Heat Kernels for Information-Sensitive Pricing Kernels

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Motivation: information-based asset pricing

In the asset pricing approach proposed in the paper

D. C. Brody, L. P. Hughston & A. Macrina (2008) Information-Based Asset Pricing. International Journal of Theoretical and Applied Finance Vol. 11, 107-142,

asset prices fluctuate due to the flow of incomplete information about the asset's future cash flows. The setup of this approach is as follows:

The financial market is modelled by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q})$, where $\mathbb{Q}$ is the risk-neutral measure.

Let a cash flow X_T, occurring at a fixed future date T, be modelled by a random variable with *a priori* density $q(x)$.

Let $\{L_{tT}\}_{0\leq t\leq T}$ be a Markov process that is used to model incomplete information about X_T. We call such a process an "information process".

The market filtration $\{\mathcal{F}_t\}$ is generated by $\{L_{tT}\}$, that is $\mathcal{F}_t = \sigma\left(\{L_{sT}\}_{0\leq s\leq t}\right)$.

To ensure that the cash flow X_T is $\mathcal{F}_T$-measurable, we require that $L_{TT} = G(X_T)$ for some invertible function $G(x)$.

This measurability condition justifies the use of random bridge processes for the construction of information processes.

A variety of such time-inhomogeneous Markov processes which are driven by "underlying" Lévy processes can be constructed explicitly.

For a detailed construction of these processes and their application to the modelling of financial information we refer to:

E. Hoyle, L. P. Hughston & A. Macrina (2009) Lévy Random Bridges and the Modelling of Financial Information. www.arXiv.org, No. 0912.3652

Another usual ingredient of information-based asset pricing is a deterministic discount bond system $\{P_{tT}\}_{0\leq t\leq T}$.

Under $\mathbb{Q}$ the price S_t of an asset with cash flow X_T at T is expressed by

$$S_t \;=\; P_{tT}\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, \mathcal{F}_t\right]. \tag{1}$$

Since $\mathcal{F}_t$ is generated by a time-inhomogeneous Markov process $\{L_{sT}\}_{0\leq s\leq t}$, we have that

$$S_t = P_{tT}\mathbb{E}^{\mathbb{Q}}\left[X_T \,|\, L_{tT}\right]. \tag{2}$$

The conditional expectation can be worked out by use of the Bayes formula.

We shall further develop information-based asset pricing by constructing stochastic discount bond systems.

Information-sensitive pricing kernels

Stochastic discount bond systems can be constructed by modelling the pricing kernel (stochastic discount factor) that we denote $\{\pi_t\}_{0\leq t}$.

The price P_{tT} at time t of a discount bond with unit payoff at maturity T is

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{P}}\left[\pi_T \,|\, \mathcal{F}_t\right]}{\pi_t}, \tag{3}$$

where $\mathbb{P}$ is the real probability measure.

Next we model the pricing kernel $\{\pi_t\}$ and the filtration $\{\mathcal{F}_t\}$ following the scheme of information-based asset pricing.

We fix a time $U > T$ and introduce a random variable X_U with real probability density $p(x)$.

The random variable X_U may represent a macroeconomic factor (e.g. the GDP level of a country at time U) revealed at time U.

Suppose that at time $t < U$, market investors have access only to incomplete information about the macroeconomic factor X_U.

We model this incomplete information by an information process $\{L_{tU}\}$ with the property that $L_{UU} = G(X_U)$, where $G(x)$ is an invertible function.

So we assume that the market filtration is given by $\mathcal{F}_t = \sigma\left(\{L_{sU}\}_{0\le s\le U}\right)$.

Let the bond price process $\{P_{tT}\}_{0\le t\le T<U}$ be adapted to $\{\mathcal{F}_t\}$.

We consider a pricing kernel $\{\pi_t\}$ that is modelled by a function of the value L_{tU} at time t, and possibly time t:

$$\pi_t := \pi(t, L_{tU}). \tag{4}$$

The function $\pi(t,x)$ shall be chosen such that the pricing kernel is guaranteed to be a positive supermartingale.

Armed with the models for the pricing kernel and the market filtration, the price P_{tT} of the discount bond is

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{P}}\left[\pi(T, L_{TU}) \,|\, L_{tU}\right]}{\pi(t, L_{tU})}. \tag{5}$$

Here we have recalled that the market filtration $\{\mathcal{F}_t\}$ is generated by $\{L_{tU}\}$ which is taken to be a time-inhomogeneous Markov process.

To obtain explicit models for the bond price P_{tT}, we need to explicitly construct (i) pricing kernel models and (ii) information processes $\{L_{tU}\}$.

One method to construct information-based pricing kernels is presented in:

L. P. Hughston & A. Macrina (2009) Pricing Fixed-Income Assets in an Information-Based Framework. www.arXiv.org, No. 0911.1610.

However, in this talk, we consider another method.

Weighted heat kernel approach with time-inhomogeneous Markov processes

Heat kernel methods for the construction of pricing kernels are proposed in:

J. Akahori, Y. Hishida, J. Teichmann & T. Tsuchiya (2009) A Heat Kernel Approach to Interest Rate Models. www.arXiv.org, No. 0910.5033

In their paper, they make use of time-homogeneous Markov processes.

However, it is possible to modify this heat kernel method in order to construct a weighted heat kernel approach for time-inhomogeneous Markov processes $\{L_{tU}\}$.

This work is included in:

J. Akahori & A. Macrina (2010) Heat Kernel Interest Rate Models with Time-Inhomogeneous Markov Processes. (Working paper)

In what follows, we present the main results.

We consider a time-inhomogeneous Markov process $\{L_{tU}\}_{0\leq t\leq U}$, and introduce a real-valued measurable function $p(u,t,x)$.

A so-called propagator $\{p(u,t,L_{tU})\}$ associated with the process $\{L_{tU}\}$ has the property that

$$\mathbb{E}\left[p\left(u,t,L_{tU}\right)\,|\,L_{sU}\right]=p\left(u+t-s,s,L_{sU}\right), \tag{6}$$

for $s<t$, $0<u$, and $0<u+t<U$.

An example of a propagator is

$$p(u,t,L_{tU})=\mathbb{E}\left[F(u+t,L_{u+t,U})\,|\,L_{tU}\right], \tag{7}$$

where $F(t,x)$ is taken to be a measurable positive function.

Next we introduce a so-called weight function $w(t,u)$ that is positive and measurable, and has the property that

$$w(t,u-s)\leq w(t-s,u), \tag{8}$$

where $t,u\in[0,U)$ and $s\leq t\wedge u$.

A weighted heat kernel is then defined by

$$g(t, L_{tU}) = \int_0^{U-t} p(u, t, L_{tU})\, w(t, u)\, \mathrm{d}u, \tag{9}$$

for $0 \leq t < U < \infty$. In the case of the propagator (7), we have

$$g(t, L_{tU}) = \int_0^{U-t} \mathbb{E}\left[F(u+t, L_{u+t,U}) \,|\, L_{tU}\right]\, w(t, u)\, \mathrm{d}u. \tag{10}$$

It can be proved that $\{g(t, L_{tU})\}$ is a positive supermartingale by showing that

$$\begin{aligned} \mathbb{E}\left[g(t, L_{tU}) \,|\, L_{sU}\right] &= \int_{t-s}^{U-s} p(s, u, L_{sU})\, w(t, u-t+s)\, \mathrm{d}u & (11) \\ &\leq \int_0^{U-s} p(s, u, L_{sU})\, w(s, u)\, \mathrm{d}u, & (12) \\ &= g(s, L_{sU}), & (13) \end{aligned}$$

for $s \leq t$.

Applying the pricing kernels constructed by weighted heat kernels, the bond price P_{tT} can be expressed by

$$P_{tT} = \frac{\int_{T-t}^{U-t} p(u, t, L_{tU}) \, w(T, u - T + t) \, \mathrm{d}u}{\int_{0}^{U-t} p(u, t, L_{tU}) \, w(t, u) \, \mathrm{d}u}. \tag{14}$$

In the case that the propagator is given by (7), we have

$$P_{tT} = \frac{\int_{T-t}^{U-t} \mathbb{E}\left[F(u+t, L_{u+t,U}) \,|\, L_{tU}\right] \, w(T, u - T + t) \, \mathrm{d}u}{\int_{0}^{U-t} \mathbb{E}\left[F(u+t, L_{u+t,U}) \,|\, L_{tU}\right] \, w(t, u) \, \mathrm{d}u} \tag{15}$$

Explicit formulae for the bond price are obtained by specifying the functions $F(t,x)$ and $w(t,u)$, and the information process $\{L_{tU}\}$.

Bond pricing with Brownian bridge information

We consider the case where $\{L_{tU}\}_{0\leq t\leq U}$ is a so-called Brownian bridge information process:

$$L_{tU} = \sigma X_U\, t + \beta_{tU}, \tag{16}$$

where X_U is taken to be independent of the Brownian bridge process $\{\beta_{tU}\}$.

We assume that the market filtration is defined by

$$\mathcal{F}_t = \sigma\left(\{L_{sU}\}_{0\leq s\leq t}\right). \tag{17}$$

Remarks:
(i) $\{L_{tU}\}$ is a time-inhomogeneous $\{\mathcal{F}_t\}$-Markov process.
(ii) X_U is $\mathcal{F}_U$-measurable.
(iii) The information flow rate σ in (16) is constant.
(iv) $\mathrm{Var}[\beta_{tU}] = t(U-t)/U$.

We recall that the bond price can be expressed by

$$P_{tT} = \frac{\mathbb{E}\left[\pi(T, L_{TU}) \middle| L_{tU}\right]}{\pi(t, L_{tU})}. \tag{18}$$

In order to work out the conditional expectation, we assume with no loss of generality, that

$$\pi(t, L_{tU}) = M_t \, g(t, L_{tU}), \tag{19}$$

where the $(\mathbb{P}, \{\mathcal{F}_t\})$-martingale $\{M_t\}$ is defined by

$$\frac{\mathrm{d}M_t}{M_t} = -\frac{\sigma U}{U-t} \, \mathbb{E}\left[X_U \,|\, L_{tU}\right] \mathrm{d}W_t, \tag{20}$$

for $0 \le t < U$ and where

$$W_t = L_{tU} + \int_0^t \frac{L_{sU}}{U-s} \, \mathrm{d}s - \sigma U \int_0^t \frac{1}{U-s} \, \mathbb{E}\left[X_U \,|\, L_{sU}\right] \mathrm{d}s. \tag{21}$$

The martingale $\{M_t\}$ induces a change of measure from $\mathbb{P}$ to the so-called bridge measure $\mathbb{B}$ under which $\{L_{tU}\}$ has the distribution of a Brownian bridge.

The bond price can thus be expressed as follows:

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{P}}\left[\pi\left(T, L_{TU}\right) \mid L_{tU}\right]}{\pi(t, L_{tU})}, \tag{22}$$

$$= \frac{\mathbb{E}^{\mathbb{P}}\left[M_T\, g(T, L_{TU}) \middle| L_{tU}\right]}{M_t\, g(t, L_{tU})}, \tag{23}$$

$$= \frac{\mathbb{E}^{\mathbb{B}}\left[g(T, L_{TU}) \middle| L_{tU}\right]}{g(t, L_{tU})}. \tag{24}$$

We emphasize that the pricing kernel $\{\pi(t, L_{tU})\}$ is a $\mathbb{P}$-supermartingale if $\{g(t, L_{tU})\}$ is a supermartingale under $\mathbb{B}$ (and vice versa).

We now may make use of

$$g(t, L_{tU}) = \int_0^{U-t} \mathbb{E}^{\mathbb{B}}\left[F(u+t, L_{u+t,U}) \mid L_{tU}\right]\, w(t,u)\, \mathrm{d}u, \tag{25}$$

and hence of

$$P_{tT} = \frac{\int_{T-t}^{U-t} \mathbb{E}^{\mathbb{B}}\left[F(L_{t+u,U}) \,|\, L_{tU}\right] w(T, u-T+t)\, \mathrm{d}u}{\int_{0}^{U-t} \mathbb{E}^{\mathbb{B}}\left[F(L_{u+t,U}) \,|\, L_{tU}\right] w(t,u)\, \mathrm{d}u}. \tag{26}$$

The information process $\{L_{tU}\}_{0\le t<U}$ has the distribution of a Brownian bridge under $\mathbb{B}$ so that the conditional expectation can be worked out explicitly.

Example: quadratic family

Let $F(x) = x^2$, and $w(t,u) = U - t - u$.

A calculation shows that

$$E^{\mathbb{B}}\left[(L_{u+t,U})^2 \;|\; L_{tU}\right] = \frac{u(U-t-u)}{U-t} + \left(\frac{U-t-u}{U-t}\right) L_{tU}^2. \tag{27}$$

With this intermediate result at hand, we can write the weighted heat kernel process as follows:

$$g(t, L_{tU}) = \int_0^{U-t} \mathbb{E}^{\mathbb{B}}\left[F(L_{u+t,U}) \,|\, L_{tU}\right] w(t,u)\,\mathrm{d}u, \tag{28}$$

$$= \int_0^{U-t} \left[\frac{u(U-t-u)}{U-t} + \left(\frac{U-t-u}{U-t}\right) L_{tU}^2\right] (U-t-u)\mathrm{d}u. \tag{29}$$

The integral in the expression for the weighted heat kernel can be calculated in closed form, so that we obtain the supermartingale

$$g(t, L_{tU}) = \frac{1}{12}(U-t)^3 + \frac{1}{4}(U-t)^2\, L_{tU}^2. \tag{30}$$

The bond price P_{tT} at time t, derived in this example, is thus given by

$$P_{tT} = \frac{\frac{1}{12}(U-T)^3 + \frac{1}{4}\frac{(T-t)(U-T)^3}{(U-t)} + \frac{1}{4}\frac{(U-T)^4}{(U-t)^2}\, L_{tU}^2}{\frac{1}{12}(U-t)^3 + \frac{1}{4}(U-t)^2\, L_{tU}^2}. \tag{31}$$

The simulation of the bond price is straightforward since the process $\{L_{tU}\}$,

$$L_{tU} = \sigma\, X_U\, t + \beta_{tU}, \tag{32}$$

is Gaussian conditional on the outcome of the underlying economic factor X_U.

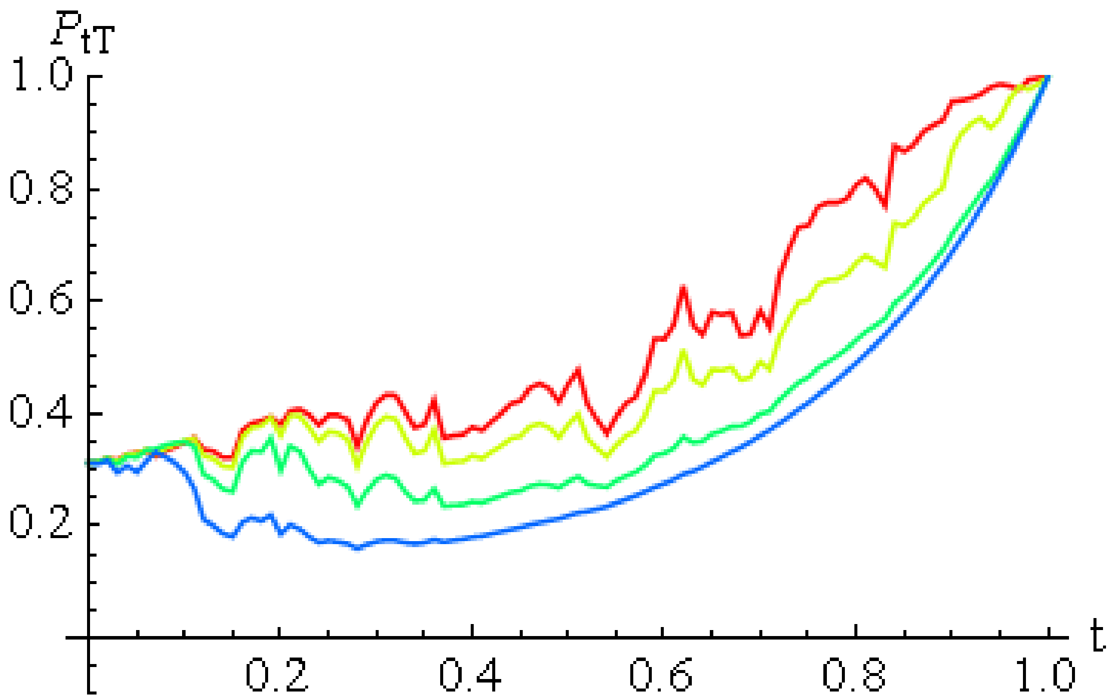

Figure 1. Sovereign discount bond price process. The parameters are chosen to be $T = 1$ and $U = 2$ for constant $X_U = 1$. We take $\sigma_U = 0.025$ (red), $\sigma_U = 0.5$ (yellow), $\sigma_U = 1.75$ (green), and $\sigma_U = 5$ (blue).

The short rate process $\{r_t\}$ can be worked out by calculating the instantaneous forward rate associated with the bond price $\{P_{tT}\}_{0\leq t\leq T<U}$.

The result is:

$$r(t, L_{tU}) = \frac{L_{tU}^2}{\frac{1}{4}(U-t)\left[\frac{1}{3}(U-t) + L_{tU}^2\right]}, \tag{33}$$

for $0 \leq t < U$. We emphasize that this is a positive interest rate model.

The market price of risk $\{\lambda_t\}$ associated with the quadratic family is

$$\lambda(t, L_{tU}) = \frac{\sigma U}{U-t}\mathbb{E}^{\mathbb{P}}\left[X_U \,|\, L_{tU}\right] - \frac{\frac{1}{2}(U-t)^2 L_{tU}}{\frac{1}{12}(U-t)^3 + \frac{1}{4}(U-t)^2 L_{tU}^2}. \tag{34}$$

Example: exponential quadratic family

Let $F(x) = \exp\left(\frac{1}{2}\,\gamma_{t+u}\,x^2\right)$, where γ_{t+u} is deterministic.

In this case the propagator takes the form

$$\mathbb{E}^{\mathbb{B}}\left[\exp\left(\tfrac{1}{2}\,\gamma_{t+u}\,L^2_{t+u,U}\right)\,|\,L_{tU}\right] = \frac{1}{\sqrt{1-u\,\gamma_{t+u}\,a_{t+u}}}\exp\left(\frac{\gamma_{t+u}\,a^2_{t+u}}{2\,(1-u\,\gamma_{t+u}\,a_{t+u})}L^2_{tU}\right), \qquad (35)$$

where $a_{t+u} = (U-t-u)/(U-t)$.

By setting $\gamma_{t+u} = (U-t-u)^{-1}$, and by choosing the weight function to be

$$w(t,u) = (U-t-u)^{\eta-\frac{1}{2}} \qquad (\eta > \tfrac{1}{2}), \qquad (36)$$

we obtain an analytical expression for the supermartingale $\{g(t, L_{tU})\}$:

$$g(t, L_{tU}) = \frac{1}{\eta-\frac{1}{2}}\,(U-t)^{\eta}\exp\left(\frac{L^2_{tU}}{2(U-t)}\right). \qquad (37)$$

The supermartingale (37) leads to a deterministic bond price, even though the related pricing kernel is stochastic.

However we can modify slightly the supermartingale $\{g(t, L_{tU})\}$:

Let $f_0(t)$ and $f_1(t)$ be positive, decreasing and differentiable functions.

Consider the supermartingale

$$\tilde{g}(t, L_{tU}) = f_0(t) + f_1(t)(U - t)^{\gamma} \exp\left(\frac{L_{tU}^2}{2(U - t)}\right). \tag{38}$$

Then the associated bond price system has stochastic dynamics:

$$P_{tT} = \frac{f_0(T) + f_1(T)(U - T)^{\gamma - 1/2}(U - t)^{1/2} \exp\left(\frac{L_{tU}^2}{2(U-t)}\right)}{f_0(t) + f_1(t)(U - t)^{\gamma} \exp\left(\frac{L_{tU}^2}{2(U-t)}\right)}, \tag{39}$$

for $t \in [0, U)$ and $u \in [0, U - t]$.

We note that further examples can be constructed: a semi-analytic formula is obtained for the exponential linear family defined by $F(x) = \exp(-\mu\, x)$.

Credit-risky discount bonds with stochastic discounting

In the last part of this talk, we give an idea how the information-based asset pricing framework can be extended so as to incorporate stochastic discounting.

In particular we aim at generalizing the information-based credit-risk models presented in

D. C. Brody, L. P. Hughston & A. Macrina (2007) Beyond hazard rates: a new framework for credit-risk modelling. In *Advances in Mathematical Finance, Festschrift Volume in Honour of Dilip Madan*, edited by R. Elliott, M. Fu, R. Jarrow & J.-Y. Yen. Birkhäuser, Basel and Springer, Berlin.

We proceed as follows:

We fix two dates T and U, where $T < U$, and attach two independent factors X_T and X_U to these dates.

The payoff of the credit-risky bond is modelled by making use of the random variable X_T.

We assume that X_T is a discrete random variable that takes values in $\{x_0, x_1, \ldots, x_n\}$ with a priori probabilities $\{p_0, p_1, \ldots, p_n\}$, where

$$0 \leq x_0 < x_1 < \ldots < x_{n-1} < x_n \leq 1. \tag{40}$$

We assume that the economic factor X_U is a continuous random variable.

With X_T and X_U we associate the independent information processes $\{L_{tT}\}_{0\leq t\leq T}$ and $\{L_{tU}\}_{0\leq t\leq U}$ defined by

$$L_{tT} = \sigma_1 \, t \, X_U + \beta_{tU}, \qquad\qquad L_{tU} = \sigma_2 \, t \, X_T + \beta_{tT}. \tag{41}$$

The market filtration $\{\mathcal{F}_t\}$ is generated by $\{L_{tT}\}$ and $\{L_{tU}\}$:

$$\mathcal{F}_t = \sigma\left(\{L_{sT}\}_{0\leq s\leq t}, \{L_{sU}\}_{0\leq s\leq t}\right) \tag{42}$$

The price B_{tT} at $t \leq T$ of a defaultable discount bond with payoff H_T at $T < U$ is given by

$$B_{tT} = \frac{\mathbb{E}^{\mathbb{P}}[\pi_T H_T \,|\, \mathcal{F}_t]}{\pi_t}. \tag{43}$$

Let the pricing kernel $\{\pi_t\}$ be defined by

$$\pi_t = M_t \, g(t, L_{tU}) \tag{44}$$

and let the payoff of the credit-risky bond be given by

$$H_T = H\left(X_T, L_{TU}\right). \tag{45}$$

The formula for the price B_{tT} of the credit-risky bond is then worked out by applying the weighted heat kernel approach for the pricing kernel, and eventually by specifying the payoff function $H(X_T, L_{TU})$.

However, we leave this task for another time...

Meantime these results can be found in:

A. Macrina & P. A. Parbhoo (2009) Security Pricing with Information-Sensitive Discounting. In: Recent Advances in Financial Engineering 2009. Proceedings of the KIER-TMU International Workshop on Financial Engineering 2009 (World Scientific).