Implied volatility from Local Volatility

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Outline

- Local volatility in terms of implied volatility.
- Implied volatility in terms of local volatility.
  - The BBF approximation.
  - The heat kernel approach.
  - BBF to higher orders.
- An exact path-integral representation for implied volatility
  - The most-likely-path approximation.
- Parameter averaging
- Numerical tests with a realistic volatility surface.
- Summary and conclusions.

Objective

Given a local volatility process

$$\frac{dS}{S} = \sigma(S,t)\,dW_t,$$

with $\sigma(S,t)$ depending only on the underlying level S and the time t, we want to compute implied volatilities $\sigma_{BS}(K,T)$ such that

$$C_{BS}(S,K,\sigma_{BS}(K,T),T) = \mathbb{E}\left[(S_T - K)^+\right]$$

or in words, we want to efficiently compute implied volatility from local volatility.

- This can of course be done with numerical PDE
  - but numerical PDE is slow,
  - too slow for efficient calibration to implied vols.

Motivations

- The condition for no static arbitrage can be simply expressed as the non-negativity of local variance.
  - It's very hard in general to eliminate static arbitrage in a given parameterization of the implied volatility surface.
- Knowing how to get implied volatility from local volatility helps us get accurate approximations to implied volatility in more complex models such as SABR.
  - Efficient calibration of complex models becomes practical.

Local volatility in terms of implied volatility

Define the Black-Scholes implied total variance:

$$w(K,T) := \sigma_{BS}^2(K,T)\,T$$

In terms of the log-strike $k := \log K/F$ and the local variance $v_L := \sigma^2(K,T)$, the Dupire equation becomes

$$\frac{\partial C}{\partial T} = \frac{v_L}{2}\left\{\frac{\partial^2 C}{\partial k^2} - \frac{\partial C}{\partial k}\right\}$$

Then, by taking derivatives of the Black-Scholes formula and simplifying, we obtain equation (1.10) in [3]:

$$v_L = \frac{\frac{\partial w}{\partial T}}{\left(1 - \frac{k}{2\,w}\,\frac{\partial w}{\partial k}\right)^2 - \frac{1}{4}\,\left(\frac{1}{4} + \frac{1}{w}\right)\,\left(\frac{\partial w}{\partial k}\right)^2 + \frac{1}{2}\,\frac{\partial^2 w}{\partial k^2}} \tag{1}$$

Special Case: No Skew

If the skew $\frac{\partial w}{\partial k}$ is zero, (1) reduces to

$$v_L = \frac{\partial w}{\partial T}$$

In this special case, the local variance reduces to the forward Black-Scholes implied variance. The solution is of course

$$w\left(T\right) = \int_0^T v_L\left(t\right)\, dt$$

Inverting the equation

- We have a formula (1) for getting local volatility from implied.
- All we need to do is to invert this formula!
  - This is certainly not easy and has not so far proved to be possible in closed-form.
- In the limit of small time however, equation (1) can been solved.

The BBF approximation

Recall equation (1) for local variance in terms of implied:

$$v_L = \frac{\frac{\partial w}{\partial T}}{\left(1 - \frac{k}{2\,w}\,\frac{\partial w}{\partial k}\right)^2 - \frac{1}{4}\,\left(\frac{1}{4} + \frac{1}{w}\right)\,\left(\frac{\partial w}{\partial k}\right)^2 + \frac{1}{2}\,\frac{\partial^2 w}{\partial k^2}}$$

Noting that $w \sim O(T)$, in the limit of small T, to leading order in T we may write

$$v_L \approx \frac{\frac{\partial w}{\partial T}}{\left(1 - \frac{k}{2\,w}\,\frac{\partial w}{\partial k}\right)^2} \tag{2}$$

Further supposing that to lowest order in T, $w \approx \sigma_{BS}(k,0)^2\,T$ and making the change of variable

$$u = \frac{1}{\sigma_{BS}(k,0)},$$

we may rewrite (2) as

$$\sigma(k,0)^2 \approx \frac{\frac{1}{u^2}}{\left(1+\frac{k}{u}\frac{\partial u}{\partial k}\right)^2}$$

or rearranging

$$\frac{\partial}{\partial k}(k\,u) = \frac{1}{\sigma(k,0)}$$

giving us the BBF approximation of Berestycki, Busca and Florent [2]:

$$\frac{1}{\sigma_{BS}(K,T)} \approx \frac{1}{\sigma_0(k)} := \frac{1}{\ln K/S_0}\int_{S_0}^{K}\frac{dS}{S\,\sigma(S,0)} = \int_0^1 \frac{d\alpha}{\sigma(\alpha\,k,0)}$$

First order term

In [5], choosing to expand $\sigma_{BS}(\cdot)$ as

$$\sigma_{BS}(k,T) = \sigma_0(k) + \sigma_1(k)\,T + O(T^2),$$

substituting into (1) and matching powers of T, Pierre Henry-Labordère (H-L) obtains the first order correction:

$$\begin{aligned}\sigma_1(k) \quad &= \quad \frac{\sigma_0(k)^3}{k^2}\left\{\ln\frac{\sqrt{\sigma(0,0)\,\sigma(k,0)}}{\sigma_0(k)}\right. \\ &\qquad \left. - \int_0^k \frac{\partial_t\sigma(y,t)|_{t=0}}{\sigma(y,0)}\,\frac{\partial}{\partial y}\left(\frac{y}{\sigma_0(y)}\right)^2 dy\right\}\end{aligned}$$

where $\sigma_0(k)$ is the lowest-order (BBF) approximation derived earlier.

Heat kernel expansion

In [4], we compute implied volatility for short times using the heat kernel expansion up to second order.

$$\sigma_{BS}(k, T) \approx \sigma_0(k) + \sigma_1(k)\, T + \sigma_2(k)\, T^2$$

The first two terms, σ_0 and σ_1 agree with BBF and H-L respectively. σ_2 is somewhat too complicated to reproduce here!

Henry-Labordère's approximation

Henry-Labordère also presents a heat kernel expansion based approximation to implied volatility in equation (5.40) on page 140 of his book [5]:

$$\sigma_{BS}(K,T) \approx \sigma_0(K)\left\{1+\frac{T}{3}\left[\frac{1}{8}\sigma_0(K)^2+\mathcal{Q}(f_{av})+\frac{3}{4}\mathcal{G}(f_{av})\right]\right\} \tag{3}$$

with

$$\mathcal{Q}(f)=\frac{C(f)^2}{4}\left[\frac{C''(f)}{C(f)}-\frac{1}{2}\left(\frac{C'(f)}{C(f)}\right)^2\right]$$

and

$$\mathcal{G}(f)=2\,\partial_t\log C(f)=2\,\frac{\partial_t\,\sigma(f,t)}{\sigma(f,t)}$$

where $C(f)=f\,\sigma(f,t)$ in our notation, $f_{av}=(S_0+K)/2$ and the term $\sigma_0(K)$ is the BBF approximation from [2].

Parameter Averaging

- Given an SDE with time-dependent parameters, the idea of parameter averaging is to optimally choose average parameters for a similar SDE with time-independent parameters and an easy-to-compute solution.
- For example, in [7], given an SDE of the form

$$dS_t = \sigma(t)\ \{b(t)\, S_t + (1 - b(t))\ S_0\}\ dW_t,$$

  Vladimir Piterbarg explains how to choose average parameters $\bar{\sigma}$ and $\bar{b}$ for the shifted-lognormal process

$$dS_t = \bar{\sigma}\ \left\{\bar{b}\, S_t + \left(1 - \bar{b}\right)\ S_0\right\}\ dW_t$$

- European options are then priced using the closed-form shifted lognormal formula with average parameters.

Quadratic Parameter Averaging

Given the time-dependent SDE

$$dX_t = \sigma(t)\left\{1 + b(t)\,(X_t - 1) + \frac{c(t)}{2}\,(X_t - 1)^2\right\};\; X_0 = 1,$$

Andersen and Hutchings [1] derive optimal choices of average parameters $\bar{\sigma}$, $\bar{b}$ and $\bar{c}$ for the quadratic SDE

$$dX_t = \bar{\sigma}\left\{1 + \bar{b}\,(X_t - 1) + \frac{\bar{c}}{2}\,(X_t - 1)^2\right\}$$

which has a known closed-form solution for a restricted set of parameters.

Optimal parameter choices

Average volatility is given by

$$\bar{\sigma}^2 = \frac{1}{T} \int_0^T \sigma(t)^2 \, dt,$$

average skew by

$$\bar{b} = \int_0^T b(t) \, w_b(t) \, dt$$

with

$$w_b(t) = \frac{\sigma(t)^2 \, \nu(t)^2}{\int_0^T \sigma(t)^2 \, \nu(t)^2 \, dt}; \; \nu(t)^2 := \int_0^t \sigma(s)^2 \, ds$$

and average curvature by

$$\bar{c} = \int_0^T c(t)\, w_c(t)\, dt$$

with

$$w_c(t) = \frac{\sigma(t)^2\, \nu(t)^4}{\int_0^T \sigma(t)^2\, \nu(t)^4\, dt}$$

Integral representation of implied volatility

As usual, we assume that the stock price S_t satisfies the SDE

$$\frac{dS_t}{S_t} = \sigma_t \, dZ_t$$

where the volatility σ_t may be random.
For fixed K and T, define the Black-Scholes gamma

$$\Gamma_{BS}(S_t, \bar{\sigma}(t)) := \frac{\partial^2}{\partial S_t^2} C_{BS}(S_t, K, \bar{\sigma}(t), T - t)$$

and further define the "Black-Scholes forward implied variance" function

$$v_{K,T}(t) = \frac{\mathbb{E}\left[\sigma_t^2 \, S_t^2 \, \Gamma_{BS}(S_t, \bar{\sigma}(t))\right]}{\mathbb{E}\left[S_t^2 \, \Gamma_{BS}(S_t, \bar{\sigma}(t))\right]} \tag{4}$$

where

$$\bar{\sigma}^2(t) := \frac{1}{T-t} \int_t^T v_{K,T}(u) \, du \tag{5}$$

Keller-Ressel and Teichmann

In [6], Martin Keller-Ressel and Josef Teichmann show by explicit construction that the forward implied variances $v_{K,T}\,(t) =: \bar{\sigma}^2_{K,T}\,(t)$ in (4) exist and give a pretty construction in terms of a state-switching process.

Specifically, consider the price process $\tilde{S}^\tau_t$ given by

$$\begin{aligned} \tilde{S}^\tau_t &= S_t \text{ for } t < \tau \\ d\tilde{S}^\tau_t &= \tilde{S}^\tau_t\,\sigma_\tau\,dW_t \text{ for } t \geq \tau. \end{aligned}$$

They show that choosing $\sigma_\tau = \bar{\sigma}_{K,T}(\tau)$ generates the market price $C(K,T)$ of the option with strike K and expiration T thus justifying our earlier terminology *"Black-Scholes forward implied variance"* for $v_{K,T}(t)$.

Path-by-path, for any suitably smooth function $f(S_t, t)$, applying Itô's Lemma, we have

$$
\begin{aligned}
f(S_T, T) - f(S_0, 0) &= \int_0^T df \\
&= \int_0^T \left\{ \frac{\partial f}{\partial S_t} \, dS_t + \frac{\partial f}{\partial t} \, dt + \frac{\sigma_t^2}{2} \, S_t^2 \, \frac{\partial^2 f}{\partial S_t^2} \, dt \right\}
\end{aligned}
$$

With $f(\cdot)$ as the Black-Scholes valuation formula $C_{BS}(\cdot)$, we obtain:

$$
\begin{aligned}
C(S_0, K, T) &= \mathbb{E}\left[(S_T - K)^+\right] \\
&= \mathbb{E}\left[C_{BS}(S_T, K, \bar{\sigma}(T), 0)\right] \\
&= C_{BS}(S_0, K, \bar{\sigma}(0), T) \\
&\quad + \mathbb{E}\left[\int_0^T \left\{ \frac{\partial C_{BS}}{\partial S_t} dS_t + \frac{\partial C_{BS}}{\partial t} dt + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 C_{BS}}{\partial S_t^2} dt \right\}\right]
\end{aligned}
$$

$C_{BS}\left(S_t, K, \bar{\sigma}(t), T-t\right)$ must satisfy the Black-Scholes equation so:

$$\frac{\partial C_{BS}}{\partial t} = -\frac{1}{2} v_{K,T}\left(t\right)\, S_t^2 \,\frac{\partial^2 C_{BS}}{\partial S_t^2}$$

Using this equation to substitute for the time derivative $\frac{\partial C_{BS}}{\partial t}$, we obtain:

$$\begin{aligned} C\left(S_0, K, T\right) &= C_{BS}\left(S_0, K, \bar{\sigma}(0), T\right) \\ &\quad + \mathbb{E}\left[\int_0^T \left\{\frac{\partial C_{BS}}{\partial S_t} dS_t + \frac{1}{2}\left\{\sigma_t^2 - v_{K,T}\left(t\right)\right\} S_t^2 \frac{\partial^2 C_{BS}}{\partial S_t^2} dt\right\}\right] \\ &= C_{BS}\left(S_0, K, \bar{\sigma}(0), T\right) \\ &\quad + \mathbb{E}\left[\int_0^T \frac{1}{2}\left\{\sigma_t^2 - v_{K,T}\left(t\right)\right\} S_t^2 \frac{\partial^2 C_{BS}}{\partial S_t^2} dt\right] \end{aligned} \tag{6}$$

The last term in (6) gives the expected realized profit on a sale of a call option at an implied volatility of $\bar{\sigma}$, delta-hedged using the deterministic forward variance function $v_{K,T}$.

From the definition (4) of $v_{K,T}(t)$, we have that

$$\mathbb{E}\left[S_t^2\,\Gamma_{BS}(S_t,\bar{\sigma}(t))\right]v_{K,T}\,(t)=\mathbb{E}\left[\sigma_t^2\,S_t^2\,\Gamma_{BS}(S_t,\bar{\sigma}(t))\right]$$

so the second term in equation (6) vanishes and $\bar{\sigma}(0)$ is the Black-Scholes implied volatility at time 0 of the option with strike K and expiration T. So

$$\sigma_{BS}(K,T)^2=\bar{\sigma}(0)^2=\frac{1}{T}\int_0^T\frac{\mathbb{E}\left[\sigma_t^2 S_t^2\Gamma_{BS}(S_t)\right]}{\mathbb{E}\left[S_t^2\Gamma_{BS}(S_t)\right]}dt \tag{7}$$

which expresses implied variance as the time-integral of expected instantaneous variance σ_t^2 under some probability measure.

Following Roger Lee, we may rewrite (7) more elegantly as

$$\sigma_{BS}(K,T)^2 = \bar{\sigma}(0)^2 = \frac{1}{T}\int_0^T \mathbb{E}^{G_t}[\sigma_t^2]\,dt \tag{8}$$

thus interpreting the definition (4) of $v(t)$ as the expectation of σ_t^2 with respect to the probability measure $\mathbb{G}_t$ defined, relative to the pricing measure $\mathbb{P}$, by the Radon-Nikodym derivative

$$\frac{d\mathbb{G}_t}{d\mathbb{P}} := \frac{S_t^2\,\Gamma_{BS}(S_t,\bar{\sigma}(t))}{\mathbb{E}\left[S_t^2\,\Gamma_{BS}(S_t,\bar{\sigma}(t))\right]}$$

Note in passing that equations (4) and (7) are circular because the gamma $\Gamma_{BS}(S_t)$ of the option on the rhs depends on $\sigma_{BS}(K,T)$ on the lhs via the forward implied variances $v_{K,T}(t)$.

Special case: Black-Scholes

Suppose $\sigma_t = \sigma(t)$, a function of t only. Then

$$v_{K,T}(t) = \frac{\mathbb{E}\left[\sigma(t)^2 S_t^2 \Gamma_{BS}(S_t)\right]}{\mathbb{E}\left[S_t^2 \Gamma_{BS}(S_t)\right]} = \sigma(t)^2$$

The forward implied variance $v_{K,T}(t)$ and the forward variance $\sigma(t)^2$ coincide. As expected, $v_{K,T}(t)$ has no dependence on the strike K or the option expiration T.

Visualizing implied volatility

Equation (7) may be rewritten in the form

$$v_{K,T}(t) = \int dS_t \, q\left(S_t; S_0, K, T\right) v_L(S_t, t)$$

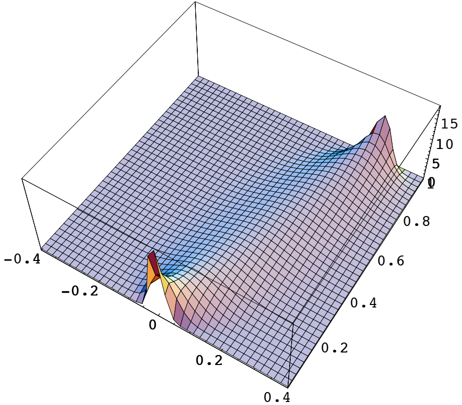
where

$$q\left(S_t, t; S_0, K, T\right) := \frac{p\left(S_t, t; S_0\right) \, S_t^2 \, \Gamma_{BS}(S_t)}{\mathbb{E}\left[S_t^2 \, \Gamma_{BS}(S_t)\right]}$$

and $v_L(S_t, t) = \mathbb{E}^P\left[\sigma_t^2 | S_t\right]$ is the local variance or alternatively in terms of $x_t := \log\left(S_t / S_0\right)$:

$$v_{K,T}(t) = \int dx_t \, q\left(x_t, t; x_T, T\right) v_L(x_t, t) \tag{9}$$

Visualizing implied volatility



The figure shows how $q(x_t, t; x_T, T)$ looks in the case of a 1 year European option struck at 1.3 with a flat 20% volatility. We see that $q(x_t, t; x_T, T)$ peaks on a line $\tilde{x}_t$ joining the stock price today with the strike price at expiration.

The most-likely-path approximation

The density $q(\cdot)$ looks roughly symmetric around the peak. Then

$$q\left(x_t, t; x_T, T\right) \approx q(\tilde{x}_t, t; x_T, T) + \frac{1}{2}\left(x_t - \tilde{x}_t\right)^2 \left.\frac{\partial^2 q}{\partial x_t^2}\right|_{x_t=\tilde{x}_t}$$

Typically $v_L(x_t, t)$ is not so far from linear in x_t in the region where $q\left(x_t, t; x_T, T\right)$ is significant so we may further write

$$v_L(x_t, t) \approx v_L(\tilde{x}_t, t) + \left(x_t - \tilde{x}_t\right) \left.\frac{\partial v_L}{\partial x_t}\right|_{x_t=\tilde{x}_t}$$

Substituting back into equation (9) gives

$$v_{K,T}\left(t\right) \approx v_L(\tilde{x}_t, t) = \sigma(\tilde{x}_t, t)^2$$

and equation (7) becomes

$$\sigma_{BS}(K, T)^2 \approx \frac{1}{T} \int_0^T \sigma(\tilde{x}_t, t)^2 \, dt \tag{10}$$

The approximate formula in words

- Equation (10) says that the Black-Scholes implied variance of an option with strike K is given approximately by the integral from valuation date ($t = 0$) to the expiration date ($t = T$) of the local variances along the path $\tilde{x}_t$ that maximizes the Brownian Bridge-like density $q\left(x_t, t; x_T, T\right)$.
- Note that in practice, it's not trivial to compute the path $\tilde{x}_t$.
  - Adil Reghai describes an efficient fixed-point algorithm to do this.

A fixed point algorithm for finding the most likely path

For each log-strike $k := \log(K/S_0)$, we approximate the most likely path $\tilde{x}_t$ as

$$\tilde{x}_t = \frac{w_t}{w_T}\,k \tag{11}$$

with

$$w_t = \int_0^t ds\,\sigma(\tilde{x}_s)^2$$

Obviously, this definition is circular. We solve by starting with the straight line

$$\tilde{x}_t = \frac{t}{T}\,k$$

as our initial guess and iterating until the path doesn't change. This iteration is extremely fast in practice.

Adil Reghai's formulation

In [8], Adil Reghai approximates the most-likely-path a little differently as

$$\begin{aligned}\tilde{S}_t &\approx \mathbb{E}\left[S_t | S_T = K\right] \\ &\approx S_0 \left(\frac{K}{S_0}\right)^{\alpha} \exp\left\{\frac{w_t\left(w_T - w_t\right)}{2\,w_T}\right\}\end{aligned} \tag{12}$$

with

$$\alpha = \frac{w_t}{w_T}$$

as before.

How well do these approximations work?

We consider the following explicit local volatility models:

- The square-root CEV model:

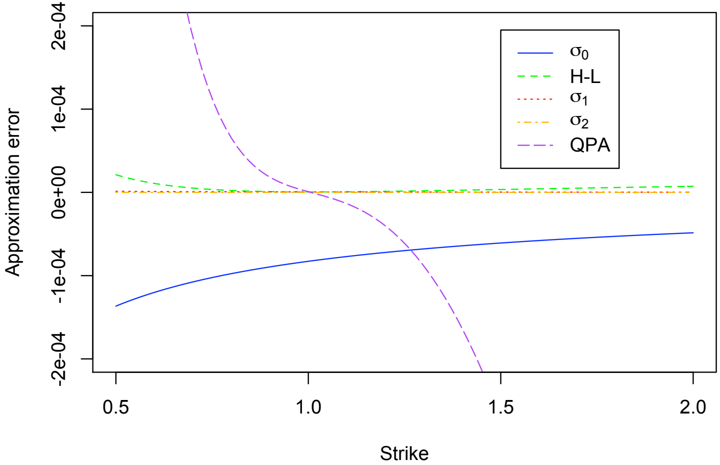
$$dS_t = e^{-\lambda\, t}\, \sigma\, \sqrt{S_t}\, dW_t$$

- The quadratic model:

$$dS_t = e^{-\lambda\, t}\, \sigma\, \left\{1 + \psi\,(S_t - 1) + \frac{\gamma}{2}\,(S_t - 1)^2\right\}\, dW_t$$

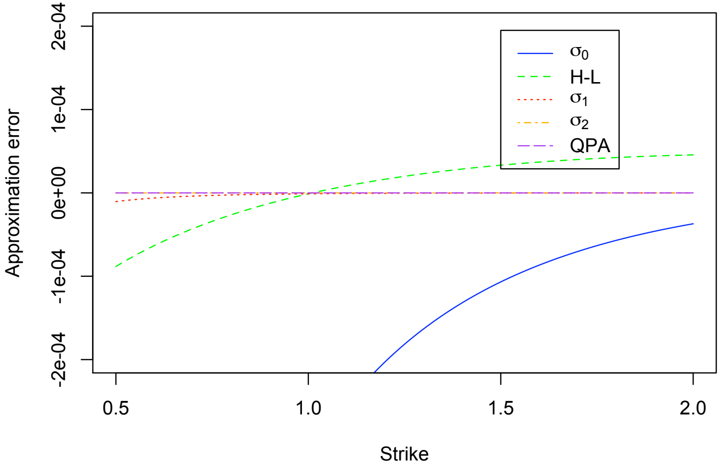
- Parameters are: $\sigma = 0.2$, $\psi = -0.5$ and $\gamma = 0.1$. In each case $S_0 = 1$ and $T = 1$.
- $\lambda = 0$ gives a time-homogeneous local volatility surface and $\lambda = 1$ a time-inhomogeneous one.
- We compare implied volatilities from the approximations and the closed-form solution.

Time-homogeneous Square Root CEV



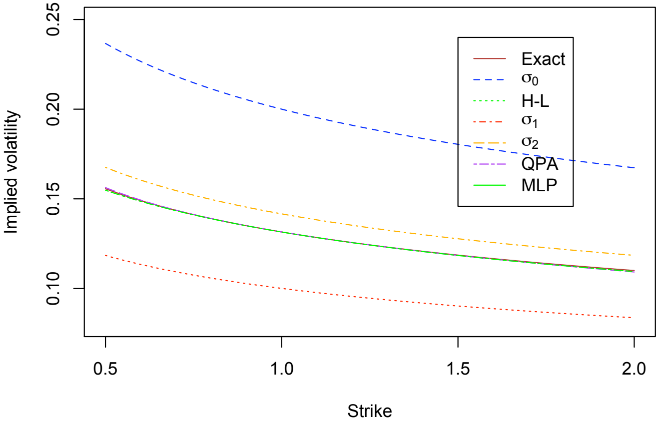
Note that all errors are tiny! Even BBF is a great approximation.

Time-homogeneous Quadratic Model



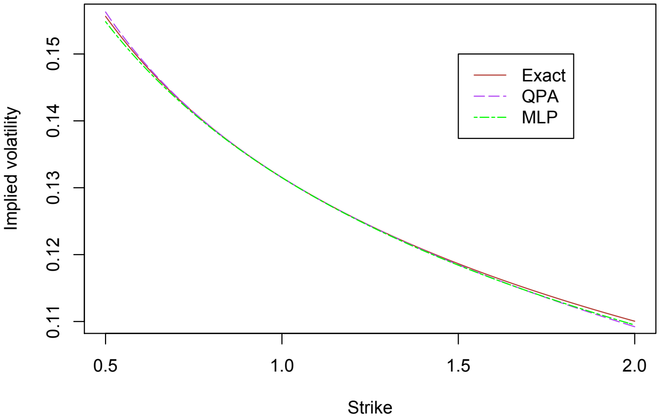
Of course, in this case, quadratic parameter averaging (QPA) is exact.

Time-inhomogeneous Square Root CEV



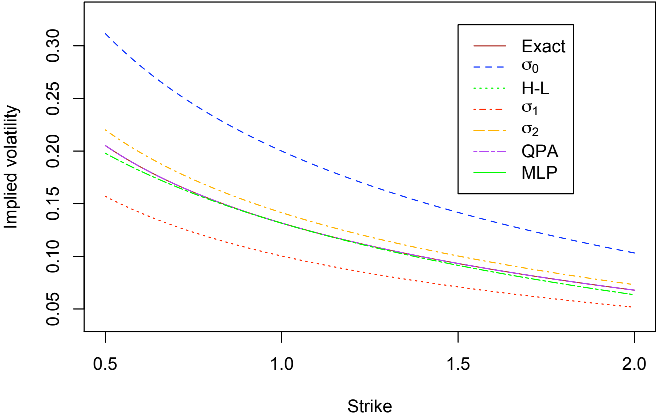
Both quadratic parameter averaging (QPA) and most-likely-path (MLP) are almost exact. The difference between the two MLP formulations is negligible.

Time-inhomogeneous Square Root CEV: zoomed view



Both quadratic parameter averaging (QPA) and most-likely-path (MLP) are almost exact.

Time-inhomogeneous Quadratic Model



Of course, in this case, quadratic parameter averaging (QPA) is exact. MLP is very close!

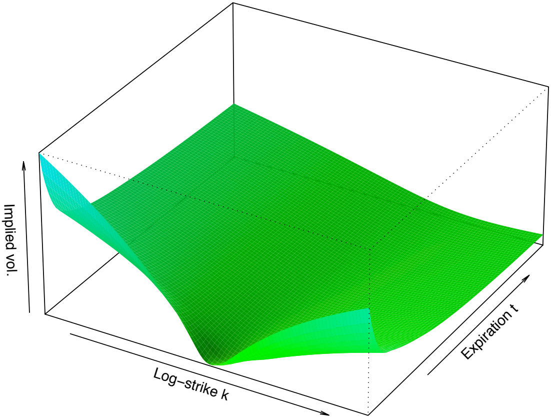
Observations so far

- The small-time expansion of [4] is definitely a big improvement over BBF in the time-inhomogeneous case.
  - The expansions depend on derivatives of the local volatility function at $t = 0$ and so are unlikely to be applicable to a realistic volatility surface.
  - These expansions do permit more accurate closed-form implied volatility approximations for simple models.
- However, the most-likely-path approximation (MLP) and quadratic parameter averaging (QPA) are both much more accurate.
  - QPA has the further advantage of permitting closed-form approximations to implied volatilities.
  - MLP is an iterative procedure so closed-form approximations using MLP are not realistic.

- How do MLP and QPA fare with a realistic volatility surface?

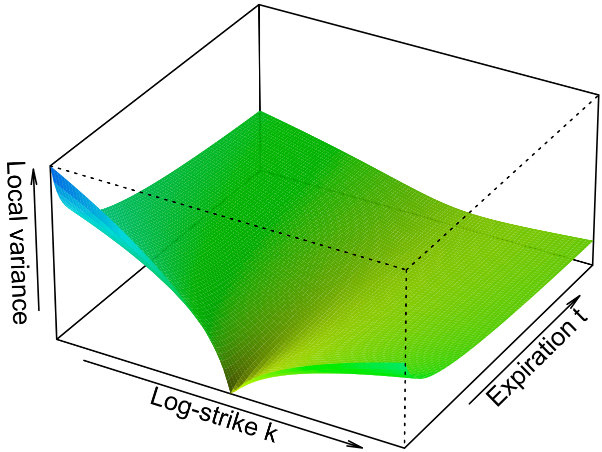
Figure 3.2: 3D plot of volatility surface

Here's a 3D plot of the volatility surface as of September 15, 2005:



$k := \log K/F$ is the log-strike and t is time to expiry.

3D plot of approximate local volatility surface



Local volatility surface parameterization

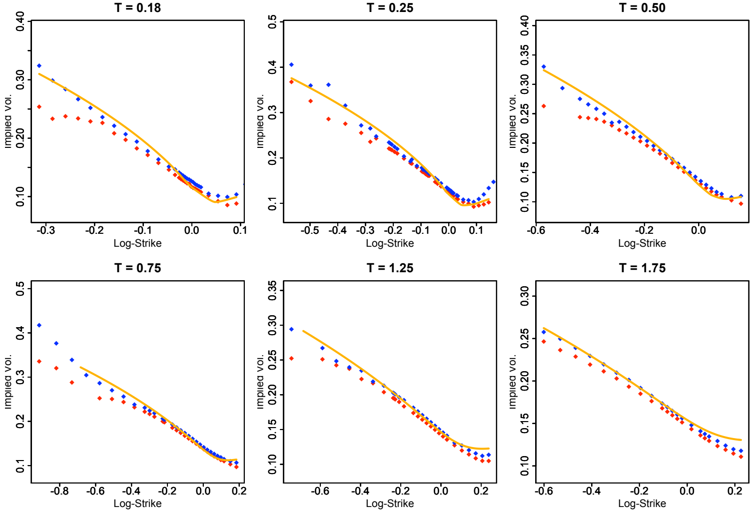
$$\sigma^2(k,t) = a + b\left\{\rho\left(\frac{k}{\sqrt{t}} - m\right) + \sqrt{\left(\frac{k}{\sqrt{t}} - m\right)^2 + \sigma^2\, t}\right\}$$

with

$$\begin{aligned} a &= 0.0012 \\ b &= 0.1634 \\ \sigma &= 0.1029 \\ \rho &= -0.5555 \\ m &= 0.0439 \end{aligned}$$

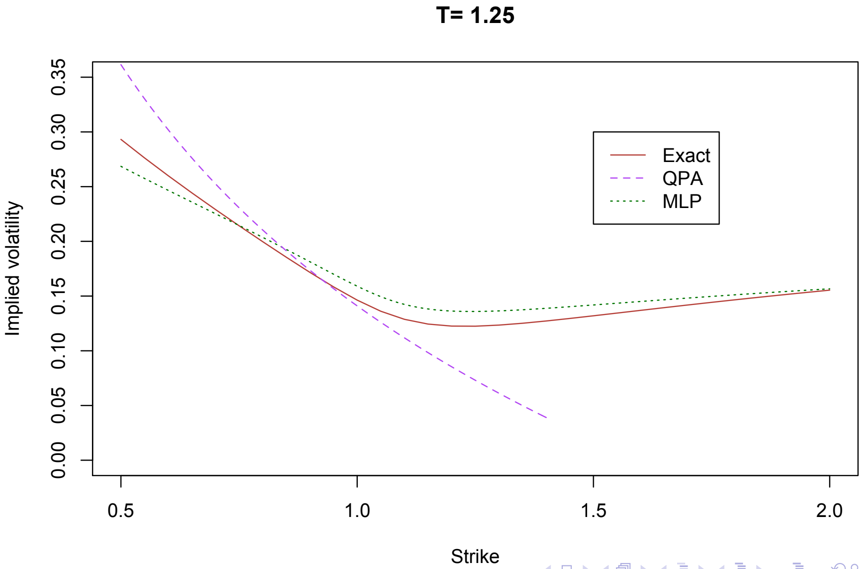
- This surface is singular at $t = 0$ so small-time expansions won't work.
- Each slice is SVI.

Picture for the sceptical



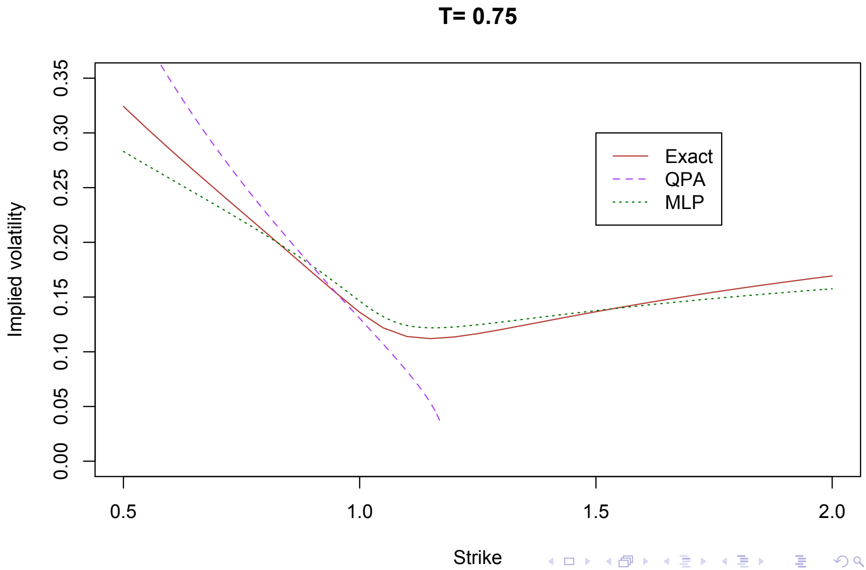
Orange lines are from PDE computations, red and blue points are bid and offered vols respectively. Fits are not too bad!

One slice of the implied volatility surface



The exact implied volatility smile is from a PDE computation

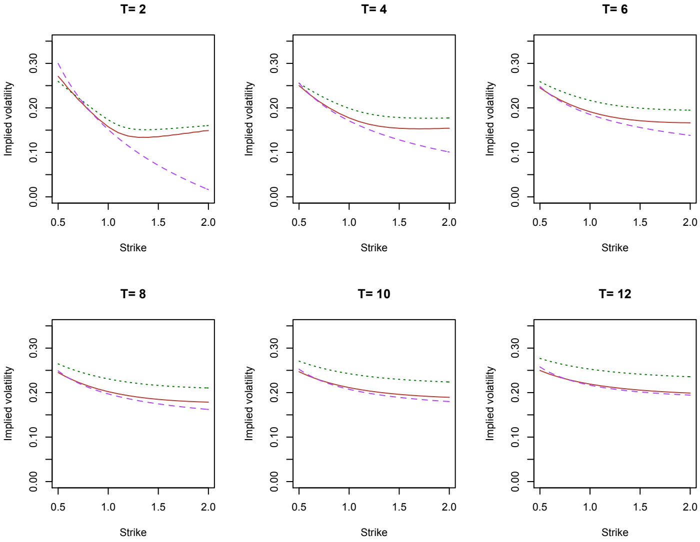
Another slice of the implied volatility surface



Further observations

- MLP is again very close, even with this realistic parameterization of the volatility surface.
  - In addition, MLP is very fast.
- QPA is totally off.
  - Parameter averaging is thrown off by singular behavior at $t = 0$.
  - The shape of the volatility smile is constrained by the form of the quadratic volatility model.
- MLP is the definite winner so far!

Performance with longer-dated smiles



Summary

- Small-time expansions are useful for generating closed-form expressions for implied volatility from simple models but cannot be applied to realistic local volatility surfaces.
- Parameter averaging seems to have problems reproducing shorter-dated smiles when the volatility surface is singular at $T = 0$, as is probably always the case in practice.
    - There is always a small chance of a large (non-diffusive) move over any given short time interval and the volatility surface should reflect this.
    - Markovian projection plus parameter averaging remains a viable technique for fitting a given (relatively simple) model to a given implied volatility surface.
- Parameter averaging can only ever return the parameters of the simple proxy model whose parameters are being approximated; realistic volatility smiles cannot be resolved.

Summary II

- The most-likely-path technique is both easy to implement and efficient in generating good numerical implied volatility approximations for shorter-dated smiles.
  - Because the MLP technique is numerical, it cannot be used for generating closed-form implied volatility approximations for a given model.
  - However, MLP can be used for fitting low-dimensional parameterizations of the local volatility surface to a given implied volatility surface, ensuring no static arbitrage by construction.
- Parameter averaging seems to outperform most-likely-path for longer-dated smiles.
  - As T increases, approximating the expectation in the integral representation (7) by the value at one point becomes less and less of a good approximation.