Matching Statistics of an Itô Process by a Process of Diffusion Type

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Outline

- Volatility smile fitting by mixing log-normal distributions.
- Corollary: A local volatility model with mixture marginals.
- Corollary: A "local volatility" model for stock and running maximum.
- Theorem.

Mixing log-normal distributions

The Black-Scholes-(Merton) model assumes that the underlying asset price has a log-normal distribution under a "risk-neutral" (martingale) probability measure at the option expiration date T. It has been proposed to instead assume that the distribution is a mixture of log-normals.

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Why use a mixture of log-normals?

- Empirical reason: Mixture of two log-normals fits the volatility smile.[1]

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Why use a mixture of log-normals?

- Empirical reason: Mixture of two log-normals fits the volatility smile.[1]
- Computational reason: The mixture of log-normals gives prices and Greeks (sensitivities) that are mixtures of Black-Scholes prices and Greeks.

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- If we don't model the evolution, we cannot build successful trading strategies. Trading strategies generate profits and losses over time.
- If we don't model the evolution, we cannot price path-dependent options. The price of a path-dependent option depends on the joint distribution of the underlying asset at multiple time points.

A mixture of models is not a model.

Assume

$$0 < v_1 < v_2.$$

Consider a "model" with

$$dS_t = rS_t\,dt + \sigma_0 S_t\,dW_t, \quad 0 \le t \le T,$$

where

$$\mathbb{P}\{\sigma_0^2 = v_1\} = \frac{1}{2}, \quad \mathbb{P}\{\sigma_0^2 = v_2\} = \frac{1}{2}.$$

We set the value of σ_0 at time zero, and then the risk-neutral distribution of $S(T)$ is a mixture of log-normals with volatilities $\sqrt{v_1}$ and $\sqrt{v_2}$.

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Immediately after time zero, we can determine σ_0 from the observed returns, and we no longer have a mixture model.

A local volatility model with mixture marginals.

At time 0 choose a volatility σ_0 with

$$\mathbb{P}\{\sigma_0^2 = v_1\} = \frac{1}{2}, \quad \mathbb{P}\{\sigma_0^2 = v_2\} = \frac{1}{2}.$$

Use this volatility throughout to obtain a process S. To simplify the presentation, we assume $r = 0$: $dS_t = \sigma_0 S_t \, dW_t$.

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Choose a time-partition

$$\Pi : 0 = T_0 < T_1 < T_2 < \cdots < T_n = T.$$

At each time T_i, compute

$$p_1(T_i, s) = \mathbb{P}\{\sigma_0^2 = v_1 | S_{T_i} = s\}, \quad p_2(T_i, s) = \mathbb{P}\{\sigma_0^2 = v_2 | S_{T_i} = s\}.$$

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Construct a second process S^Π recursively. On $[T_0, T_1)$ use the volatility σ_0 chosen above. At each subsequent time T_i, redraw the volatility according to

$$\mathbb{P}\{(\sigma^\Pi_{T_i})^2 = v_1\} = p_1(T_i, S^\Pi_{T_i}), \quad \mathbb{P}\{(\sigma^\Pi_{T_i})^2 = v_2\} = p_2(T_i, S^\Pi_{T_i}),$$

and use it on $[T_i, T_{i+1})$.

Relationship between S and S^{Π}

- We set $\sigma_0^{\Pi} = \sigma_0$.
- We use volatility σ_0 to generate S_t, $0 \le t \le T$.
- We use volatility σ_0^{Π} to generate S_t^{Π}, $0 \le t \le T_1$.
- Therefore, $S_t = S_t^{\Pi}$ for $0 \le t \le T_1$.

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- We use volatility σ_0^{Π} to generate S_t^{Π}, $0 \leq t \leq T_1$.
- Therefore, $S_t = S_t^{\Pi}$ for $0 \leq t \leq T_1$.
- At time T_1, we choose a new $\sigma_{T_1}^{\Pi}$ so that $(S_{T_1}, \sigma_0) \stackrel{\mathcal{D}}{=} (S_{T_1}^{\Pi}, \sigma_{T_1}^{\Pi})$.
- We use volatility $\sigma_{T_1}^{\Pi}$ to continue S_t^{Π}, $\quad T_1 \leq t \leq T_2$.
- Therefore, $(S_t, \sigma_0) \stackrel{\mathcal{D}}{=} (S_t^{\Pi}, \sigma_t^{\Pi})$, $T_1 \leq t \leq T_2$.

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- We use volatility σ_0^Π to generate S_t^Π, $0 \le t \le T_1$.
- Therefore, $S_t = S_t^\Pi$ for $0 \le t \le T_1$.
- At time T_1, we choose a new $\sigma_{T_1}^\Pi$ so that $(S_{T_1}, \sigma_0) \overset{\mathcal{D}}{=} (S_{T_1}^\Pi, \sigma_{T_1}^\Pi)$.
- We use volatility $\sigma_{T_1}^\Pi$ to continue S_t^Π, $\quad T_1 \le t \le T_2$.
- Therefore, $(S_t, \sigma_0) \overset{\mathcal{D}}{=} (S_t^\Pi, \sigma_t^\Pi)$, $T_1 \le t \le T_2$.
- At time T_2, we choose a new $\sigma_{T_2}^\Pi$ so that $(S_{T_2}, \sigma_0) \overset{\mathcal{D}}{=} (S_{T_2}^\Pi, \sigma_{T_2}^\Pi)$.
- We use volatility $\sigma_{T_2}^\Pi$ to continue S_t^Π, $T_2 \le t \le T_3$.
- Therefore, $(S_t, \sigma_0) \overset{\mathcal{D}}{=} (S_t^\Pi, \sigma_t^\Pi)$, $T_2 \le t \le T_3$.

A local volatility model with mixture marginals.

Properties of S^{Π}.

- For each t, S_t and S_t^{Π} have the same distribution, and so . . .
- European calls on S have the same prices as European calls on S^{Π}.
- S^{Π} has piecewise constant volatility.
- Immediately after each T_i, observation of S^{Π} reveals the volatility being used on $[T_i, T_{i+1})$, but not the volatilities that will be used after time T_{i+1}.

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Take the limit. Recall

$$0 = T_0 < T_1 < T_2 < \cdots < T_n = T.$$

Let $n \to \infty$ so that $\max_i |T_{i+1} - T_i| \to 0$. It can be shown that S^{Π} converges to a process $S^{\ell v}$ ("S local volatility") satisfying

$$dS_t^{\ell v} = \sigma(t, S_t^{\ell v}) S_t^{\ell v} dW_t, \quad 0 \le t \le T,$$

where

$$\sigma^2(t, s) = \mathbb{E}\big[\sigma_0^2 \big| S_t = s\big] = \frac{v_1 \pi_1(t, s) + v_2 \pi_2(t, s)}{\pi_1(t, s) + \pi_2(t, s)}$$

and $\pi_i(t, s)$ is the log-normal distribution corresponding to time t and volatility v_i. This is a new argument for a known result.[2]

[2] Brigo, D. and Mercurio, F. A mixed-up smile, Risk, September 2000.

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Corollary (to the theorem at the end)

Assume

$$dS_t = \sigma_t S_t dW_t, \quad 0 \le t \le T,$$

where σ_t can be an adapted, time-varying process satisfying $\mathbb{E} \int_0^T \sigma_t^2 \, dt < \infty$. Then there exists a function $\sigma(t, s)$ and a weak solution to the stochastic differential equation

$$dS_t^{\ell v} = \sigma(t, S_t^{\ell v}) S_t^{\ell v} \, dW_t, \quad 0 \le t \le T,$$

such that for each $t \ge 0$, the random variables S_t and $S_t^{\ell v}$ have the same distribution. Furthermore, for Lebesgue-almost-every $t \in [0, T]$, the "local volatility" function $\sigma^2(t, s)$ is a version of $\mathbb{E}\left[\sigma_t^2 \middle| S_t = s\right]$.

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- The corollary generalizes a known result[3] by removing the assumption that σ_t must be bounded away from zero and bounded above. These boundedness conditions are violated in many models, e.g., Heston stochastic volatility model.

[3] I. Gyöngy (1986) Mimicking the one-dimensional marginal distributions of processes having an Itô differential, *Prob. Theory and Related Fields* **71**, 501–516.

[4] B. Dupire (1994) Pricing with a smile, *Risk* **7**, 18–20.

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- The corollary generalizes a known result[3] by removing the assumption that σ_t must be bounded away from zero and bounded above. These boundedness conditions are violated in many models, e.g., Heston stochastic volatility model.
- If there is a transition density for $S_t^{\ell v}$, then one can compute $\sigma(t, s)$ using a formula due to Dupire[4].

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Disclaimer

Under the assumptions in the Corollary, the equation

$$dS_t^{\ell v} = \sigma(t, S_t^{\ell v}) S_t^{\ell v} \, dW_t,$$

can have more than one solution. We have examples of this built around $\sigma(t, s)$ taking the value zero. We do not yet have a general theorem that guarantees uniqueness of the solution to the stochastic differential equation.

Nonuniqueness

Let $X_0 = 0$ and $dX(t) = \sigma_t \, dW_t$, where

$$\sigma_t = I_{(1,\infty)}(t) I_{\{W_1 > 0\}}.$$

The solution is

$$X_t = I_{(1,\infty)}(t) I_{\{W_1 > 0\}} (W_t - W_1).$$

We have $\sigma(t, x) = 0$ for $0 \le t \le 1$, and for $t > 1$,

$$\sigma^2(t, x) = \mathbb{E}[\sigma_t^2 | X_t = x] = \begin{cases} 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Both $X_t^{(1)} \equiv 0$ and

$$X^{(2)}(t) = I_{(1,\infty)}(t)(W_t - W_1)$$

are solutions of $dX_t^{\ell v} = \sigma(t, X_t^{\ell v}) dW_t$. The weak solution we want is $X^{(1)}$ with probability $\frac{1}{2}$ and $X^{(2)}$ with probability $\frac{1}{2}$.

Beyond local volatility.

What about path dependent options?

The price of a knock-out call is

$$B(0, S_0; \sigma^2) = \mathbb{E}\big[(S_T - K)^+ I_{\{M_T \le B\}}\big],$$

where

$$M_T = \max_{0 \le u \le T} S_u.$$

From the reflection principle for Brownian motion, we have an explicit formula for $B(0, S_0; \sigma^2)$ when the volatility of S is a constant σ.

Beyond local volatility.

At time 0 choose a volatility σ_0 with

$$\mathbb{P}\{\sigma_0^2 = v_1\} = \frac{1}{2}, \quad \mathbb{P}\{\sigma_0^2 = v_2\} = \frac{1}{2}.$$

Use this volatility throughout to obtain a process S. Then the knock-out call price is

$$\frac{1}{2}B(0, S_0; v_1) + \frac{1}{2}B(0, S_0; v_2).$$

This is nice analytic formula, but it is based on a nonsensical dynamic model.

Beyond local volatility.

We could instead use the local volatility model

$$dS_t^{\ell v} = \sigma(t, S_t^{\ell v}) S_t^{\ell v} \, dW_t,$$

where

$$\sigma^2(t,s) = \mathbb{E}\left[\sigma_0^2 \middle| S_t = s\right] = \frac{v_1 \pi_1(t,s) + v_2 \pi_2(t,s)}{\pi_1(t,s) + \pi_2(t,s)}$$

and $\pi_i(t,s)$ is the log-normal distribution corresponding to time t and squared volatility v_i.

- For each $t \geq 0$, the *random variable* $S_t^{\ell v}$ has the same distribution as the *random variable* S_t.
- But the *paths* of the process $S^{\ell v}$ do not have the same distribution as the *paths* of the process S.
- In particular,

$$\mathbb{E}\left[(S_T^{\ell v} - K)^+ I_{\{M_T^{\ell v} \leq B\}}\right] \neq \frac{1}{2} B(0, S_0; v_1) + \frac{1}{2} B(0, S_0; v_2).$$

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At time 0 choose a volatility σ_0 with

$$\mathbb{P}\{\sigma_0^2 = v_1\} = \frac{1}{2}, \quad \mathbb{P}\{\sigma_0^2 = v_2\} = \frac{1}{2}.$$

Use this volatility throughout to obtain a process S. Define $M_t = \max_{0 \le u \le t} S_u$.

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Use this volatility throughout to obtain a process S. Define $M_t = \max_{0 \le u \le t} S_u$. Choose a time-partition

$$\Pi : 0 = T_0 < T_1 < T_2 < \cdots < T_n = T.$$

At each time T_i, compute

$$p_k(T_i, s, m) = \mathbb{P}\{\sigma_0^2 = v_k | S_{T_i} = s, M_{T_i} = m\}, \quad k = 1, 2.$$

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Construct a second process S^Π recursively. On $[T_0, T_1)$ use the volatility σ_0 chosen above. At each subsequent time T_i, redraw the volatility according to

$$\mathbb{P}\{(\sigma_{T_i}^\Pi)^2 = v_k\} = p_k(T_i, S_{T_i}^\Pi, M_{T_i}^\Pi), \quad k = 1, 2,$$

and use it on $[T_i, T_{i+1})$, where $M_{T_i}^\Pi = \max_{0 \le u \le T_i} S_u^\Pi$.

Beyond local volatility.

Properties of (S^Π, M^Π).

- For each t, (S_t, M_t) and (S_t^Π, M_t^Π) have the same distribution, and so . . .
- Barrier options on S have the same prices as barrier options on S^Π, i.e., they are a mixture of Black-Scholes prices.
- S^Π has piecewise constant volatility.
- Immediately after each T_i, observation of S^Π reveals the volatility being used on $[T_i, T_{i+1})$, but not the volatilities that will be used after time T_{i+1}.

Let $n \to \infty$ so that $\max_i |T_{i+1} - T_i| \to 0$. It can be shown that S^Π converges to a process $S^{\ell v}$ satisfying

$$dS_t^{\ell v} = \sigma(t, S_t^{\ell v}, M_t^{\ell v}) S_t^{\ell v} dW_t,$$

where

$$\sigma^2(t, s, m) = \mathbb{E}\left[\sigma_0^2 \middle| S_t = s, M_t = m\right].$$

Beyond local volatility.

Corollary (to the theorem at the end)

Assume

$$dS_t = \sigma_t S_t dW_t, \quad 0 \le t \le T,$$

where σ_t *can be an adapted, time-varying process satisfying* $\mathbb{E} \int_0^T \sigma_t^2 \, dt < \infty$. *Define*

$$M_t \triangleq \max_{0 \le u \le t} S_u.$$

Then there exists a function $\sigma(t, s, m)$ *and a weak solution to the stochastic differential equation*

$$dS_t^{\ell v} = \sigma(t, S_t^{\ell v}, M_t^{\ell v}) S_t^{\ell v} \, dW_t,$$

where

$$M_t^{\ell v} \triangleq \max_{0 \le u \le t} S_u^{\ell v},$$

such that for each $t \ge 0$, *the pair of random variables* $(S_t^{\ell v}, M_t^{\ell v})$ *has the same distribution as the pair* (S_t, M_t).

The theorem at the end.

Let C^d denote the space of continuous functions from $[0, \infty)$ to $\mathbb{R}^d$.

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Define three operators mapping $C^d \times [0, \infty)$ to C^d:

- Shift operator: $\Theta(x, t) \triangleq x(t + \cdot)$,
- Stopping operator: $\nabla(x, t) \triangleq x(t \wedge \cdot)$,
- Difference operator: $\Delta(x, t) \triangleq x(t + \cdot) - x(t)$.

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- Difference operator: $\Delta(x, t) \triangleq x(t + \cdot) - x(t)$.

We say $\Phi \colon C^d \to C^d$ is an updating function if

- Initiation: $\Phi_0(x) = x(0)$,
- Non-anticipativity: $\nabla(\Phi(x), t) = \nabla\Big(\Phi\big(\nabla(x, t)\big), t\Big)$,
- "Markov" property: $\Theta\big(\Phi(x), t\big) = \Phi\big(\Phi_t(x) + \Delta(x, t)\big)$.

Theorem (Brunick[5])

Given

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s, \quad t \geq 0,$$

where $\mathbb{E} \int_0^t \left(\|\mu_s\| + \|\sigma_s \sigma_s^T\| \right) ds < \infty$ *for all* $t \geq 0$*. Let* $Z = \Phi(X)$*. For Lebesgue-almost-every* t*, there are versions*

$$\widehat{\mu}(t, Z_t) = \mathbb{E}[\mu_t | Z_t], \quad \widehat{\sigma}(t, Z_t) \widehat{\sigma}^T(t, Z_t) = \mathbb{E}[\sigma_t \sigma_t^T | Z_t],$$

and a weak solution

$$\begin{aligned} \widehat{X}_t &= \widehat{X}_0 + \int_0^t \widehat{\mu}(s, \widehat{Z}_s) \, ds + \int_0^t \widehat{\sigma}(s, \widehat{Z}_s) \, dW_s, \\ \widehat{Z} &= \Phi(\widehat{X}), \end{aligned}$$

such that $\widehat{Z}_t \stackrel{\mathcal{D}}{=} Z_t$ *for every* $t \geq 0$*.*

[5] G. Brunick (2008) A weak existence result with application to the financial engineer's calibration problem, Ph.D. dissertation, Carnegie Mellon University.

Extended partition Π

- Canonical space C^d.
- Canonical filtration $\{\mathcal{F}_t\}_{t \geq 0}$.
- $0 = T_0 \leq T_1 \leq \cdots \leq T_n$, a sequence of finite stopping times.
- $\{\mathcal{G}_i\}_{i=1}^n$, a collection of σ-fields with $\mathcal{G}_i \subset \mathcal{F}_{T_i}$ for every i.
- $$T_{i+1} - T_i \in \mathcal{G}_i \vee \sigma\big(\Delta(X, T_i)\big)$$
- $$\mathcal{H}_{i+1} \triangleq \mathcal{G}_i \vee \sigma\Big(\nabla\big(\Delta(X, T_i), T_{i+1}\big)\Big)$$
- $$\mathcal{G}_{i+1} \subset \mathcal{H}_{i+1}$$

Concatenation of measures

Theorem

Let $\mathbb{P}$ be a probability measure on C^d. Then there exists a unique measure $\mathbb{P}^{\otimes\Pi}$ on C^d such that

- $$\mathbb{P}^{\otimes\Pi}[A] = \mathbb{P}[A] \quad \forall A \in \mathcal{H}_i, \forall i,$$

- $$\mathbb{P}^{\otimes\Pi}[B|\mathcal{F}_{T_i}] = \mathbb{P}[B|\mathcal{G}_i] \quad \forall B \in \mathcal{H}_{i+1}, \forall i,$$

  i.e., every $\mathbb{P}$-version of $\mathbb{P}[B|\mathcal{G}_i]$ is a $\mathbb{P}^{\otimes}$-version of $\mathbb{P}^{\otimes\Pi}[B|\mathcal{F}_{T_i}]$.

Convergence

- Construct a sequece Π_n of extended partitions with $\|\Pi_n\| \to 0$.
- Define $C = \langle X \rangle$.
- Under $\mathbb{P}$, X and $XX^T - C$ are local martingales.
- Show that under each $\mathbb{P}^{\otimes \Pi_n}$, X and $XX^T - C$ are local martingales.
- Recall the assumption that $\mathbb{E}\|C_T\| < \infty$.
- Show that $\mathbb{E}^{\otimes \Pi_n}\|C_T\| = \mathbb{E}\|C_T\|$. Use this to conclude that the collection of probability measures $\{\mathbb{P}^{\otimes \Pi_n}\}_{n=1}^{\infty}$ is tight.[6]
- Tightness implies convergence along a subsequence. Call the limiting measure $\mathbb{P}^{\infty}$.
- Show that $C_t = \int_0^t \widehat{\sigma}(u, \Phi_u(X))\widehat{\sigma}(u, \Phi_u(X))^T du$, where $\widehat{\sigma}(t, s) = \mathbb{E}^{\infty}[\sigma_t \sigma_t^T | \Phi_t(X) = s]$.

[6] R. Rebolledo, La méthode des martingales appliquée à l'étude de la convergence en loi de processus, *Mémoires de la Société Mathématique de France* **62**, 1–125, 1979.