Target Volatility Option Pricing

Giuseppe Di Graziano*

Deutsche Bank AG, London

Workshop on Financial Derivatives and Risk Management
Fields Institute, Toronto

Joint work with Lorenzo Torricelli

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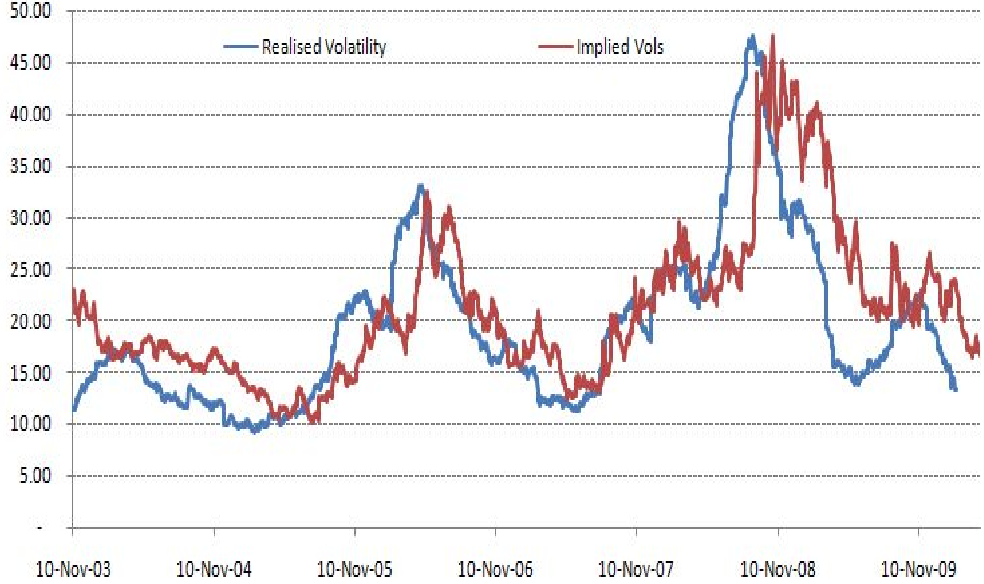
*e-mail = giuseppe.di-graziano@db.com.

Figure 1: 3M Implied vs Realized Vols for Gold



1. Motivation

- Volatility is the key ingredient of option pricing:

  1. In efficient markets, option prices reflect the market expected volatility.
  2. What if most dealers are short volatility?
  3. What if risk management procedures in banks impose limits on the volatility exposure?
  4. What if there is no volatility market for the underlying? (e.g. proprietary trading strategies/indices).
  5. In periods of market stress option prices can be prohibitively high. Is there a way to take exposure to the underlying in an option without "over-paying" ?

- Target Volatility Options (TVOs) are a partial answer to the problems above. They are tyically used as follows:

  1. to express a joint view on the performance of the asset and its volatility.
  2. to cheapen the price of an option in the same way as Asian, Barrier and other Exotics.
  3. to control the risk of the underlying strategy.
  4. to allow dealers to buy/sell options on an underlying with no (or illiquid) volatility market.

2. Notation and assumptions

- The asset price is characterized by the following dynamics,

$$dS_t = \sigma_t S_t dW_t, \tag{1}$$

  where W_t is a standard Brownian motion.

- We shall set interest rates to zero.
- The stochastic volatility process σ_t is assumed to be independent from W_t.
- Let X_t be the rescaled log-price,

$$X_t \equiv \log(S_t/S_0), \tag{2}$$

  so that ...

$$\langle X \rangle_t \equiv \int_0^t \sigma_u^2 du. \tag{3}$$

- Assume that for any t, $\langle X \rangle_t$ is strictly positive.
- We are interested of calculating quantities of the type:

$$C_t^{TV}(K) \equiv E_t \left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} (S_T - K)^+ \right], \tag{4}$$

where $\bar{\sigma}$ is the Target Volatility.

3. TVO pricing via Taylor expansion

- Suppose one wishes to calculate the price of an ATM Call TVO at inception:

$$C_0^{TV}(S_0) \equiv E_0\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_T}}(S_T - S_0)^+\right] \tag{5}$$

- Using the independence assumption and Bachelier approximation formula, we have:

$$\begin{aligned}
C_0^{TV}(S_0) &= E_0\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_T}}E[(S_T - K)^+ \mid \mathcal{F}^\sigma]\right] \\
&= E_0\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_T}}C^{BS}(S_0, S_0, \langle X\rangle_T)\right] \\
&\approx S_0 E_0\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_T}}\sqrt{\frac{\langle X\rangle_T}{2\pi}}\right] \\
&= S_0\bar{\sigma}\sqrt{\frac{T}{2\pi}} \\
&\approx C^{BS}(S_0, S_0, \bar{\sigma}^2 T).
\end{aligned}$$

- The TVO price is approximately the Black-Scholes price of a vanilla call with implied volatility $\bar{\sigma}$.

- The idea above can be extended to a generic strike K and valuation time t.
- Expanding the Black-Scholes price around the strike K, we have

$$C(K) = C(S_0) + C^{(1)}(K - S_0) + \sum_{n=0}^{\infty} C^{(n+2)}(S_0) \frac{(K - S_0)^{n+2}}{(n+2)!} \tag{6}$$

- Derivatives higher than the second, can be deduced using the formula

$$C^{n+2}(S_0) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\hat{\sigma}^2}{8}\right) \frac{P_n(d^-)}{S_0^{n+1} \hat{\sigma}^{n+1}} (-1)^n. \tag{7}$$

- Here $P_n(d^-)$ satisfies the following recursive equation,

$$P_n(d^-) = (d^- + n\hat{\sigma}) P_{n-1}(d^-) - P'_{n-1}(d^-), \tag{8}$$

where

$$\hat{\sigma} = \sigma\sqrt{t}, \tag{9}$$

and

$$d^- \equiv \frac{\log(S_0/K)}{\sigma\sqrt{t}} - \frac{\sigma\sqrt{t}}{2} \tag{10}$$

- Solving the equation above, rearranging things slightly and putting $\hat{\sigma}$ as a common factor, we have:

$$\begin{aligned} C(K) &= S_0\{N\left(\frac{\hat{\sigma}}{2}\right) - N\left(-\frac{\hat{\sigma}}{2}\right)\} \\ &- N\left(-\frac{\hat{\sigma}}{2}\right)(K - S_0) \\ &+ \exp(-\hat{\sigma}^2/8)\lim_n \sum_{j=0}^{g(n)} \hat{\sigma}^{-(1+2j)} W^{n,j}(K), \end{aligned}$$

where

$$W^{n,j}(K) \equiv \frac{1}{\sqrt{2\pi}} \sum_{k=2j}^{n} (-1)^k C(f(k)-j,k) \frac{(K-S_0)^{k+2}}{S_0^{k+1}(k+2)!}, \tag{11}$$

$C(j,n)$ is the jth term of the polynomial P_n, and

$$f(k) = \begin{cases} \frac{k}{2}, & \text{k even;} \\ \frac{k-1}{2}, & \text{k odd.} \end{cases} \tag{12}$$

and

$$g(n) = \begin{cases} -n-1, & \text{n even;} \\ -n, & \text{n odd.} \end{cases} \tag{13}$$

- It is convenient express the volatility terms as functionals of exponential of the variance

$$\frac{1}{\sqrt{x}} N\left(-\frac{\sqrt{x}}{2}\right) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-(z+1/8)x}}{\sqrt{z+1/8}} dz, \tag{14}$$

and

$$x^{-r} = \frac{1}{r\Gamma(r)} \int_0^\infty e^{-z^{1/r} x} dz, \tag{15}$$

- Letting $\hat{\sigma} = \sqrt{\langle X \rangle_T}$ in the expansion above and substituting it in the TVO pricing formula

$$C_0^{TV}(K) = E_0 \left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} C^{BS}(S_0, K, \langle X \rangle_T) \right] \tag{16}$$

we obtain,

$$C_0^{TV}(K) \approx \bar{\sigma}\sqrt{T} \left[\frac{2S_0}{\sqrt{\pi}} I_0^{1/2,0} + \frac{S_0 + K}{2\sqrt{\pi}} \Phi_0^{1,1/8} + \sum_{j=0}^{g(n)} \tilde{W}^{n,j}(K) I_0^{j+1,1/8} \right],$$

where the following quantities have been defined

$$\begin{aligned} I_0^{r,a} &\equiv \int_0^\infty E_0\left[e^{\lambda^{r,a}(z)\langle X\rangle_T}\right]dz \\ \Phi_0^{r,a} &\equiv \int_0^\infty \frac{E_0\left[e^{\lambda^{1,1/8}(z)\langle X\rangle_T}\right]}{\sqrt{z+1/8}}dz, \end{aligned}$$

$$\lambda^{r,a}(z) \equiv -(z^{1/r}+a), \tag{17}$$

and

$$\tilde{W}^{n,j} \equiv \frac{W^{n,j}(K)}{(j+1)\Gamma(j+1)}. \tag{18}$$

- Alternatively, for short dated maturities, we can use the Bachelier approximation for the ATM term,

$$C_0^{TV}(K) \approx S_0\bar{\sigma}\sqrt{\frac{T}{\pi}} + \bar{\sigma}\sqrt{T}\left[\frac{K - S_0}{2\sqrt{\pi}}\Phi_0^{1,1/8} + \sum_{j=0}^{g(n)} \tilde{W}^{n,j}(K) I_0^{j+1,1/8}\right]. \tag{19}$$

- The expressions above can be calculated in closed-form for a variety of models (e.g. affine stochastic volatility models).

4. TVO pricing via Taylor expansion, $t > 0$

- The pricing problem at $t > 0$ is similar, but some symmetry is lost:

$$C_t(K) \equiv E_t\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_T}}(S_T - K)^+\right] \tag{20}$$

$$= E_t\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\epsilon_t + \langle X\rangle_T - \langle X\rangle_t}} C^{BS}(S_t, K, \langle X\rangle_T - \langle X\rangle_t)\right], \tag{21}$$

where $\epsilon_t \equiv \langle X\rangle_t$.

- The Black-Scholes equation can again be expanded via Taylor ... but this time we have to deal with objects of the following form:

$$q_1(x) \equiv \frac{N(-\sqrt{x}/2)}{\sqrt{\epsilon + x}}, \tag{22}$$

$$q_2(x) \equiv \frac{x^{-(j+1/2)}}{\sqrt{\epsilon + x}}. \tag{23}$$

- We could consider the numerator and denominator in (22) and (23) as separate objects and write them in terms of their integral representation.
- However, this leads to convergence issues when one tries to derive a robust price for the claim
- A less elegant but more effective solution is to use a Taylor expansion in ϵ.
- Some algebra shows that

$$\frac{N(-\frac{\sqrt{x}}{2})}{\sqrt{\epsilon+x}} \approx \frac{N(-\frac{\sqrt{\epsilon+x}}{2})}{\sqrt{\epsilon+x}} + \frac{e^{-(\epsilon+x)/8}}{\sqrt{2\pi}} \sum_{j=0}^{n} \omega(j,n)(\epsilon+x)^{-(j+1)} + \mathrm{O}(n+1) \tag{24}$$

where

$$\omega(j,n) \equiv \sum_{k=j}^{n-1} (-1)^{k+1} \gamma(j,k+1) \frac{\epsilon^{k+1}}{k+1!} \tag{25}$$

and $\gamma(j,k)$ is the jth coefficient of the kth derivative of the Taylor expansion.

- Similarly, we have

$$\frac{x^{-(j+1/2)}}{\sqrt{\epsilon+x}} \approx \sum_{k=0}^{n} \zeta(k,j)(\epsilon+x)^{-(j+k+1)} + \mathrm{O}(n+1). \tag{26}$$

- In the expression above, we have defined:

$$\zeta(k,j) = \frac{(-1)^k \epsilon^k}{k!} \prod_{i=0}^{k-1} (j+i+1/2). \tag{27}$$

- These results can used in the pricing equation of the TVO at time t. The final result is a linear combination of terms of the form

$$\begin{aligned} I_t^{r,a,b} &= \int_0^\infty e^{-(z^{1/r}+b)\epsilon_t} E_t\left[e^{\lambda^{r,a}(\langle X\rangle_T - \langle X\rangle_t)}\right] dz, \\ \Phi_t^{1,a} &= \int_0^\infty \frac{e^{-(z+a)\epsilon_t}}{\sqrt{z+a}} E_t\left[e^{\lambda^{1,a}(\langle X\rangle_T - \langle X\rangle_t)}\right] dz, \end{aligned}$$

  for some non-negative real constants a and b and the coefficient

$$\lambda^{r,a} \equiv -(z^{1/r}+a). \tag{28}$$

- As usual, we can compute a model dependent price, for the TVO, but we can also do better ...

5. TVO pricing using Laplace Transforms, the $t > 0$ case

- Let's consider the pricing problem of a put TVO, where the pay-off is expressed in terms of the log-strike k and the log-terminal value s_T

$$P_t(k) \equiv E_t\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_T}}(e^k - e^{s_T})^+\right] \tag{29}$$

- For any complex α such that $\mathrm{Re}(\alpha) > 1$, the Laplace transform of $P(k)$ is equal to

$$\begin{aligned}\hat{P}_t(\alpha) &\equiv \int_0^\infty e^{-\alpha k} P_t(k) dk \\ &= \bar{\sigma}\sqrt{T} S_t^{1-\alpha} E_t\left[\frac{1}{\sqrt{\epsilon_t + \langle X\rangle_T - \langle X\rangle_t}} \frac{e^{(1-\alpha)(X_T - X_t)}}{\alpha(\alpha-1)}\right].\end{aligned}$$

- The denominator admits the familiar representation

$$\frac{1}{\sqrt{\epsilon + x}} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2(\epsilon + x)} dz. \tag{30}$$

- Using the independence of σ and W_t together with Fubini's theorem, we can write $\hat{P}(\alpha)$ in terms of S_t and the quadratic variation

$$\hat{P}_t(\alpha) = 2\bar{\sigma}\sqrt{\frac{T}{\pi}}S_t^{1-\alpha}\int_0^\infty \frac{e^{-z^2\langle X\rangle_t}E_t\left[e^{\lambda_{z,\alpha}(\langle X\rangle_T - \langle X\rangle_t)}\right]}{\alpha(\alpha-1)}dz \tag{31}$$

where $\lambda_{z,\alpha} = -(z^2 + \alpha(1-\alpha))$

- If we don't mind model dependence, we can calculate explicitly the quantities inside the expectation for a variety stochastic volatility models (e.g., affine models).

- The price of the TVO options can be obtained by numerically inverting the closed form Laplace transform (31).

6. Robust pricing...

- Carr-Lee provide a way to express the exponential of the quadratic variation in terms of the terminal value of the underlying. For any complex λ we have:

$$E_t[e^{\lambda(\langle X\rangle_T-\langle X\rangle_t)}]=E_t\left[e^{(X_T-X_t)p(\lambda)}\right]=E_t\left(\frac{S_T}{S_t}\right)^{p(\lambda)}, \tag{32}$$

where

$$p(\lambda)=\frac{1}{2}\pm\sqrt{\frac{1}{4}+2\lambda}. \tag{33}$$

- Breeden-Litzenberger and Carr-Madan show that sufficiently smooth pay-off can be expressed as porfolio of call and put pay-offs:

$$f(S)=f(k)+f^{'}(k)[S-k]+\int_k^{\infty}f^{''}(x)(S-x)^+dx+\int_0^k f^{''}(x)(x-S)^+dx \tag{34}$$

6.1. Robust pricing using Laplace transform

- We can use the result by Carr and Lee, to express the Laplace transform of the TVO in terms of X_T

$$\hat{P}_t(\alpha) = 2\bar{\sigma}\sqrt{\frac{T}{\pi}} S_t^{1-\alpha} \int_0^\infty \frac{e^{-z^2\langle X\rangle_t} E_t\left[e^{p^{\pm}_{z,\alpha}(X_T - X_t)}\right]}{\alpha(\alpha-1)} dz \tag{35}$$

, where

$$p^{\pm}_{z,\alpha} = 1/2 \pm \sqrt{1/4 - 2(z^2 + \alpha(1-\alpha))}. \tag{36}$$

- Inverting the Laplace transform and applying Fubini's theorem twice, we can derive two alternative representations for the price of the TVO:

$$P_t(k) = \frac{\bar{\sigma}\sqrt{T}}{\pi^{3/2}} E_t\left[\int_0^\infty e^{-z^2\langle X\rangle_t} \int_{-\infty}^{\infty} e^{(a+iu)k} \frac{S_t^{1-a-iu}}{(a+iu)(a+iu-1)} \left(\frac{S_T}{S_t}\right)^{p^{\pm}_{z,\alpha}} dudz\right], \tag{37}$$

or equivalently

$$P_t(k) = \frac{4e^{ak}\bar{\sigma}\sqrt{T}}{\pi^{3/2}} E_t \left[\int_0^\infty e^{-z^2\langle X\rangle_t} \int_0^\infty \mathrm{Re}\left(\frac{S_t^{1-a-iu}(S_T/S_t)^{p^{\pm}_{z,\alpha}}}{(a+iu)(a+iu-1)}\right) cos(uk) du dz \right], \tag{38}$$

where we have set $\alpha = a + iu$.

- To achieve pricing robustness, we would like to express the price of a TVO as a weighted portfolio of quoted vanilla calls $C_t^M(K)$and puts $P_t^M(K)$ plus some other term.

- To this end, we can define the function $f(S)$ as follows and use it in the Carr-Madan representation

$$f(S) \equiv \frac{4e^{ak}\bar{\sigma}\sqrt{T}}{\pi^{3/2}} \int_0^\infty e^{-z^2\langle X\rangle_t} \int_0^\infty \mathrm{Re}\left(\frac{S_t^{1-a-iu}(S/S_t)^{p^{\pm}_{z,\alpha}}}{(a+iu)(a+iu-1)}\right) cos(uk) du dz. \tag{39}$$

- In particular, if we set the ATM strike S_t as the separator between calls and puts and take the t-conditional expectation of $f(S_T)$, we obtain:

$$P_t(k) = \frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_t}}(K - S_t)^+ + \int_{S_t}^\infty f''(x) C_t^M(x) dx + \int_0^{S_t} f''(x) P_t^M(x) dx. \tag{40}$$

- ...which does not exist, because the integral in the second derivative of $f(S)$ does not converge.

6.2. Robust pricing via Taylor

- The Carr-Lee formula can be used to express $I_t^{\cdot}$ and $\Phi_t^{\cdot}$ in the Taylor expansion for $t > 0$ as a function of S_T:

$$\begin{aligned} I_t^{r,a,b} &= \int_0^\infty e^{-(z^{1/r}+b)\epsilon_t} E_t\left[e^{\lambda^{r,a}(\langle X\rangle_T - \langle X\rangle_t)}\right] = E_t \int_0^\infty e^{-(z^{1/r}+b)\epsilon_t} \mathrm{Re}\left(\frac{S_T}{S_t}\right)^{p^{r,a}(z)} dz \\ \Phi_t^{1,a} &= \int_0^\infty \frac{e^{-(z+a)\epsilon_t}}{\sqrt{z+a}} E_t\left[e^{\lambda^{1,a}(\langle X\rangle_T - \langle X\rangle_t)}\right] dz = E_t \int_0^\infty \mathrm{Re}\frac{e^{-(z+a)\epsilon_t}}{\sqrt{z+a}} \mathrm{Re}\left(\frac{S_T}{S_t}\right)^{p^{1,a}(z)} dz, \end{aligned}$$

  where

$$p^{r,a} \equiv 1/2 \pm \sqrt{1/4 - 2z^{1/r} - 2a}. \tag{41}$$

- Since the price the price of the TVO is a linear combination of quantities like $I_t^{\cdot}$ and $\Phi_t^{\cdot}$, we only need to express the latter terms as a combination of traded calls and puts.

- In particular, we can define the functions

$$\begin{aligned}
\tilde{I}^{r,a,b}(S) &\equiv \int_0^\infty e^{-(z^{1/r}+b)\epsilon_t}\mathrm{Re}\left(\frac{S}{S_t}\right)^{p^{r,a}(z)} dz \\
\tilde{\Phi}(S)^{1,a} &\equiv \int_0^\infty \frac{e^{-(z+a)\epsilon_t}}{\sqrt{z+a}}\mathrm{Re}\left(\frac{S}{S_t}\right)^{p^{1,a}(z)} dz.
\end{aligned}$$

- $I^{r,a,b}(S)$ and $\Phi(S)^{1,a}(S)$ are twice differentiable in S and are well defined in S_t.
- We can thus use the Carr-Madan representation to express the price of a TVO as a function of traded instruments.

7. Numerical Results

- In the following numerical examples we assumed the following:

$$dS_t = v_t^{1/2} S_t dW_t, \quad S_0 = 100, \tag{42}$$

where the instantaneous variance satisfies the the CIR SDE:

$$dv_t = \kappa(\theta - v_t)dt + \eta v_t^{1/2} dZ_t, \quad v_0 = 0.2. \tag{43}$$

Table 1: **An overview of the performance of the different methods, maturity T=1.**

Strike	60	80	100	120	140
Taylor n=4	9.8764	6.3718	3.9578	2.4086	1.4275
Laplace transform	9.7790	6.3622	3.9565	2.4135	1.4719
Bernstein polynomial n=30	10.3676	6.6147	3.9558	2.3117	1.4011
Monte Carlo	9.7550	6.3512	3.9557	2.4132	1.4682

7.1. Taylor expansion

Figure 2: Value of the TVO as a function of the time to maturity. K=110.

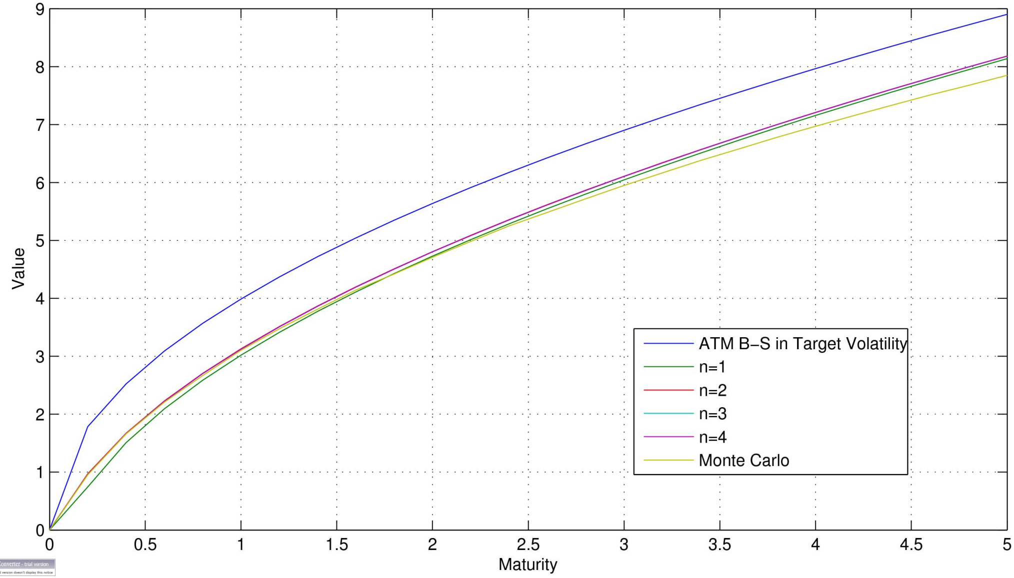


Table 2: **Taylor prices for the TVO for T=1 using Bachelier approximation for the ATM**

Strike	ATM value	n=1	n=2	n=3	n=4	Monte Carlo
90	3.9878	4.9568	5.0656	5.0711	5.0709	5.0550
95	3.9878	4.4723	4.4995	4.5002	4.5002	4.4714
100	3.9878	3.9878	3.9878	3.9878	3.9878	3.9566
105	3.9878	3.5032	3.5305	3.5298	3.5298	3.4985
110	3.9878	3.0187	3.1276	3.1221	3.1219	3.0898

7.2. Laplace transform

Figure 3: Laplace transform Call TVO prices as a function of the strike, T=0.5.

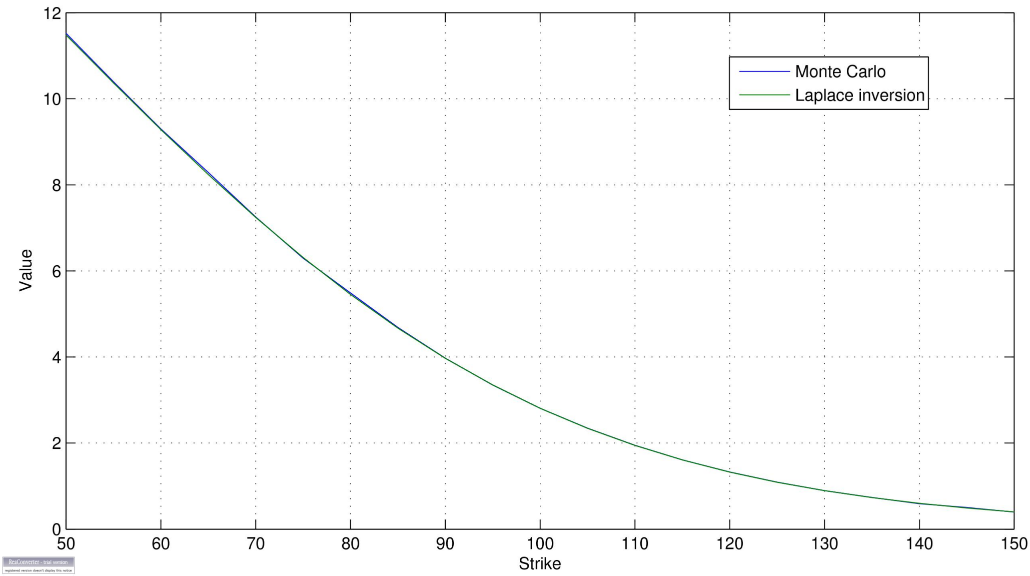


Figure 4: Laplace transform Call TVO prices as a function of the strike, T=3.

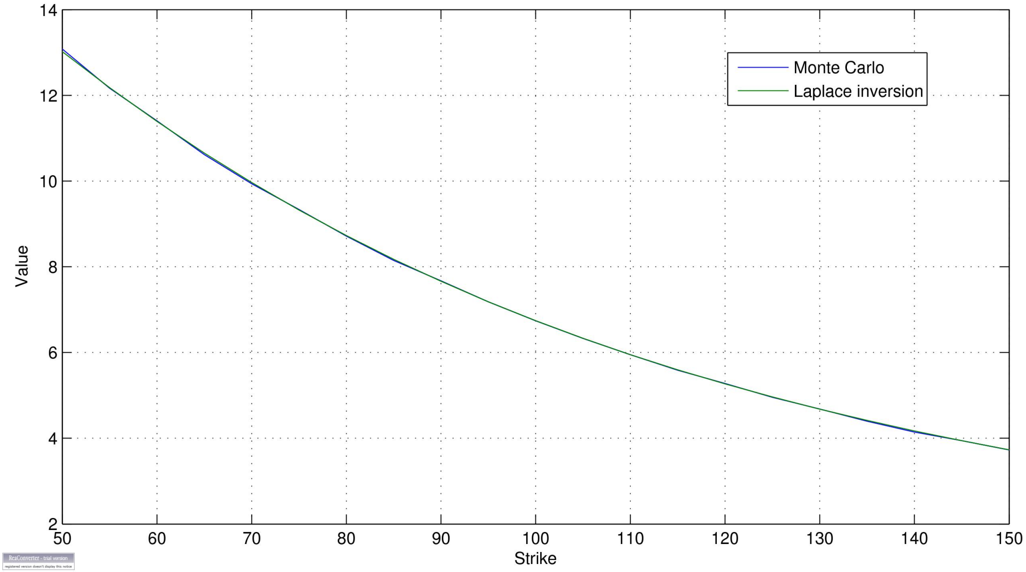


Figure 5: Laplace Transform ATM prices for increasing maturity.

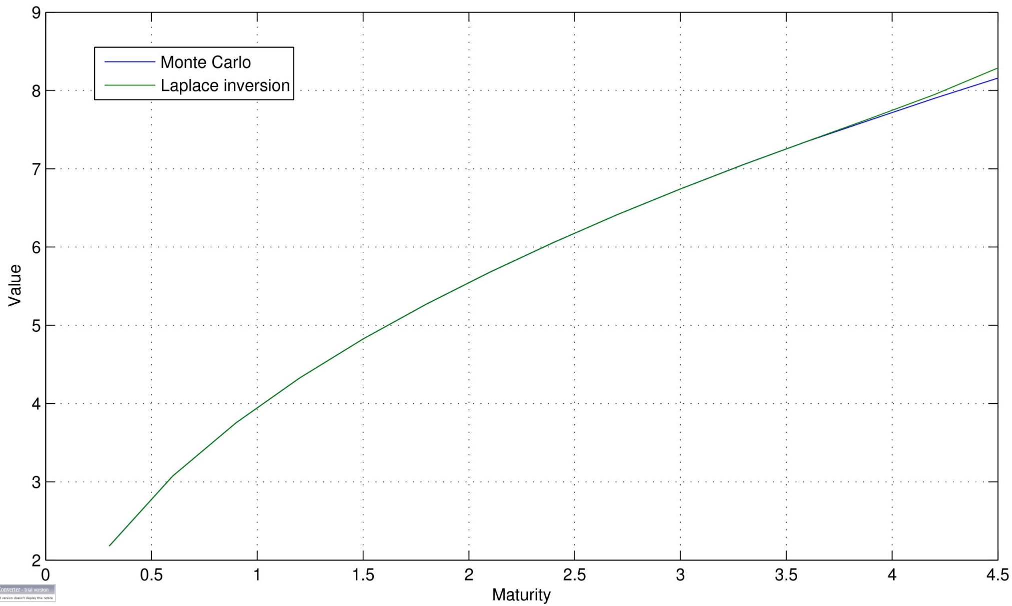


Table 3: **Laplace transform vs Monte Carlo simulation, T=3.**

Strike	Inversion	Monte Carlo
90	7.6715	7.6619
95	7.1843	7.1846
100	6.7389	6.7417
105	6.3286	6.3308
110	5.9473	5.9495

8. What's next?

- Hedging ... W.I.P.
- Non zero correlation case.
- Target volatility indices:

$$\begin{aligned}\frac{\Delta I_t}{I_t} &= \frac{\bar{\sigma}\sqrt{\Delta T}}{\sqrt{\langle X\rangle_{t+\Delta t} - \langle X\rangle_t}}\frac{\Delta S_t}{S_t} && (44)\\ &\approx \bar{\sigma}\Delta W_t && (45)\end{aligned}$$

- The price of call option on a TVO index is thus approximately

$$E_0[(I_T - K)^+] = C^{BS}(I_0, K, \bar{\sigma}) + \mathrm{Err}(volvol, \rho, \Delta t) \quad (46)$$