is (1 or 2)-isogenous
to a twisted Edwards curve.

about elliptic curves
t points of order 4?

about elliptic curves
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uing project (B.–L.):
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$4\mathbf{Z}$	$\frac{5}{6}p$	0
all	$2p$	$\frac{2}{3}p$

Different statistic

Bad news:
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Statistics for many $p \in 1 +$
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Curves	total	odd	2odd	4odd
orig	$\frac{1}{24}p$	0	0	0
compl	$\frac{1}{2}p$	0	0	$\frac{1}{4}p$
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twist	$\frac{5}{6}p$	0	0	$\frac{5}{12}p$
$4\mathbf{Z}$	$\frac{5}{6}p$	0	0	$\frac{5}{12}p$
all	$2p$	$\frac{2}{3}p$	$\frac{1}{2}p$	$\frac{5}{12}p$

Different statistics for $3 +$

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Statistics for many $p \in 1 + 4\mathbf{Z}$,
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Curves	total	odd	2odd	4odd	8odd
orig	$\frac{1}{24}p$	0	0	0	0
compl	$\frac{1}{2}p$	0	0	$\frac{1}{4}p$	$\frac{1}{8}p$
Ed	$\frac{2}{3}p$	0	0	$\frac{1}{4}p$	$\frac{3}{16}p$
twist	$\frac{5}{6}p$	0	0	$\frac{5}{12}p$	$\frac{3}{16}p$
$4\mathbf{Z}$	$\frac{5}{6}p$	0	0	$\frac{5}{12}p$	$\frac{3}{16}p$
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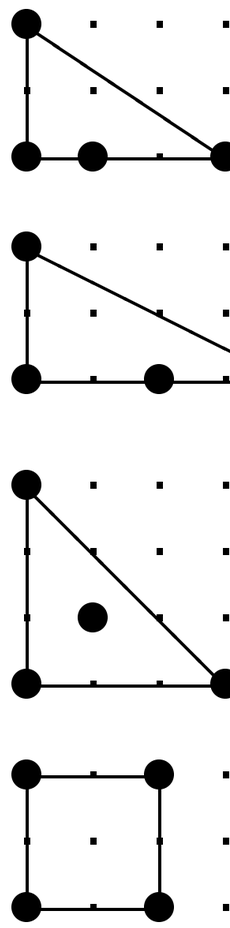

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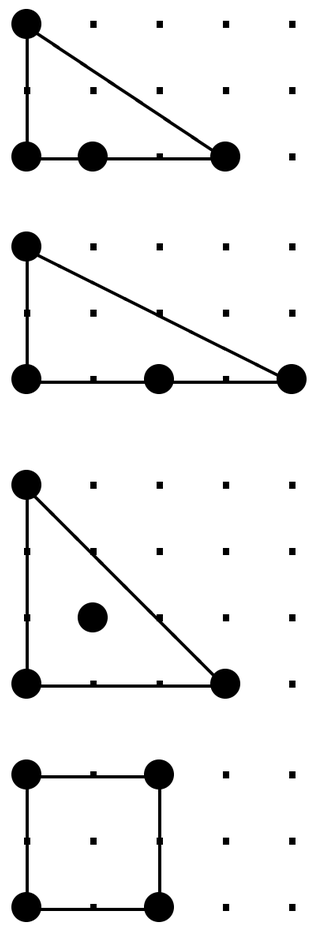
Some Newton po



Short W
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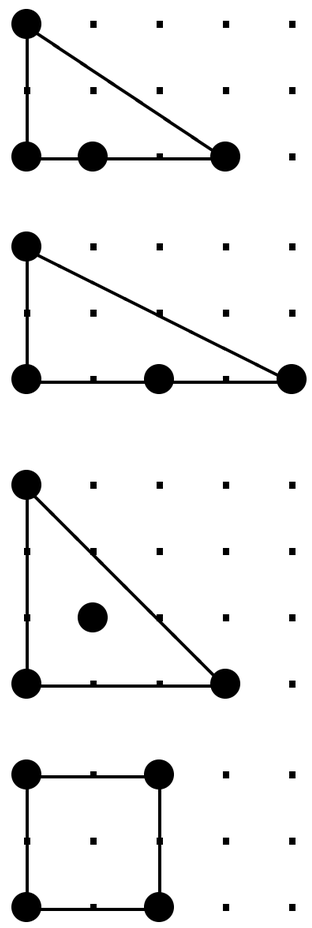
Some Newton polygons



Short Weierstrass

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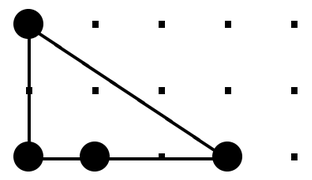
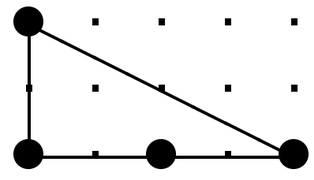

Short Weierstrass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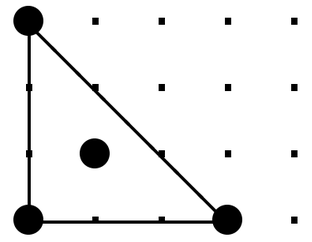
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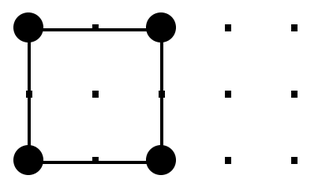
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s	total	odd	2odd	4odd	8odd
	$\frac{1}{24}p$	0	0	0	0
	$\frac{1}{2}p$	0	0	$\frac{1}{4}p$	$\frac{1}{8}p$
	$\frac{2}{3}p$	0	0	$\frac{1}{4}p$	$\frac{3}{16}p$
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	$2p$	$\frac{2}{3}p$	$\frac{1}{2}p$	$\frac{5}{12}p$	$\frac{3}{16}p$

nt statistics for $3 + 4\mathbf{Z}$.

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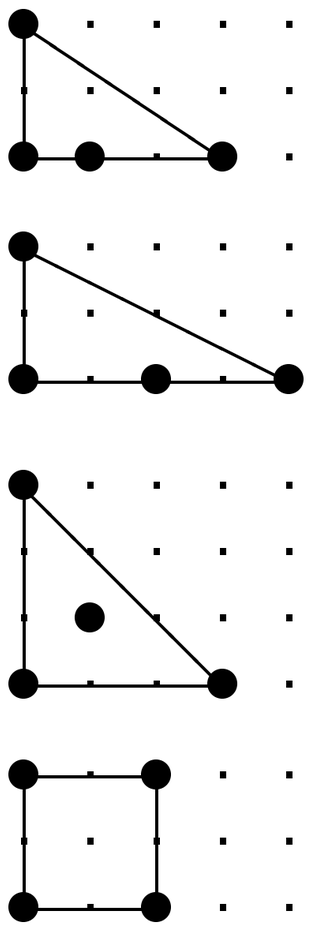

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How to

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$x \leftrightarrow y$

Curve s

$d_{11}xy$

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ny $p \in 1 + 4\mathbf{Z}$,
rs $(j(E), \#E)$:

2odd	4odd	8odd
0	0	0
0	$\frac{1}{4}p$	$\frac{1}{8}p$
0	$\frac{1}{4}p$	$\frac{3}{16}p$
0	$\frac{5}{12}p$	$\frac{3}{16}p$
0	$\frac{5}{12}p$	$\frac{3}{16}p$
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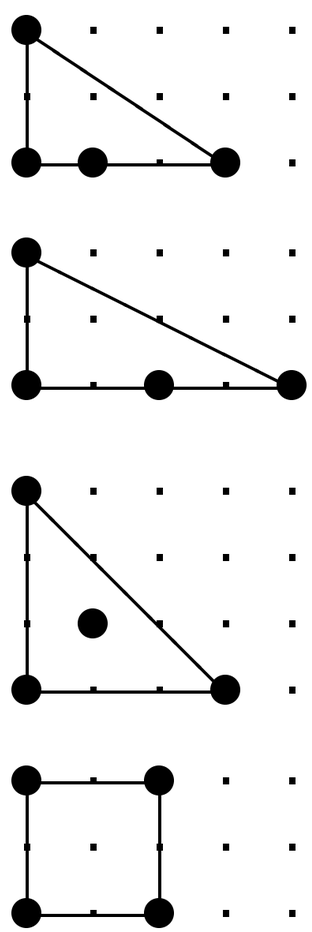

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Design decision:
quadratic in x ar

Design decision:
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d_{20} d

d_{10} d

d_{00} d

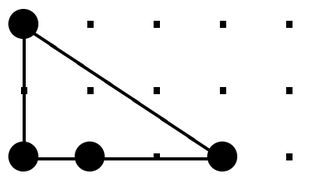
Curve shape d_{00}
$d_{11}xy + d_{20}(x^2$
$d_{21}xy(x + y) +$

$- 4\mathbf{Z}$,
$\#E)$:

d	8odd
0	0
p	$\frac{1}{8}p$
p	$\frac{3}{16}p$
p	$\frac{3}{16}p$
p	$\frac{3}{16}p$

$4\mathbf{Z}$.

Some Newton polygons



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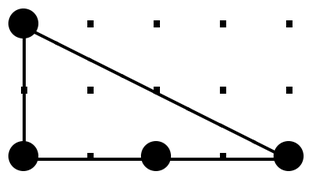

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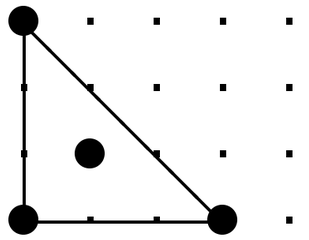

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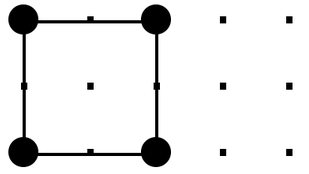

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How to generalize Edwards

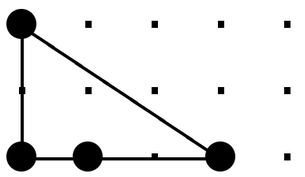
Design decision: want quadratic in x and in y.

Design decision: want $x \leftrightarrow y$ symmetry.

$$\begin{matrix} d_{20} & d_{21} & d_{22} \\ d_{10} & d_{11} & d_{21} \\ d_{00} & d_{10} & d_{20} \end{matrix}$$

Curve shape $d_{00} + d_{10}(x +$
$d_{11}xy + d_{20}(x^2 + y^2) +$
$d_{21}xy(x + y) + d_{22}x^2y^2 =$

Some Newton polygons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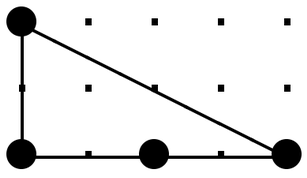
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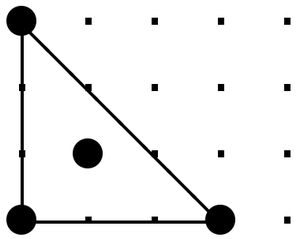
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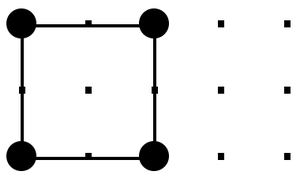
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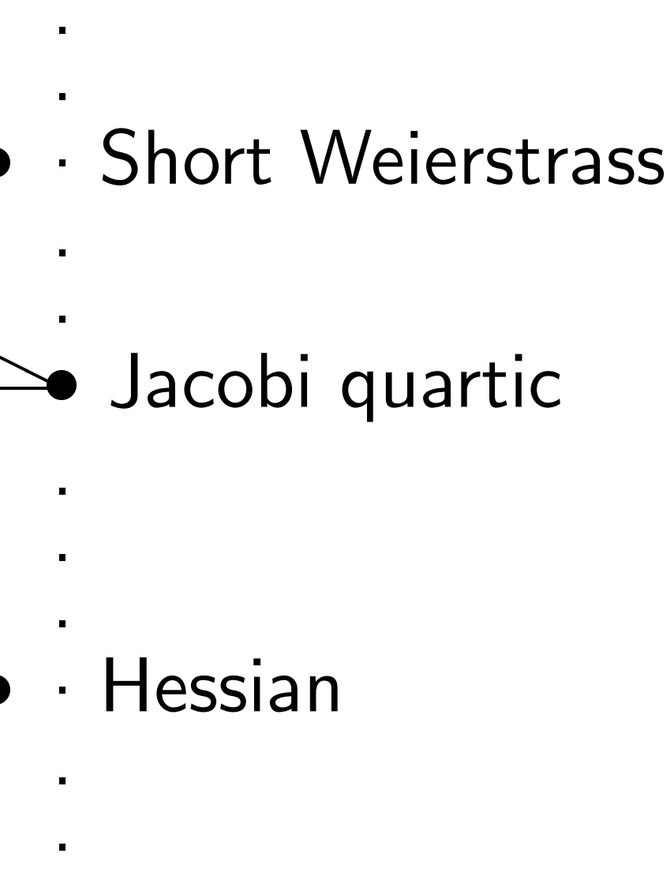
Curve shape $d_{00} + d_{10}(x + y) + d_{11}xy + d_{20}(x^2 + y^2) + d_{21}xy(x + y) + d_{22}x^2y^2 = 0$.

Newton polygons



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$d_{11}XY$
$d_{21}XY$

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$$d_{00} \quad d_{10} \quad d_{20}$$

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$d_{21}xy(x + y) + d_{22}x^2y^2 = 0.$

Suppose that d_{22}

$$d_{20} \quad d$$

$$d_{10} \quad d$$

$$d_{00} \quad d$$

Genus $1 \Rightarrow (1, 1)$
interior point $\Rightarrow$

Homogenize:
$d_{00}Z^3 + d_{10}(X +$
$d_{11}XYZ + d_{20}(X$
$d_{21}XY(X + Y) =$

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$d_{21}xy(x + y) + d_{22}x^2y^2 = 0.$

Suppose that $d_{22} = 0$:

$$\begin{array}{ccc} d_{20} & d_{21} & \cdot \\ \\ d_{10} & d_{11} & d_{21} \\ \\ d_{00} & d_{10} & d_{20} \end{array}$$

Genus 1 $\Rightarrow (1, 1)$ is an
interior point $\Rightarrow d_{21} \neq 0$.

Homogenize:
$d_{00}Z^3 + d_{10}(X + Y)Z^2 +$
$d_{11}XYZ + d_{20}(X^2 + Y^2)Z$
$d_{21}XY(X + Y) = 0.$

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decision: want
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decision: want
symmetry.

$$d_{20} \quad d_{21} \quad d_{22}$$

$$d_{10} \quad d_{11} \quad d_{21}$$

$$d_{00} \quad d_{10} \quad d_{20}$$

shape $d_{00} + d_{10}(x + y) +$
$+ d_{20}(x^2 + y^2) +$
$x + y) + d_{22}x^2y^2 = 0.$

Suppose that $d_{22} = 0$:

$$d_{20} \quad d_{21} \quad .$$

$$d_{10} \quad d_{11} \quad d_{21}$$

$$d_{00} \quad d_{10} \quad d_{20}$$

Genus $1 \Rightarrow (1, 1)$ is an
interior point $\Rightarrow d_{21} \neq 0$.

Homogenize:
$d_{00}Z^3 + d_{10}(X + Y)Z^2 +$
$d_{11}XYZ + d_{20}(X^2 + Y^2)Z +$
$d_{21}XY(X + Y) = 0.$

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$$d_{11} \quad d_{21}$$

$$d_{10} \quad d_{20}$$

$$+ d_{10}(x+y) +$$

$$+ y^2) +$$

$$d_{22}x^2y^2 = 0.$$

Suppose that $d_{22} = 0$:

$$d_{20} \quad d_{21} \quad .$$

$$d_{10} \quad d_{11} \quad d_{21}$$

$$d_{00} \quad d_{10} \quad d_{20}$$

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$d_{11}XYZ + d_{20}(X^2+Y^2)Z +$
$d_{21}XY(X+Y) = 0.$

Points at ∞ are
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Study $(1:0:0)$
$y = Y/X$, $z = Z$
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$d_{00}z^3 + d_{10}(1+$
$d_{11}yz + d_{20}(1+$
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Suppose that $d_{22} = 0$:

$$\begin{array}{ccc} d_{20} & d_{21} & \cdot \\ d_{10} & d_{11} & d_{21} \\ d_{00} & d_{10} & d_{20} \end{array}$$

Genus 1 $\Rightarrow (1,1)$ is an interior point $\Rightarrow d_{21} \neq 0$.

Homogenize:
$d_{00}Z^3 + d_{10}(X+Y)Z^2 +$
$d_{11}XYZ + d_{20}(X^2+Y^2)Z +$
$d_{21}XY(X+Y) = 0.$

Points at ∞ are $(X:Y:0$
with $d_{21}XY(X+Y) = 0$:
$(1:0:0)$, $(0:1:0)$, $(1:-$

Study $(1:0:0)$ by setting
$y = Y/X$, $z = Z/X$
in homogeneous curve equa
$d_{00}z^3 + d_{10}(1+y)z^2 +$
$d_{11}yz + d_{20}(1+y^2)z +$
$d_{21}y(1+y) = 0.$

Nonzero coefficient of y
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Suppose that $d_{22} = 0$:

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$$\begin{matrix} d_{10} & d_{11} & d_{21} \end{matrix}$$

$$\begin{matrix} d_{00} & d_{10} & d_{20} \end{matrix}$$

Genus 1 $\Rightarrow (1,1)$ is an interior point $\Rightarrow d_{21} \neq 0$.

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$d_{11}XYZ + d_{20}(X^2+Y^2)Z +$
$d_{21}XY(X+Y) = 0.$

Points at ∞ are $(X:Y:0)$
with $d_{21}XY(X+Y) = 0$: i.e.,
$(1:0:0)$, $(0:1:0)$, $(1:-1:0)$.

Study $(1:0:0)$ by setting
$y = Y/X$, $z = Z/X$
in homogeneous curve equation:
$d_{00}z^3 + d_{10}(1+y)z^2 +$
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Points at ∞ are $(X : Y : 0)$
with $d_{21}XY(X+Y) = 0$: i.e.,
$(1:0:0)$, $(0:1:0)$, $(1:-1:0)$.

Study $(1:0:0)$ by setting
$y = Y/X$, $z = Z/X$
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$d_{11}yz + d_{20}(1+y^2)z +$
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Points at ∞ are $(X : Y : 0)$
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$(1 : 0 : 0)$, $(0 : 1 : 0)$, $(1 : -1 : 0)$.

Study $(1 : 0 : 0)$ by setting
$y = Y/X$, $z = Z/X$
in homogeneous curve equation:
$d_{00}z^3 + d_{10}(1 + y)z^2 +$
$d_{11}yz + d_{20}(1 + y^2)z +$
$d_{21}y(1 + y) = 0$.

Nonzero coefficient of y
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Addition law cannot be complete
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Points at ∞ are
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Study $(1 : 0 : 0)$
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Points at ∞ are $(X : Y : 0)$
with $d_{21}XY(X+Y) = 0$: i.e.,
$(1 : 0 : 0)$, $(0 : 1 : 0)$, $(1 : -1 : 0)$.

Study $(1 : 0 : 0)$ by setting
$y = Y/X$, $z = Z/X$
in homogeneous curve equation:
$d_{00}z^3 + d_{10}(1+y)z^2 +$
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Study $(1 : 0 : 0)$ again:
$d_{00}z^4 + d_{10}(1+y)z^3 +$
$d_{11}yz^2 + d_{20}(1+y^2)z^2 +$
$d_{21}y(1+y)z + d_{22}y^2 = 0.$

Coefficients of $1, y, z$ are 0
so $(1 : 0 : 0)$ is singular.

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Study $(1 : 0 : 0)$ by setting
$y = Y/X$, $z = Z/X$
in homogeneous curve equation:
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$_{21}XY(X+Y) = 0$: i.e.,

$0)$, $(0 : 1 : 0)$, $(1 : -1 : 0)$.

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Coefficients of $1, y, z$ are 0
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So we require $d_{22} \neq 0$.

Points at ∞ are $(X : Y : 0)$
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Study $(1 : 0 : 0)$ again:
$d_{00}z^4 + d_{10}(1 + y)z^3 +$
$d_{11}yz^2 + d_{20}(1 + y^2)z^2 +$
$d_{21}y(1 + y)z + d_{22}y^2 = 0.$

Coefficients of $1, y, z$ are 0
so $(1 : 0 : 0)$ is singular.

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$d_{21}u(1 + uz) +$

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So we require $d_{22} \neq 0$.

Points at ∞ are $(X : Y : 0)$
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$d_{00}z^4 + d_{10}(1 + y)z^3 +$
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Coefficients of $1, y, z$ are 0
so $(1 : 0 : 0)$ is singular.

Put $y = uz$, divide by z^2
to blow up singularity:

$d_{00}z^2 + d_{10}(1 + uz)z +$
$d_{11}uz + d_{20}(1 + u^2z^2) +$
$d_{21}u(1 + uz) + d_{22}u^2 = 0$

Substitute $z = 0$ to find
points above singularity:
$d_{20} + d_{21}u + d_{22}u^2 = 0.$

We require the quadratic
$d_{20} + d_{21}u + d_{22}u^2$
to be irreducible in k.
Special case: complete Edw
$1 - du^2$ irreducible in k.

So we require $d_{22} \neq 0$.

Points at ∞ are $(X : Y : 0)$
with $d_{22}X^2Y^2 = 0$: i.e.,
$(1 : 0 : 0)$, $(0 : 1 : 0)$.

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$d_{11}yz^2 + d_{20}(1 + y^2)z^2 +$
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Put $y = uz$, divide by z^2
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$d_{00}z^2 + d_{10}(1 + uz)z +$
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Put $y = uz$, divide by z^2
to blow up singularity:

$d_{00}z^2 + d_{10}(1 + uz)z +$
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In particular d_{20}

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Design decision:
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$d_{00} = 0$; $d_{10} \neq 0$

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Put $y = uz$, divide by z^2
to blow up singularity:

$d_{00}z^2 + d_{10}(1 + uz)z +$
$d_{11}uz + d_{20}(1 + u^2z^2) +$
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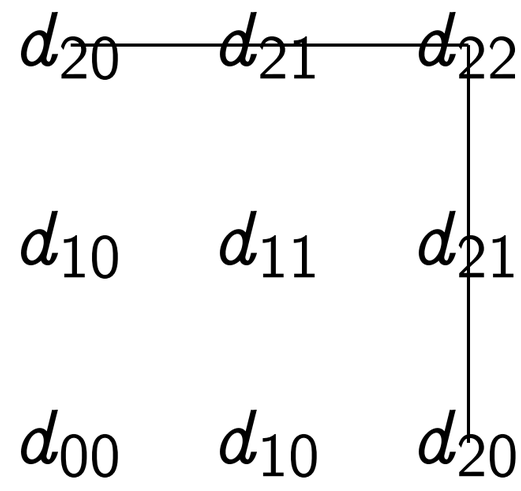
to be irreducible in k.
Special case: complete Edwards,
$1 - du^2$ irreducible in k.

In particular $d_{20} \neq 0$:



Design decision: Explore
a deviation from Edwards.
Choose neutral element (0,
$d_{00} = 0$; $d_{10} \neq 0$.

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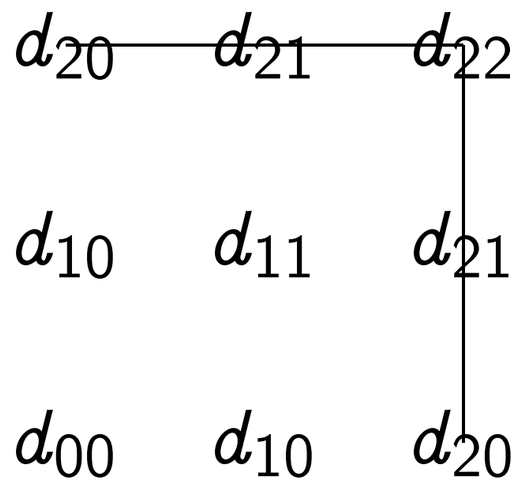
Put $y = uz$, divide by z^2
to blow up singularity:

$d_{00}z^2 + d_{10}(1 + uz)z +$
$d_{11}uz + d_{20}(1 + u^2z^2) +$
$d_{21}u(1 + uz) + d_{22}u^2 = 0.$

Substitute $z = 0$ to find
points above singularity:
$d_{20} + d_{21}u + d_{22}u^2 = 0.$

We require the quadratic
$d_{20} + d_{21}u + d_{22}u^2$
to be irreducible in k.
Special case: complete Edwards,
$1 - du^2$ irreducible in k.

In particular $d_{20} \neq 0$:



Design decision: Explore
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Choose neutral element $(0, 0)$.
$d_{00} = 0$; $d_{10} \neq 0$.

Can vary neutral element.
Warning: bad choice can produce
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$$+ d_{10}(1+uz)z +$$
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$d_{21}u + d_{22}u^2 = 0.$

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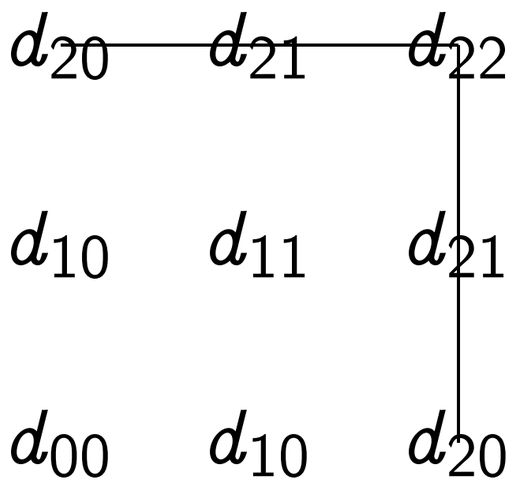

Design decision: Explore a deviation from Edwards.

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$d_{00} = 0$; $d_{10} \neq 0$.

Can vary neutral element.

Warning: bad choice can produce surprisingly expensive negation.

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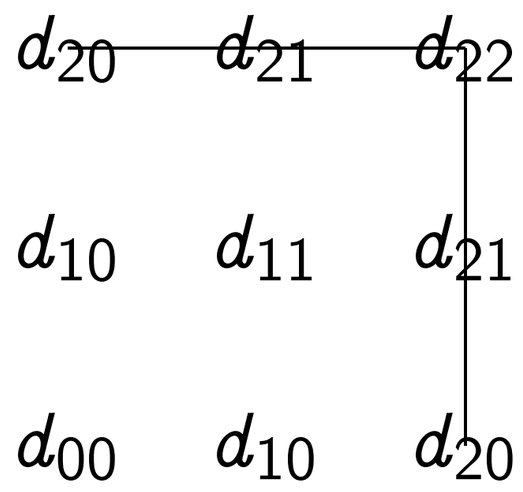
In particular $d_{20} \neq 0$:



Design decision: Explore a deviation from Edwards.

Choose neutral element $(0, 0)$.

$d_{00} = 0$; $d_{10} \neq 0$.

Can vary neutral element.

Warning: bad choice can produce surprisingly expensive negation.

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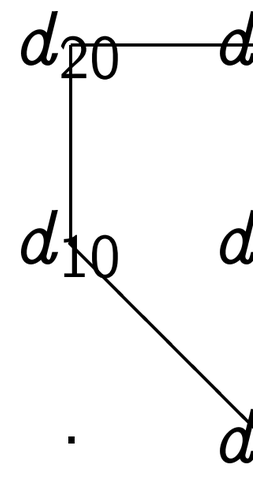

By scaling x, y

and scaling curve

can limit d_{10}, d_{11}

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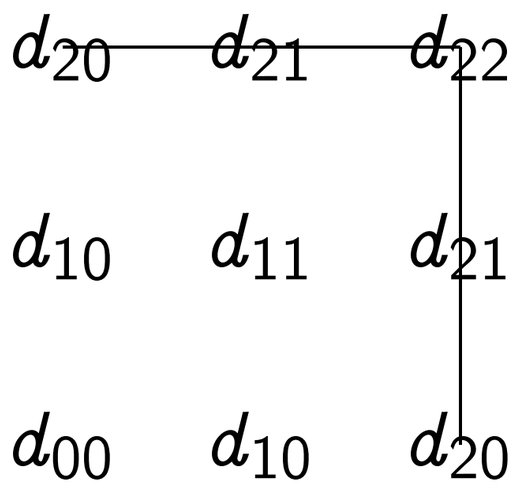
In particular $d_{20} \neq 0$:



Design decision: Explore
a deviation from Edwards.
Choose neutral element $(0, 0)$.
$d_{00} = 0$; $d_{10} \neq 0$.

Can vary neutral element.
Warning: bad choice can produce
surprisingly expensive negation.

Now have a Newton polyg
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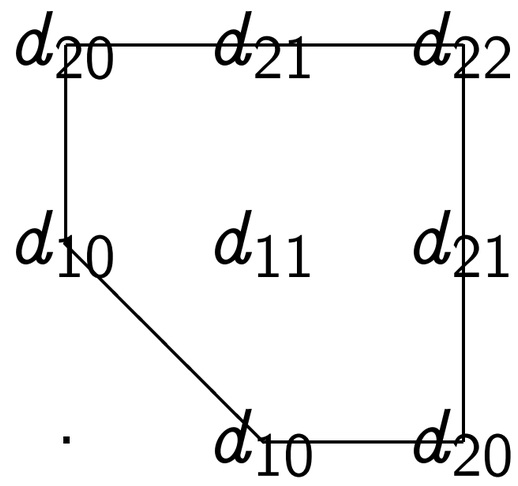
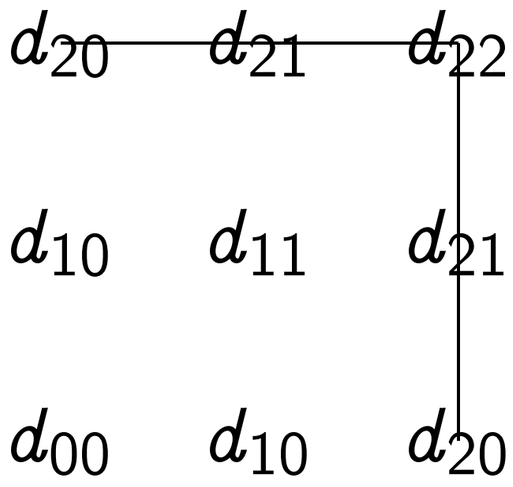


By scaling x, y
and scaling curve equation
can limit $d_{10}, d_{11}, d_{20}, d_{21}$,
to three degrees of freedon

In particular $d_{20} \neq 0$:



Design decision: Explore
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Choose neutral element $(0, 0)$.
$d_{00} = 0$; $d_{10} \neq 0$.

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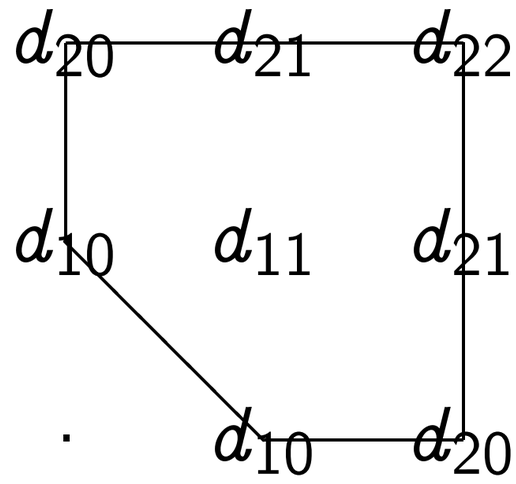
Now have a Newton polygon
for generalized Edwards curves:



By scaling x, y
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can limit $d_{10}, d_{11}, d_{20}, d_{21}, d_{22}$
to three degrees of freedom.

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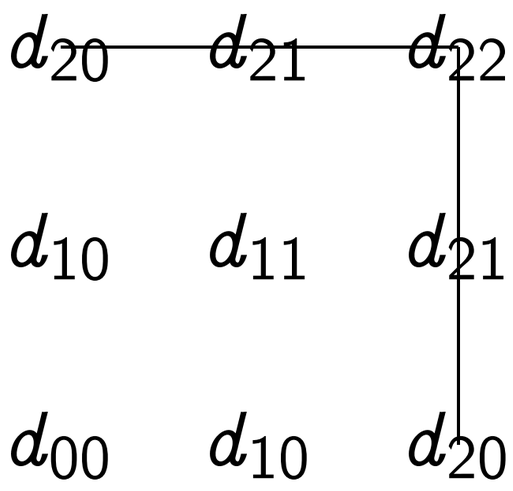

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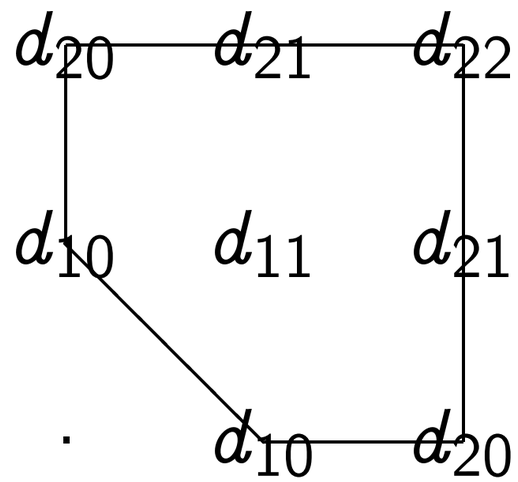
Now have a Newton polygon
for generalized Edwards curves:



By scaling x, y
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can limit $d_{10}, d_{11}, d_{20}, d_{21}, d_{22}$
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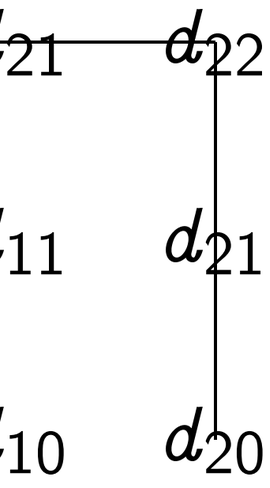

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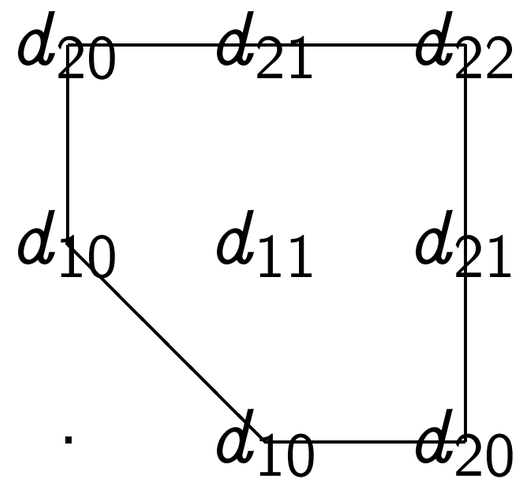

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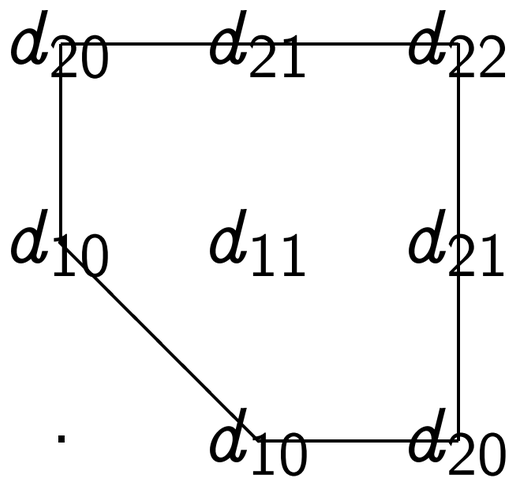

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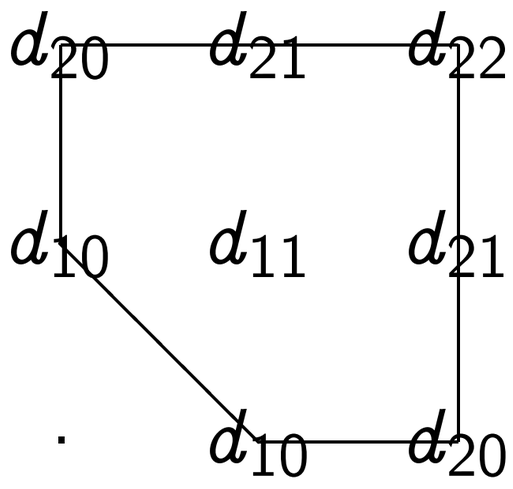

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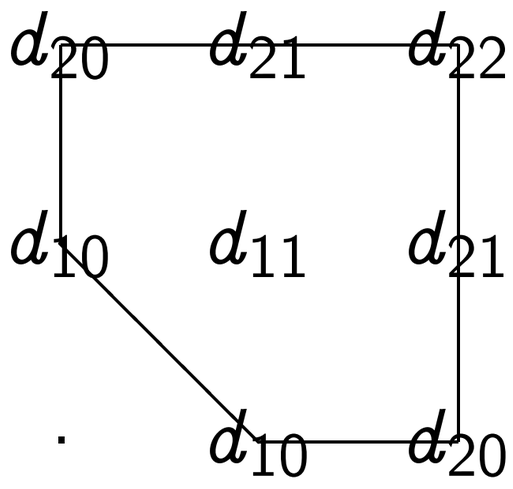

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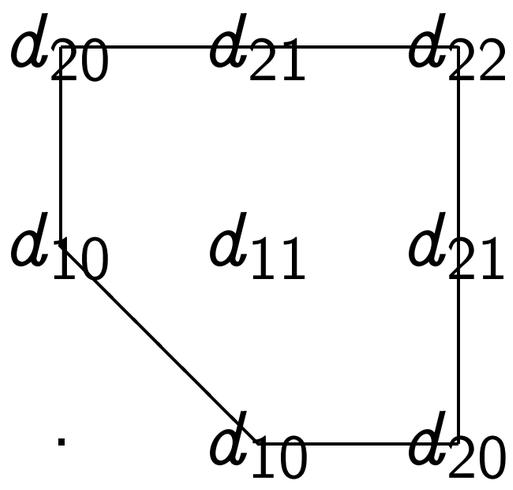

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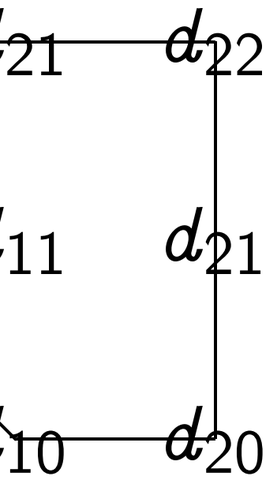

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Consider, e.g., th
$x^2 + y^2 = x + y$
with $d = -1$ and
$t = \begin{matrix} 7875101804111725 \\ 7671764645385450 \\ 1395651 \end{matrix}$
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Consider, e.g., the curve
$x^2+y^2 = x+y+txy+d$
with $d=-1$ and

$$t = \begin{array}{r} 787510180411172525454209999 54 \\ 767176464538545060814630202 84 \\ 13956511758592017 99 \end{array}$$

over $\mathbf{F}_p$ where $p = 2^{256} -$
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$$a_6 = \begin{array}{r} 410583637251521421293261297 80 \\ 047268409114441015993725554 83 \\ 52563140394674012 91 \end{array}$$

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An addition law for
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An easy Magma script uses
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Are those laws nice? No!
Find lower-degree laws by
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$x^2 + y^2 = x + y$
$v^2 - (t+2)uv +$
$u^3 - (t+2)$
i.e. $v^2 - (t+2)$
$(u^2 - d)(u$

$u = (dxy + t +$

$v = \frac{((t+2)^2 -}{(t+2)xy +}$

Assuming $t+2$ s
only exceptional
$(0,0)$, mapping t

Inverse: $x = v/($
$y = ((t+2)u -$

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$c_2 +$

$y_1y_2)$

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$)x_2y_2)$

$2 +$

$x_1x_2)$

.

$)y_2x_2)$

Note on computing addition laws:
An easy Magma script uses Riemann–Roch to find addition law given a curve shape.

Are those laws nice? No!
Find lower-degree laws by Monagan–Pearce algorithm, ISSAC 2006; or by evaluation at random points on random curves.

Are those laws complete? No!
But always seems easy to find complete addition laws among low-degree laws where denominator constant term $\neq 0$.

Birational equivalence from

$x^2 + y^2 = x + y + txy + d$

$v^2 - (t+2)uv + dv =$

$u^3 - (t+2)u^2 - du +$

i.e. $v^2 - (t+2)uv + dv =$

$(u^2 - d)(u - (t+2))$

$u = (dxy + t + 2)/(x + y)$

$$v = \frac{((t+2)^2 - d)x}{(t+2)xy + x + y}.$$

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Inverse: $x = v/(u^2 - d)$;
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Completeness

$$x_3 = \frac{x_1 + x_2 + (}{(x_1 - y_1)(x}$$
$$\frac{dx_1^2(x_2y_1 +}{1 - 2dx_1x_2}$$
$$dx_1^2(x_2 + y$$

$$y_3 = \frac{y_1 + y_2 + (}{(y_1 - x_1)(y}$$
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$$1 - 2dx_1x_2y_2 - dx_1^2(x_2 + y_2 + (t-2$$

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Can denominators be 0?

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$^2 = x + y + txy + dx^2y^2$ to
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k is a f
d, t, x_1
d is no
$27d \neq$
$x_1^2 + y_1^2$
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lence from
$+txy+dx^2y^2$ to
$-dv=$
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$2)/(x+y)$;
$-d)x$
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$dx_1^2(x_2+y_2+(t$

n
x^2y^2 to

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:

);

not:

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Completeness

$$x_3 = \frac{\begin{array}{l} x_1 + x_2 + (t-2)x_1x_2 + \\ (x_1 - y_1)(x_2 - y_2) + \\ dx_1^2(x_2y_1 + x_2y_2 - y_1y_2) \end{array}}{\begin{array}{l} 1 - 2dx_1x_2y_2 - \\ dx_1^2(x_2 + y_2 + (t-2)x_2y_2) \end{array}};$$

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Proof:
$1 - 2d$
$dx_1^2(x_2$

$(t-2)x_1x_2 +$
$x_2 - y_2) +$
$- x_2y_2 - y_1y_2)$
$\frac{}{}$;
$y_2 -$
$_2 + (t-2)x_2y_2)$

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Note that $x_1 \neq 0$.

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Note that $x_1 \neq 0$.

Use curve equation$_2$ to see that

$(1 - dx_1x_2y_2)^2 = dx_1^2(x_2 - y_2)^2$.

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Note that $x_1 \neq 0$.

Use curve equation$_2$ to see that

$(1-dx_1x_2y_2)^2 = dx_1^2(x_2-y_2)^2.$

By hypothesis d is non-square

so $x_1^2(x_2-y_2)^2 = 0$

and $(1-dx_1x_2y_2)^2 = 0.$

Only if d is a square!

Theorem: Assume that

k is a field with $2 \neq 0$;

$d, t, x_1, y_1, x_2, y_2 \in k$;

d is not a square in k;

$27d \neq (2-t)^3$;

$x_1^2+y_1^2 = x_1+y_1+tx_1y_1+dx_1^2y_1^2$;

$x_2^2+y_2^2 = x_2+y_2+tx_2y_2+dx_2^2y_2^2$.

Then $1 - 2dx_1x_2y_2 -$

$dx_1^2(x_2+y_2+(t-2)x_2y_2) \neq 0.$

By $x \leftrightarrow y$ symmetry

also $1 - 2dy_1y_2x_2 -$

$dy_1^2(y_2+x_2+(t-2)y_2x_2) \neq 0.$

Proof: Suppose that

$1 - 2dx_1x_2y_2 -$

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m: Assume that

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$, y_1, x_2, y_2 \in k$;

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$(2-t)^3$;

$= x_1 + y_1 + t x_1 y_1 + d x_1^2 y_1^2$;

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$- 2 d x_1 x_2 y_2 -$

$+ y_2 + (t-2) x_2 y_2) \neq 0$.

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Curve equation$_1$

$1 + y_1^2/x_1^2 =$

$1/x_1 + y_1(1/x_1^2 +$

$-dx_1^2y_1^2;$
$-dx_2^2y_2^2.$

$) \neq 0.$

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Curve equation$_1$ times $1/x$
$1 + y_1^2/x_1^2 =$
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Substitute $2x_2^2 = 2x_2 + tx_2^2 + dx_2^4$:
$(1 - dy_1x_2^2)^2 = d(x_2 - y_1)^2.$

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Now $2x_2^2 = 2x_2 + tx_2^2 + x_2$
so $3 = (2-t)x_2$ so $27d = (2-t)^3.$
Contradiction.

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