Dupire's equation for bubbles

Erik Ekström and Johan Tysk
Uppsala University

The Fields Institute
Toronto

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Financial Bubbles

Financial bubbles have been studied extensively over the last few years. It has been suggested to use models in which the underlying discounted price process is a *strict* local martingale under the pricing measure. Such models are known to exhibit several anomalies.

The Dupire Equation (the standard case)

The Dupire equation is a forward equation for the call option price C as a function of the strike price K and the time to maturity T. If the underlying stock price process follows a local volatility model, then the call option price satisfies

$$\begin{cases} C_T(K,T) = \mathcal{L}C(K,T) & \text{for } (K,T) \in (0,\infty)^2 \\ C(K,0) = (x-K)^+, \end{cases}$$

where $\mathcal{L}$ is the second order differential operator

$$\mathcal{L} = \frac{\sigma^2(K,T)}{2}\frac{\partial^2}{\partial K^2} - (r-q)K\frac{\partial}{\partial K} - q.$$

Here r is the interest rate, q is the continuous dividend yield, x is the current stock price and σ is a local volatility function that grows at most linearly in the spatial variable.

Failure of Dupire's formula for bubbles

It is easy to check that the Dupire equation fails in its usual form described above for models with bubbles if option prices are given by risk-neutral valutation.

Main results

In the present paper we consider Dupire type equations for such local volatility models with bubbles. In our main result, we show that if option prices are given by risk-neutral valuation, then the Dupire equation for call options contains extra terms.
Surprisingly, the corresponding equation for put options does not contain these extra terms, and is therefore perhaps better suited for calibration issues.
We show that the option price is the unique classical solution of the Dupire equation with a bounded distance to the pay-off function.

Alternative prices

Several alternative definitions of the price of an option, for which the discounted option prices process is merely a local martingale, have been suggested in the literature. Examples of such alternative prices are motivated by intermediate collateral requirements. We formulate a Dupire type equation also for such prices.

Financial model

We let the risk free rate be a constant $r \geq 0$ and we assume that the stock pays a continuous dividend yield $q \geq 0$. Under the risk neutral measure, the stock price process X is modeled by

$$\begin{cases} dX(t) = (r-q)X(t)\,dt + \sigma(X(t),t)\,dB(t) \\ X(0) = x, \end{cases} \tag{1}$$

where σ is a given local volatility function and B is a standard Brownian motion. The current stock price $x > 0$ denotes throughout the paper a given constant. If the boundary state zero can be reached in finite time, then we assume that zero is an absorbing barrier for the process X. By Ito's formula, the process $e^{-(r-q)t}X(t)$ is a local martingale, but not necessarily a martingale.

Hypothesis

The volatility function $\sigma : (0, \infty) \times [0, \infty) \to (0, \infty)$ *is continuous. Moreover, it is locally Hölder(1/2) in the first variable, i.e. for any compact set* $[D^{-1}, D] \times [0, D]$ *there exists a constant* C *such that* $|\sigma(x, t) - \sigma(y, t)| \leq C|x - y|^{1/2}$ *for all* $(x, y, t) \in [D^{-1}, D]^2 \times [0, D]$*. Furthermore, there exists a constant* A *such that* $\sigma(x, t) \leq A$ *for* $x \leq 1$.

Remark

We allow for example models in which $\sigma(x,t)$ *grows at least like* $x^{1+\eta}$ *for large* x*, where* $\eta > 0$*. In fact, in any such model, the process* $e^{-(r-q)t}X(t)$ *is a strict local martingale. Also note that, regardless of the growth rate of* σ *at infinity, equation* (1) *has a unique solution that exists for all* $t \geq 0$*. Indeed, the linear bound at infinity is usually used to avoid exploding solutions; however, in the present context the process* $e^{-(r-q)t}X(t)$ *is automatically a supermartingale and hence does not explode.*

Prices as risk neutral expected values

We mainly study the discounted expected values

$$C(K, T) := e^{-rT} E(X(T) - K)^+ \tag{2}$$

and

$$P(K, T) := e^{-rT} E(K - X(T))^+ \tag{3}$$

for different non-negative values of the strike price K and maturity dates T. By construction, defining option prices as discounted expected values implies that the corresponding discounted derivative price processes are martingales (and not merely local martingales). These discounted expected values coincide with the smallest initial fortune needed to superreplicate the corresponding option. We will refer to these expected values as the prices of call options and put options, respectively.

Main Result

Assume that Hypothesis 1 holds. Then the call price $C(K,T)$ is the unique bounded classical solution of the equation

$$\begin{cases} C_T = \mathcal{L}C + qm + m_T & \text{for } (K,T) \in (0,\infty)^2 \\ C(K,0) = (x-K)^+ \\ C(0,T) = m(T), \end{cases} \tag{4}$$

where $m(T) = e^{-rT}EX(T)$. The put price $P(K,T)$ is a classical solution of

$$\begin{cases} P_T = \mathcal{L}P & \text{for } (K,T) \in (0,\infty)^2 \\ P(K,0) = (K-x)^+ \\ P(0,T) = 0. \end{cases} \tag{5}$$

Moreover, P is the unique classical solution of (5) satisfying

$$(e^{-rT}K - e^{-qT}x)^+ \le P(K,T) \le e^{-rT}K \tag{6}$$

for all $(K,T) \in (0,\infty)^2$.

Equation (5) can formally be viewed as a pricing (Black-Scholes) equation for a *call* option if we regard K as the spot price of an underlying asset. Consequently, one solution of (5) is given by the stochastic representation

$$\tilde{P}(K,T) = e^{-qT}E(k(T)-x)^+,$$

where k is the diffusion process

$$\begin{cases} dk(t) = -(r-q)k(t)\,dt + \sigma(k(t),T-t)\,dB(t) \\ k(0) = K \end{cases} \tag{7}$$

absorbed at 0. Indeed, it follows from [2] that $\tilde{P}$ is a classical solution to (5). In fact, it is the smallest non-negative solution, so

$$\tilde{P}(K,T) \le P(K,T). \tag{8}$$

In the case of bubbles, $\tilde{P}$ is typically not convex in the spatial variable. However, it follows directly from (2) and (3) that the functions C and P are convex in the strike price K. Accordingly, the inequality (8) may be strict, and $\tilde{P}$ does not necessarily coincide with P for models with bubbles. Thus there is no uniqueness of solutions to equation (5) in the class of functions of at most linear growth. (If the volatility σ satisfies a linear bound at infinity, then P and $\tilde{P}$ coincide.)

It has been suggested that the set of admissible portfolios is restricted so that the hedging portfolio satisfies a collateral requirement at all times before maturity. In the present setting, this requirement means that the hedging portfolio for a call option should be worth at least

$$\alpha(e^{-q(T-t)}X(t) - K)^{+}$$

at each instant $t \in [0, T]$, where $\alpha \in [0, 1]$ is some given constant.

The smallest initial value of a superreplicating portfolio satisfying this collateral requirement is given by

$$C^{\alpha}(K,T) = C(K,T) + \alpha(xe^{-qT} - m(T)). \tag{9}$$

Since the pay-off of a put option is bounded, it is not natural to impose collateral conditions on the hedging portfolio in this case. We therefore refrain from considering alternative prices for put options. The proof of the following result directly follows from the main theorem and (9).

Theorem

The smallest superreplicating price $C^\alpha(K, T)$, in the presence of collateral requirements as described above, is the unique bounded classical solution of the equation

$$\begin{cases} C^\alpha_T = \mathcal{L} C^\alpha + (1-\alpha)(qm + m_T) & \textit{for } (K,T) \in (0,\infty)^2 \\ C^\alpha(K,0) = (x-K)^+ \\ C^\alpha(0,T) = (1-\alpha)m(T) + \alpha x e^{-qT}. \end{cases} \tag{10}$$

Remark

The price corresponding to $\alpha = 1$ is the one that behaves most like the price when there is no bubble in the underlying. For example, note that in this case the usual Dupire equation is obtained.

Calibration

The above results may be used for calibration of models from given option prices. The existence of a local volatility consistent with observed option data is closely related to the problem of finding a Markov process with the same distributional properties as a given stochastic process. Assume that prices $\tilde{C}(K, T)$ of call options are given (or more realistically, that $\tilde{C}(K, T)$ is constructed from a discrete set of observed prices using some suitable method of interpolation).

If α is specified to be 1, then define

$$\sigma(K,T) = \sqrt{\frac{2(\tilde{C}_T(K,T) + (r-q)K\tilde{C}_K(K,T) + q\tilde{C}(K,T))}{\tilde{C}_{KK}(K,T)}}. \tag{11}$$

Assuming that σ satisfies Hypothesis 1, the corresponding call option prices $C^1(K,T)$ can be calculated according to (9), or equivalently by solving (10).

If $\tilde{C}(K,T)$ is bounded and satisfies the boundary conditions $\tilde{C}(K,0) = (x-K)^+$ and $\tilde{C}(0,T) = xe^{-qT}$, then by uniqueness of bounded solutions to (10) we have $\tilde{C} \equiv C^1$. Thus we have found a local volatility model which is consistent with the given market data and with the given collateral requirement corresponding to $\alpha = 1$. The case of a general $\alpha \neq 1$ can be treated similarly by inserting observed option prices (and their derivatives) in (10), and then solving for σ.

We briefly discuss why the proof of Dupire's equation in a standard setting where the underlying is a martingale is not directly applicable in the strict local martingale setting. To do this, assume for simplicity that $r = q = 0$, and let $p(y,t) = P(X_t \in dy)/dy$ denote the density of X (assuming that this density exists). Then

$$\begin{aligned} C(K,T) &:= E(X(T)-K)^+ = \int_K^\infty (y-K)p(y,T)\,dy \qquad (12) \\ &= \int_K^\infty \int_y^\infty p(z,T)\,dzdy, \end{aligned}$$

where the last equality is justified by integration by parts since X_T has a finite mean.

Differentiating (12) with respect to T and using the forward equation for p, we get

$$\begin{aligned} C_T(K,T) &= \int_K^\infty \int_y^\infty \frac{1}{2}(\sigma^2(z,T)p(z,T))_{zz}\, dzdy \qquad (13) \\ &= \frac{1}{2}\sigma^2(K,T)p(K,T) = \frac{1}{2}\sigma^2(K,T)C_{KK}(K,T), \end{aligned}$$

provided that the out-integrated terms vanish. However, these terms *do not* vanish if X is a strict local martingale since the density in such a case does not decay rapidly at infinity, and thus the standard argument fails to generalise.

Step 1: Processes close to Geometric Brownian motion

First assume that σ satisfies the bounds

$$D^{-1}x \leq |\sigma(x,t)| \leq Dx \tag{14}$$

for some constant $D > 0$ and has bounded derivatives of all orders. By Ito's formula, the process $Y(t) := \ln X(t)$ satisfies

$$dY(t) = \beta_Y(Y(t),t)\,dt + \sigma_Y(Y(t),t)\,dB(t),$$

where

$$\beta_Y(y,t) := -\frac{\sigma^2(e^y,t)}{2e^{2y}} + r - q$$

and

$$\sigma_Y(y,t) := \frac{\sigma(e^y,t)}{e^y}.$$

The process Y is a diffusion on the real line with the drift and the volatility possessing bounded derivatives of all orders, and the volatility is bounded from below. Consequently, Y has a smooth transition density

$$p_Y(z, T) := P(Y(T) \in dz)/dz$$

which satisfies the forward equation

$$(p_Y)_T = (\frac{\sigma_Y^2}{2} p_Y)_{zz} - (\beta_Y p_Y)_z,$$

and $p_Y(y, T)$ and its derivatives decay like $o(e^{-|y|})$ for large $|y|$. It follows that also the process X has a smooth density $p(y, T) = P(X(T) \in dy)/dy$ which satisfies

$$p_T = (\frac{\sigma^2}{2} p)_{yy} - ((r-q)yp)_y.$$

Now, since

$$\begin{aligned} P(K,T) &= e^{-rT}E(K-X(T))^{+} = e^{-rT}\int_0^K (K-y)p(y,T)\,dy \\ &= e^{-rT}\int_0^K \int_0^y p(z,T)\,dzdy \end{aligned}$$

by integration by parts, the put price $P(K,T)$ is smooth on $(0,\infty)^2$.

Straightforward differentiation shows that

$$
\begin{aligned}
P_T(K,T) &= e^{-rT}\int_0^K\int_0^y p_T(z,T)\,dzdy - rP(K,T) \\
&= e^{-rT}\int_0^K\int_0^y \Big(\frac{\sigma^2(z,T)}{2}p(z,T)\Big)_{zz} - (r-q)(zp(z,T))_z\,dzdy \\
&\quad -rP(K,T) \\
&= \frac{\sigma^2(K,T)}{2}e^{-rT}p(K,T) - e^{-rT}\int_0^K (r-q)yp(y,T)\,dy - rP(K,T) \\
&= \frac{\sigma^2(K,T)}{2}e^{-rT}p(K,T) - (r-q)Ke^{-rT}\int_0^K p(y,T)\,dy \\
&\quad +(r-q)e^{-rT}\int_0^K\int_0^y p(z,T)\,dz - rP(K,T) \\
&= \frac{\sigma^2(K,T)}{2}e^{-rT}p(K,T) - (r-q)Ke^{-rT}\int_0^K p(y,T)\,dy \\
&\quad -qP(K,T).
\end{aligned}
$$

Since $P_K(K,T) = e^{-rT}\int_0^K p(y,T)\,dy$ and $P_{KK}(K,T) = e^{-rT}p(K,T)$, we find that

$$P_T(K,T) = \frac{\sigma^2(K,T)}{2}P_{KK}(K,T) - (r-q)KP_K(K,T) - qP(K,T). \tag{15}$$

Step 2: General volatilites close to zero

Step 2. Next we carry out an approximation argument to remove the bound (14) for small values of the underlying. Thus we assume that σ, in addition to Hypothesis 1, satisfies

$$0 < \sigma(x,t) \leq D(1+x) \tag{16}$$

for all $(x,t) \in (0,\infty) \times [0,\infty)$, and we assume that zero is an absorbing boundary for the corresponding solution X of (1). Let $\{\sigma_n\}_{n=1}^{\infty}$ be a sequence of volatilities such that

- $\sigma_n(x,t) \to \sigma(x,t)$ as $n \to \infty$ for all (x,t),
- each σ_n satisfies the bound (14) for some constant $D_n > 0$ and has bounded derivatives of all orders,
- σ_n satisfies the upper bound in (16) uniformly in n, and has a Hölder norm (in the spatial variable) which is bounded on compact subsets of $(0,\infty)^2$ uniformly in n.

Let X^n be the solution of (1) with σ replaced by σ_n, and let P^n be defined by

$$P^n(K,T) = e^{-rT}E(K - X^n(T))^+.$$

It follows that $P^n(K,T) \to P(K,T)$ as $n \to \infty$ for each $(K,T) \in [0,\infty) \times [0,T]$. By Step 1 above, each P_n satisfies

$$P^n_T(K,T) = \frac{\sigma_n^2(K,T)}{2} P^n_{KK}(K,T) - (r-q)KP^n_K(K,T) - qP^n(K,T)$$

on $(0,\infty)^2$. Since the functions $P^n(K,T)$ are locally bounded uniformly in n, interior Schauder estimates imply that P^n has derivatives P^n_K, P^n_{KK} and P^n_T that are locally bounded, uniformly in n.

Moreover, these derivatives are locally Hölder$(1/2)$ continuous (with respect to the parabolic distance) with Hölder norms that are bounded uniformly in n. By the Arzela-Ascoli theorem, the sequence $\{P^n\}_{n=1}^{\infty}$ has a subsequence $\{P^{n_k}\}_{k=1}^{\infty}$ such that P^n and its derivatives P^n_K, P^n_{KK} and P^n_T converge locally uniformly to a function $\tilde{P}$ and its corresponding derivatives. Clearly, by uniqueness of limits we have $\tilde{P} = P$. Since σ_n converges to σ, the limit function P satisfies (15).

Step 3: General volatilities

Now we consider the general case of a volatility σ that merely satisfies the requirements in Hypothesis 1. Let $\{\sigma_n\}_{n=1}^{\infty}$ be a sequence of volatilities satisfying Hypothesis 1 with a Hölder norm that is bounded on compacts uniformly in n. Moreover, we assume that $\sigma_n(x,t) = \sigma(x,t)$ for $x \leq n$ and that the growth assumption (16) holds for constants D_n. Let X^n be the corresponding stock price process. Since σ_n coincides with σ on $(0,n) \times [0,\infty)$, the random variables $X^n(T)$ converge almost surely to $X(T)$. Thus $P^n(K,T)$ converges to $P(K,T)$ by bounded convergence. Another application of the interior Schauder estimates shows that P solves (15).

Remark

The boundedness of the pay-off function $y \mapsto (K-y)^+$ *of a put option is essential in the argument for the convergence of* P^n *to* P *used in Step 3. Note that the corresponding call prices* C^n *do not converge to* C *in general. Indeed,* $C^n(K,T) \geq (x-K)^+$*, whereas* $C(K,T)$ *may be strictly smaller than* $(x-K)^+$ *for certain values of* K *and* T*. Also note that dominated convergence cannot be applied to prove* $C^n \to C$ *since the random variable* $X_T^* := \sup_n X^n(T)$ *is not necessarily integrable.*

Step 4: Boundary behaviour

Since $e^{-(r-q)t}X(t)$ is a supermartingale, it follows from Jensen's inequality that

$$\begin{aligned} P(K,T) &= e^{-rT}E(K-X(T))^+ \geq e^{-rT}(K-EX(T))^+ \\ &\geq (e^{-rT}K - xe^{-qT})^+. \end{aligned}$$

On the other hand, we clearly have $P(K,T) \leq e^{-rT}K$. It follows that P is continuous up to the boundary $K=0$ and that $P(0,T)=0$. Moreover, since the paths of X are continuous, we have that $X(T) \to x$ as $T \downarrow 0$. Therefore, another application of bounded convergence shows that $P(K,T)$ is continuous up to the initial boundary $T=0$, and $P(K,0)=(K-x)^+$. This finishes the proof that the put option price P is a classical solution of (5) that satisfies (6).

Step 5: Uniqueness in appropriate classes

Next we apply maximum principle techniques to prove that P is the unique classical solution of (5) that satisfies (6). To do that, assume that P^1 and P^2 both satisfy (5) and (6). Then $F(K,T) := P^1(K,T) - P^2(K,T)$ is a bounded classical solution of

$$\begin{cases} F_T(K,T) = \mathcal{L}F(K,T) & \text{for } (K,T) \in (0,\infty)^2 \\ F(0,T) = 0 \\ F(K,0) = 0. \end{cases}$$

Define

$$h(K) = 1 + K,$$

and note that

$$h_T - \mathcal{L}h = rK + q > 0.$$

For $\epsilon > 0$, define

$$F^\epsilon(K,T) = F(K,T) + \epsilon h(K),$$

let $\Gamma := \{(K,T) \in [0,\infty) \times [0,\overline{T}] : F^\epsilon < 0\}$ for some $\overline{T} > 0$, and assume that $\Gamma \neq \emptyset$.

Since F is bounded and $F(0, T) = 0$, the set Γ is contained in $(D^{-1}, D) \times [0, \overline{T}]$ for some constant $D > 0$. Thus, by compactness, the infimum

$$T_0 := \inf\{T \geq 0 : (K, T) \in \overline{\Gamma} \text{ for some } K \in (0, \infty)\}$$

is attained at some point (K_0, T_0), and $F^\epsilon(K_0, T_0) = 0$ by continuity. Since $F^\epsilon(K, 0) = \epsilon h(K) > 0$, we have $T_0 > 0$. Therefore, at the point (K_0, T_0) we have

$$F^\epsilon_T(K_0, T_0) - \mathcal{L}F^\epsilon(K_0, T_0) = \epsilon(h_T - \mathcal{L}h)(K_0, T_0) > 0.$$

On the other hand, by the definition of T_0 and K_0, the function $K \mapsto F^\epsilon(K, T_0)$ has a local minimum at $K = K_0$. Consequently, the function F^ϵ satisfies $F^\epsilon = 0$, $F^\epsilon_K = 0$, $F^\epsilon_{KK} \geq 0$ and $F^\epsilon_T \leq 0$ at the point (K_0, T_0). Consequently,

$$F^\epsilon_T(K_0, T_0) - \mathcal{L}F^\epsilon(K_0, T_0) \leq 0.$$

This contradiction shows that $\Gamma = \emptyset$, so $F^\epsilon \geq 0$ on $(0, \infty) \times [0, \overline{T}]$. Since $\epsilon > 0$ and $\overline{T}$ are arbitrary, it follows that $0 \leq F = P^1 - P^2$. Interchanging the role of P^1 and P^2 yields the reverse inequality, i.e. $P^1 = P^2$.

The case of call options

Step 6. Finally, we treat the call option price C using a put-call parity relation. Taking expected values in the equality

$$(X(T) - K)^+ = (K - X(T))^+ - K + X(T)$$

we find that

$$C(K, T) = P(K, T) - e^{-rT} K + m(T). \tag{17}$$

Therefore,

$$\begin{aligned}
C_T(K,T) &= P_T(K,T) + re^{-rT}K + m_T(T) \\
&= \frac{\sigma^2(K,T)}{2} P_{KK}(K,T) - (r-q)KP_K(K,T) - qP(K,T) \\
&\quad + re^{-rT}K + m_T(T) \\
&= \frac{\sigma^2(K,T)}{2} C_{KK}(K,T) - (r-q)KC_K(K,T) - qC(K,T) \\
&\quad + qm(T) + m_T(T),
\end{aligned}$$

where we used (17), $P_K = C_K + e^{-rT}$ and $P_{KK} = C_{KK}$. The fact that C satisfies the given boundary conditions also follows from the put-call parity (17) and the boundary behaviour of P. Finally, the proof of the uniqueness of the solutions to equation (5) within the given class also shows uniqueness of solutions with a bounded difference to $e^{-rT}K$. This translates directly to uniqueness for equation (4) for bounded functions. This finishes the proof of our result.

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