Deterministic criteria for the absence of arbitrage in diffusion models

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Structure of the talk

1. Definition of various concepts of no-arbitrage (NFLVR, NGA, NRA).
2. Deterministic characterisation in diffusion models and comparison.

Free lunch with vanishing risk (FLVR)

Discounted asset price model: semimart $S = (S_t)_{t \in [0,T]}$, $T \in (0, \infty]$.
Admissible Trading strategy: predictable process $H = (H_t)_{t \in [0,T]}$
s.t. $\exists$ a constant $c_H \geq 0$ and

$$H \cdot S_t \geq -c_H \quad \text{a.s.} \quad \forall t \in [0,T].$$

Discounted wealth process with the initial capital $x \in \mathbb{R}$: $x + H \cdot S$.

The model S satisfies the *NFLVR* condition if $\overline{C} \cap L^\infty_+ = \{0\}$ where

$$C := \{g \in L^\infty \mid \exists \text{ admissible } H \text{ such that } g \leq H \cdot S_T \text{ a.s.}\}.$$

$\overline{C}$ is the closure of $C \subset L^\infty$ in the norm topology.

Financial significance and characterisation of NFLVR

FLVR in $S \Longrightarrow \exists\, g \in L^\infty_+ \backslash \{0\}$, $g_n \in C \subset L^\infty$ and attainable claims $H^n \cdot S_T$, $n \in \mathbb{N}$, such that

$$g_n \leq H^n \cdot S_T \quad \text{a.s.} \quad \text{and} \lim_{n\to\infty} \|g - g_n\|_\infty = 0.$$

Economic interpretation: the risk of H^n vanishes with increasing n

$$\lim_{n\to\infty} \left((H^n \cdot S_T) \wedge 0\right) = 0.$$

(Delbaen and Schachermayer 1998): S satisfies NFLVR iff there exists an equivalent sigma-martingale measure for S.

If S is locally bounded from below, NFLVR holds iff $\exists$ equivalent local martingale measure for S (Ansel-Stricker lemma)

Generalised arbitrage (GA)

Disc. asset price model: non-negative semimart $S = (S_t)_{t\in[0,T]}$.
Predictable trading strategies $H = (H_t)_{t\in[0,T]}$ is given by

$$H = \sum_{k=1}^{N} h_{k-1} I_{(\tau_{k-1},\tau_k]}, \text{ where } N \in \mathbb{N}, 0 \le \tau_0 \le \cdots \le \tau_N \le T$$

are stopping times, h_{k-1} are $\mathbb{R}$-valued $\mathcal{F}_{\tau_{k-1}}$-measurable. Let

$$C := \{h \in L^\infty \mid \exists H \text{ simple strategy s.t. } h \le \frac{(H\cdot S)_T}{(1+S_T)} \text{ a.s.}\}.$$

The model S satisfies *NGA* if

$$\overline{C}^* \cap L^\infty_+ = \{0\},$$

where $\overline{C}^*$ is closure of C in weak-* topology $\sigma(L^\infty, L^1)$ on L^∞.

NFLVR and NGA

FLVR: (Delbaen and Schachermayer 1994)
GA: (Sin 1996), (Yan 1998), (Cherny 2007)

Discounted asset price process: non-negative cts. semimart S

NFLVR on $[0,T] \iff \exists$ Q $\sim$ P: $(S_t)_{t\in[0,T]}$ is a Q-loc. mart.
NFLVR on $[0,\infty) \iff \exists$ Q $\sim$ P: $(S_t)_{t\in[0,\infty)}$ is a Q-loc. mart.

NGA on $[0,T] \iff \exists$ Q $\sim$ P: $(S_t)_{t\in[0,T]}$ is a Q-mart.
NGA on $[0,\infty) \iff \exists$ Q $\sim$ P: $(S_t)_{t\in[0,\infty)}$ is a Q-u.i. mart.

In particular, NGA $\Longrightarrow$ NFLVR

Setting

Bond price $\equiv 1$
Stock price $dY_t = \mu(Y_t)\,dt + \sigma(Y_t)\,dW_t$, $Y_0 = x_0 \in J := (0, \infty)$

Assumptions
(A) $\sigma(x) \neq 0 \; \forall x \in J$
(B) $1/\sigma^2 \in L^1_{\mathsf{loc}}(J)$
(C) $\mu/\sigma^2 \in L^1_{\mathsf{loc}}(J)$
(D) Y does not exit at ∞

On the contrary, Y may exit at 0. We stop Y after it reaches 0.

Inputs: functions μ and σ
Outputs: determnistic criteria for NFLVR, NGA and NRA in terms of μ and σ

Ingredients

$$\frac{\mu^2}{\sigma^4} \in L^1_{\mathsf{loc}}(J) \tag{1}$$

$$\frac{x\mu^2(x)}{\sigma^4(x)} \in L^1_{\mathsf{loc}}(0+) \tag{2}$$

$$\frac{x}{\sigma^2(x)} \notin L^1_{\mathsf{loc}}(0+) \tag{3}$$

Recall (C) $\mu/\sigma^2 \in L^1_{\mathsf{loc}}(J)$

Criteria for NFLVR and NGA in the difussion model Y

Assume (A)–(D)

Theorem 1 *NFLVR on* $[0, T] \iff$ *(a) or (b), where*
(a) (1) and (2) hold
(b) (1) and (3) hold and Y *does not exit at 0*

Corollary 2 *(Delbaen and Shirakawa 2002) If* Y *does not exit at 0: NFLVR on* $[0, T] \iff$ *(1) and (3)*

Theorem 3 *NFLVR on* $[0, \infty) \iff$ *(1), (2), and* $s(\infty) = \infty$*, where* s *denotes the scale function of* Y

Proposition 4 *NGA on* $[0, T] \iff$ *NFLVR on* $[0, T]$ *and* $x/\sigma^2(x) \notin L^1_{\mathsf{loc}}(\infty-)$

Proposition 5 *There is always GA on* $[0, \infty)$

What is behind these results?

Let $b : J \to \mathbb{R}$ Borel measurable with $b^2/\sigma^2 \in L^1_{\mathsf{loc}}(J)$ and

$$Z_t = \exp\left\{\int_0^{t\wedge\zeta} b(Y_u)\,dW_u - \frac{1}{2}\int_0^{t\wedge\zeta} b^2(Y_u)\,du\right\}, \quad t \in [0,\infty),$$

where we set $Z_t := 0$ for $t \geq \zeta$ on $\{\zeta < \infty, \int_0^\zeta b^2(Y_u)\,du = \infty\}$.

(i) When is Z a martingale?

(ii) When is $Z_T > 0$ P-a.s. for $T \in (0,\infty]$?

(iii) Can Z be defined for $b = -\mu/\sigma^2$?

(iv) Is the candidate density Z of the form above?

Proofs: (Mijatović and Urusov 2009a), (Mijatović and Urusov 2009b)

The setting for relative arbitrage (RA)

Stochastic portfolio theory (Fernholz 2002), (Fernholz and Karatzas 2008b). Assume from now on $T < \infty$.

RA on $[0, T]$: there exists a self-financing strategy with a strictly positive wealth $(V_t)_{t\in[0,T]}$ such that $V_0 = Y_0$, $V_T \geq Y_T$ a.s., and $\mathsf{P}(V_T > Y_T) > 0$

For RA we assume (A), (B), (C'), and (D')

(A) $\sigma(x) \neq 0 \ \forall x \in J$

(B) $1/\sigma^2 \in L^1_{\mathrm{loc}}(J)$

(C') $\mu^2/\sigma^4 \in L^1_{\mathrm{loc}}(J)$

(D') Y exits neither at 0 nor at ∞

Criterion for NRA

Assume (A), (B), (C'), and (D')

Recall $dY_t = \mu(Y_t)\,dt + \sigma(Y_t)\,dW_t$, $Y_0 = x_0 \in J = (0,\infty)$
Set $\overline{Z}_t := \exp\{-\int_0^t (\mu/\sigma)(Y_u)\,dW_u - (1/2)\int_0^t (\mu^2/\sigma^2)(Y_u)\,du\}$
By Itô's formula $\overline{Z}Y = (\overline{Z}_t Y_t)_{t\in[0,T]}$ is a local martingale

(Fernholz and Karatzas 2008a) and (Mijatović and Urusov 2009a):

$$\text{NRA} \iff \overline{Z}Y \text{ martingale}$$

Proposition 6 *NRA* $\iff x/\sigma^2(x) \notin L^1_{\mathsf{loc}}(\infty-)$

Proof. $d(\overline{Z}_t Y_t) = \overline{Z}_t Y_t b(Y_t)\,dW_t$ with $b(x) = \sigma(x)/x - \mu(x)/\sigma(x)$
$\overline{Z}_t Y_t = x_0 \mathcal{E}(\int_0^\cdot b(Y_u)\,dW_u)_t$ $\square$

Comparison

Assume (A), (B), (C'), and (D')

(i) NFLVR $\iff x/\sigma^2(x) \notin L^1_{\mathsf{loc}}(0+)$
(ii) NRA $\iff x/\sigma^2(x) \notin L^1_{\mathsf{loc}}(\infty-)$
(iii) NGA $\iff x/\sigma^2(x) \notin L^1_{\mathsf{loc}}(0+)$ and $x/\sigma^2(x) \notin L^1_{\mathsf{loc}}(\infty-)$

Thus, NFLVR and NRA are in a general position and

$$\text{NGA} \iff \text{NFLVR and NRA}$$

NFLVR & NRA	$dY_t = Y_t\,dt + Y_t\,dW_t$
NFLVR & RA	$dY_t = Y_t\,dt + Y_t^2\,dW_t$
FLVR & NRA	$dY_t = \frac{1}{Y_t}\,dt + dW_t$
FLVR & RA	$dY_t = 2\,dt + (\sqrt{Y_t} + Y_t^2)\,dW_t$

Thank you for your attention!

References

CHERNY, A. S. (2007): "General arbitrage pricing model: probability approach," *Lecture Notes in Mathematics*, 1899, 415–446.

DELBAEN, F., AND W. SCHACHERMAYER (1994): "A general version of the fundamental theorem of asset pricing," *Math. Ann.*, 300(3), 463–520.

DELBAEN, F., AND W. SCHACHERMAYER (1998): "The fundamental theorem of asset pricing for unbounded stochastic processes," *Math. Ann.*, 312(2), 215–250.

DELBAEN, F., AND H. SHIRAKAWA (2002): "No Arbitrage Condition for Positive Diffusion Price Processes," *Asia-Pacific Financial Markets*, 9, 159–168.

FERNHOLZ, D., AND I. KARATZAS (2008a): "On optimal arbitrage," *Preprint,* http://www.math.columbia.edu/~ik/OptArb.pdf.

FERNHOLZ, R. (2002): *Stochastic portfolio theory*, vol. 48 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, Stochastic Modelling and Applied Probability.

FERNHOLZ, R., AND I. KARATZAS (2008b): "Stochastic portfolio theory: an overview," *To appear in: Handbook of Numerical Analysis*.

MIJATOVIĆ, A., AND M. URUSOV (2009a): “Deterministic criteria for the absence of arbitrage in one-dimensional diffusion models,” Accepted in *Finance and Stochastics,* available at http://www.ma.ic.ac.uk/~amijatov/.

——— (2009b): “On the martingale property of certain local martingales,” *preprint, available at http://www.ma.ic.ac.uk/~amijatov/*.

SIN, C. A. (1996): *Strictly local martingales and hedge ratios in stochastic volatility models*. Cornell University, Ithaka, NY, PhD-thesis.

YAN, J.-A. (1998): “A new look at the fundamental theorem of asset pricing,” *J. Korean Math. Soc.*, 35(3), 659–673.