Model Independent Bounds for Variance Swaps

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Fields Institute, May 2010
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Variance swaps: Neuberger/Dupire

Suppose that X is continuous. Suppose we know the prices of vanilla options with maturity T.

Suppose that we wish to price a variance swap, a path-dependent security paying $\int_0^T (dX_t)^2/X_t^2 dt$.

Then, by Itô's formula

$$d \ \log X_t = \frac{dX_t}{X_t} - \frac{1}{2}\frac{(dX_t)^2}{X_t^2}$$

and

$$\int_0^T \frac{(dX_t)^2}{X_t^2} = -2\ln X_T + 2\ln X_0 + \int_0^T \frac{2}{X_t} dX_t$$

(Under path-continuity) There is a model-independent price for a variance swap, and a perfect hedge.

In Brownian terms

WLOG we may assume X_t is a martingale.

If we write $X_t = B_{A_t}$ for some continuous time-change, (and $B_0 = X_0 = 1$ [WLOG]) then

$$\begin{aligned}\int_0^T \frac{(dX_t)^2}{X_t^2} &= \int_0^T \frac{(dB_{A_t})^2}{B_{A_t}^2} = \int_0^{A_T} \frac{(dB_s)^2}{B_s^2} = \int_0^{A_T} \frac{ds}{B_s^2} \\ &= -2\ln B_{A_T} + \int_0^{A_T} \frac{2}{B_s} dB_s\end{aligned}$$

Writing τ for A_T, and if $X_T \equiv B_\tau \sim \mu$, then

$$\mathbb{E}\left[\int_0^\tau \frac{ds}{B_s^2}\right] = \mathbb{E}[-2logZ|Z \sim \mu].$$

Skorokhod embeddings

A Skorokhod embedding (for μ in B) is a stopping rule τ such that $B_\tau \sim \mu$.

Identifying models consistent with a given set of maturity T call prices is equivalent to finding solutions of the SEP.

Finding price bounds on path-dependent derivatives is equivalent to finding solutions of the SEP for which some functional of Brownian motion is maximised.

For the variance swap, this functional is

$$\mathbb{E}\left[\int_0^T \frac{(dX_t)^2}{X_t^2}\right] \equiv \mathbb{E}\left[\int_0^\tau \frac{ds}{B_s^2}\right]$$

Over embeddings τ of μ, this is independent of τ.

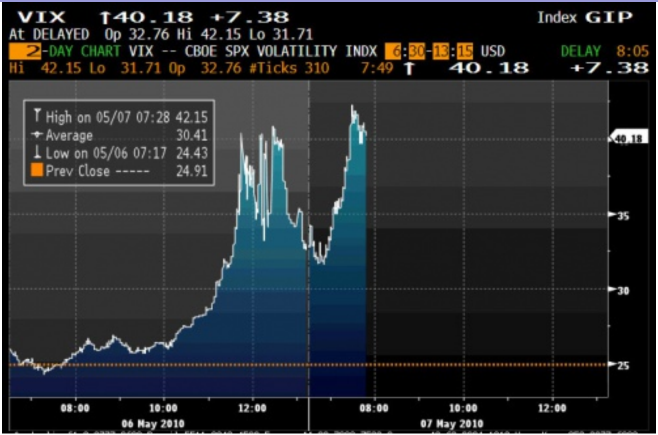

Figure: *VIX contracts; 7th May 2010*

The discontinuous case

What if we no longer assume continuity (and just assume right-continuity) of our martingale?

Following the convention in Demeterfi, Derman, Kamal and Zhou we will assume (the floating leg of) the variance swap pays $\int_0^T (dX_t)^2 / X_{t-}^2$.

The problem

Place upper/lower bounds on the price of the variance swap contract, given the prices of vanilla calls on X with maturity T.

A (seemingly) unrelated problem for Brownian motion

Let B be Brownian motion (with $B_0 = 1$), let $S_t = \sup_{u \leq t} B_u$ and $I_t = \inf_{u \leq t} B_u$.

Suppose μ is a distribution with mean 1.

The problem
Over Skorokhod embeddings τ (of μ in B) find the embedding which minimises $\int_0^\tau du\, S_u^{-2}$ (in expectation).

The intuitive solution?
Azema-Yor solution of the SEP

We have

$$d\left(1-\frac{B_t}{S_t}\right)^2 = 2\left(1-\frac{B_t}{S_t}\right)\frac{B_t}{S_t^2}dS_t - 2\left(1-\frac{B_t}{S_t}\right)\frac{1}{S_t}dB_t + dt\frac{1}{S_t^2}$$

Then,

$$\int_0^\tau dt\frac{1}{S_t^2} = \left(1-\frac{B_\tau}{S_\tau}\right)^2 + 2\int_0^\tau \left(1-\frac{B_t}{S_t}\right)\frac{1}{S_t}dB_t$$

The problem becomes to minimise $\left(1-\frac{B_\tau}{S_\tau}\right)^2$ over solutions τ of the SEP for μ.

Optimality for the SEP

Theorem

Suppose $F(b, s)$ is such that $F_s(b, s)/(s-b)$ is increasing in b for each s. Then $\mathbb{E}[F(B_\tau, S_\tau)]$ is minimised by the Perkins solution of the SEP.

The Perkins soltion takes the form $\tau = \inf\{u : B_u \notin [\xi(S_u), \eta(I_u)]\}$.

Example

If F is an incresing function of s alone.

Example

If $F(b, s) = (1 - b/s)^2$ then $F_s(b, s)/(s-b) = b/s^3$ is incresasing in b.

The variance swap revisited

We have $X_t = B_{A_t}$ but the time-change A may have jumps.

Define $J_t = \inf_{s \le t} X_s$ and $R_t = \sup_{s \le t} X_s$.

Recall $I_t = \inf_{s \le t} B_s$ and $S_t = \sup_{s \le t} B_s$.

Then $I_{A_t} \le J_t \le X_t \le R_t \le S_{A_t}$.

$$\int_0^T \frac{(dX_t)^2}{(R_{t-})^2} \le \int_0^T \frac{(dX_t)^2}{(X_{t-})^2} \le \int_0^T \frac{(dX_t)^2}{(J_{t-})^2}$$

Taking the lower bound

$$\int_0^T \frac{(dX_t)^2}{(X_{t-})^2} \ge \int_0^T \frac{(dX_t)^2}{(R_{t-})^2} \ge \int_0^T \frac{(dB_{A_t})^2}{(S_{A_{t-}})^2} \ge \int_0^{A_T} \frac{du}{(S_u)^2}$$

A lower bound on the price of a variance swap

Combining this result with the result on lower bounds for $\int_0^\tau \frac{du}{(S_u)^2}$ we have a model independent bound for the price of the variance swap in the presence of jumps.

Question

Can this bound be attained?

For the bound to be attained we must have
•(when there are jumps) $X_{t-} = R_{t-} = S_{A_{t-}} = S_{A_t}$
• (if μ has a density) $(S_\tau \geq x) \equiv (B_\tau \geq x) \cup (B_\tau < \gamma(x))$ for a monotonic decreasing function γ [The Perkins optimality property]
[Recall we have reduced the problem to a functional of B_τ and S_τ alone]

Theorem

Given μ with mean 1, then there exist $f, g : [0, T] \mapsto \mathbb{R}_+$ with f increasing, g decreasing and $f(0) = 1 = g(0)$, and $h : [0, T] \mapsto [0, 1]$ such that if $U = U[0, T]$ and $V = U[0, 1]$ then $(M_t)_{0 \leq t \leq T}$ defined by

$$M_u = \begin{cases} f(u) & u < U \\ f(U) & U \leq u, \quad V \geq h(U) \\ g(U) & U \leq u, \quad V < h(U) \end{cases}$$

satisfies

- *M is a right-continuous martingale with $M_0 = 1$ and $M_T \sim \mu$,*
- *if M has a jump at t then $M_{t-} = \sup_{s \leq t} M_s$*
- $(\sup_{s \leq T} M_s \geq f(x)) \equiv (M_T \geq f(x)) \cup (M_T < g(x))$

In particular, M is a candidate price process for which the price of the variance swap attains the model-independent lower bound.

We have $\mathbb{E}[M_u | U \geq u] = \mathbb{E}[M_{u+du} | U \geq u]$.

Then

$$\begin{aligned} f(u) &= f(u+du)\left(1 - \frac{du}{1-u}\right) \\ &\quad + f(u)\frac{du}{1-u}(1-h(u)) + g(u)\frac{du}{1-u}h(u) \\ f(u) &= f(u) + du\left(f'(u) - (f(u)-g(u))\frac{h(u)}{1-u}\right) + o(du) \end{aligned}$$

Hence $h(u) = f'(u)(1-u)/(f(u)-g(u))$.

Upper bounds

We can obtain analogous upper bounds using $I_{A_t} \leq J_t \leq X_t$ and upper bounds for $\int_0^\tau I_u^{-2} du$ over embeddings for μ in Brownian motion.

The upper bound is attained by an inversion (in space) of the construction of the optimal lower bound martingale.

The upper bound may explode if μ places too much mass near 0.

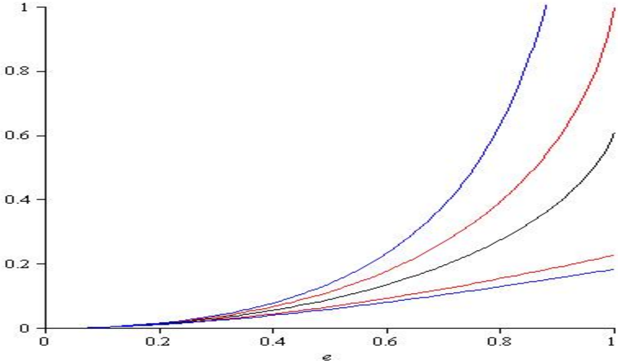

Figure: *Upper and lower bounds for the uniform case - together with the price under a continuity assumption*

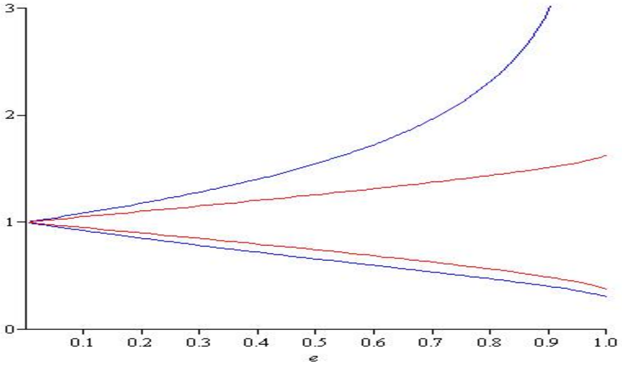

Figure: *Upper and lower bounds for the uniform case - relative to the price under the continuity assumption*

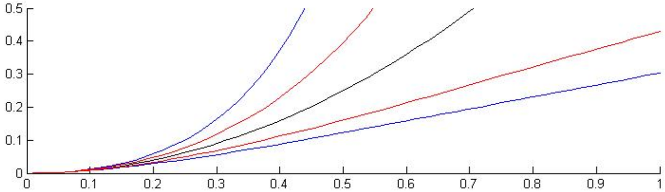

Figure: *Upper and lower bounds for the lognormal case - together with the price under the continuity assumption. Horizontal axis senotes volatility.*

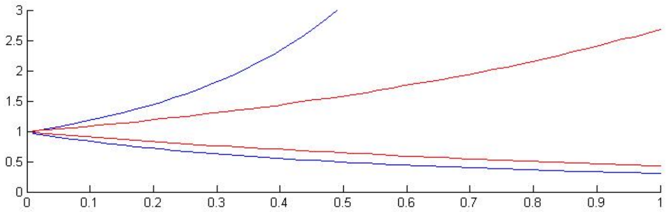

Figure: *Upper and lower bounds for the lognormal case - relative to the price under the continuity assumption. Horizontal axis senotes volatility.*

Summary

We have found bounds on the price of volatility swaps which apply accross all models consistent with an observed set of co-maturing puts and calls.

The ideas can be extended to weighted variance swaps, and forward starting variance swaps.

Can incorporate intermediate data.