Portfolio Optimization Under Partial Information With Expert Opinions

Rüdiger Frey
Universität Leipzig

Fields Institute, Toronto, May 2010

ruediger.frey@math.uni-leipzig.de
www.math.uni-leipzig.de/~frey

joint work with Ralf Wunderlich, Zwickau
and Abdel Gabih, Marrakech

1. Introduction

In practical asset management relatively little use of sophisticated quantitative techniques and dynamic models, other than for pricing/hedging derivatives

- Cultural barriers
- Unrealistic objective functions
- Transaction costs
- Model parameters: investment strategies are largely influenced by drift or growth rate of assets, whereas derivative asset analysis is only concerned with volatility

Portfolio Optimization and Drift

Drift is a key input in any optimal portfolio strategy. Consider e.g. classical Merton problem:

- $dS_t = \mu S_t dt + \sigma S_t dW_t$, short rate r
- investor maximizes $E(u(V_T))$ for $u(x) = \frac{1}{\theta}x^\theta$, $\theta < 1$ (CRRA).

Then optimal proportion of wealth in risky asset equals

$$h_t^{(0)} = \frac{1}{(1-\theta)\sigma^2}(\mu - r)$$

and $h^{(0)}$ is a key building block of optimal strategies also in more complicated models.

Portfolio Optimization and Drift cont.

Drifts are hard to estimate empirically:

- Drift estimation needs data over long time horizons (other than volatility estimation). Consider e.g.

$$dX_t = \mu dt + dW_t, \quad 0 \le t \le T.$$

  MLE-estimator for μ given by $\widehat{\mu} = \frac{1}{T}(X_T - X_0)$.

  $\Rightarrow$ Problems with stationarity

- Compounded by the fact that for typical return series drift is dominated by volatility

Implications

- Academic literature: Consider dynamic models with drift driven by unobservable factors and apply filtering techniques.
  - ⋆ Linear Gaussian models: Lakner (1998), Nagai, Peng (2002), Brendle (2006)
  - ⋆ Hidden market models: Sass, Haussmann (2004), Gabih,Wunderlich (2010), Rieder, Bäuerle (2005), Nagai, Rungaldier (2008), . . .
- Practitioners use static Black-Litterman-model (Litterman 2003) where Bayesian updating is used to combine subjective views such as “asset 1 will grow at least 5%” with empirical or implied return estimates.

In the present paper we combine the two approaches by including expert opinions in dynamic models with partial observation

2. The Model

- $(\Omega, \mathcal{G} = (\mathcal{G}_t)_{t\in[0,T]}, P)$ filtered probability space (full information)
- $N+1$ securities $(S_t^0, S_t^1, \ldots, S_t^N)$. Bond $S^0 \equiv 1$; stock price dynamics given by Black-Scholes model with random drift

$$dS_t^i = S_t^i \left(a^i(X_t)dt + \sum_{j=1}^{N} \sigma^{ij} dW_t^j \right), \quad S_0^i = s^i, \quad i = 1, \cdots, N. \quad (1)$$

- Factor process X is a a finite-state Markov chain with state space $\mathfrak{X} = \{x_1, \cdots, x_K\}$, generator matrix Q (HMM).
- Volatility $\sigma = (\sigma^{ij})_{1\le i,j\le N}$ is assumed to be a constant invertible matrix and W is $\mathcal{G}$-Brownian motion.
- Returns $dR_t = \frac{dS_t}{S_t} = a(X_t)\, dt + \sigma\, dW_t$

Investor information

Investor is not informed about factor process X_t, he only observes

- Stock price S or equivalently stock return R_t
- Expert opinions revealed at discrete time points T_n such as
  - ⋆ own views about future performance
  - ⋆ news, recommendations of analysts or rating agencies

$\Longrightarrow$ Model with partial information. Investor needs to "learn" the drift from observable quantities.

Expert Opinions

Modelled by marked point process $I = (T_n, Z_n) \sim I(dt, dz)$

- At random points in time $T_n \sim$ Poi(λ) investor observes r.v. Z_n
- Z_n depends on current state X_{T_n}, density $f(X_{T_n}, z)$ $(Z_n)_n$ cond. independent given $\mathcal{F}_T^X = \sigma(X_s : s \in [0, T])$

Examples

- Absolute view. $Z_n = a(X_{T_n}) + \sigma\varepsilon_n$, $\varepsilon_n \sim N(0, 1)$.
  The view "S will grow by 5%" is modelled by the observation $Z_n = 0.05$; σ models confidence of investor
- Relative view (2 assets) $Z_n = a_1(X_{T_n}) - a_2(X_{T_n}) + \tilde{\sigma}\varepsilon_n$

Investor filtration. $\mathbb{F}$ with $\mathcal{F}_t = \sigma(R_u \colon u \le t;\ (T_n, Z_n) \colon T_n \le t)$

The optimization problem

- Admissible Strategies. (described via portfolio weights)

$$\mathcal{H}(t) = \{(h_s)_{s\in[t,T]} \in \mathbb{R}^n,\ \mathbb{F}\text{-adapted},\ \int_t^T ||h_s||^2 < \infty\}$$

- Wealth $dV_t^h = V_t^h\, h_t^\top (a(X_t)\, dt + \sigma dW_t), \qquad V_0^h = v_0$

- Utility function $U(x) = \frac{x^\theta}{\theta}$, (power utility), $\theta \in (-\infty, 1) \setminus \{0\}$; $U(x) = \log(x)$, $\theta = 0$.

- Value function. $J^h(t, v) = E_{t,v}[\, U(V_T^h)\,]$ for $h \in \mathcal{H}(t)$ and $J(t, v) = \sup\{J^h(t, v) \colon h \in \mathcal{H}(t)\}$.

- Investor problem. Find optimal strategy $h^* \in \mathcal{H}(0)$ such that $J(0, v_0) = J^{h^*}(0, v_0)$

3. Filtering and Reduction to Complete Information

HMM Filtering, only return observation

- Investor Filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ with $\mathcal{F}_t = \sigma(R_u \colon u \le t) \subset \mathcal{G}_t$
- Filter $p_t^k := P(X_t = x_k | \mathcal{F}_t)$, $1 \le k \le K$
- Innovation process. Put $\widehat{a(X_t)} := E[a(X_t)|\mathcal{F}_t] = \sum\limits_{j=1}^{K} p_t^j a_j$. Then $\widetilde{W}_t := \sigma^{-1}\big(\ R_t - \int_0^t \widehat{a(X_s)} ds\ \big)$ is an $\mathbb{F}$-BM
- HMM filter

$$
\begin{aligned}
p_0^k &= \pi^k \\
dp_t^k &= \sum_{j=1}^{K} Q^{jk} p_t^j dt + p_t^k \Big(a_k - \sum_{j=1}^{K} p_t^j a_j \Big)^{\top} (\sigma^{\top})^{-1} d\widetilde{W}_t
\end{aligned}
$$

Well-known result: Lipster-Shiriaev, Wonham, Elliott, . . .

b) HMM Filtering, including expert opinions

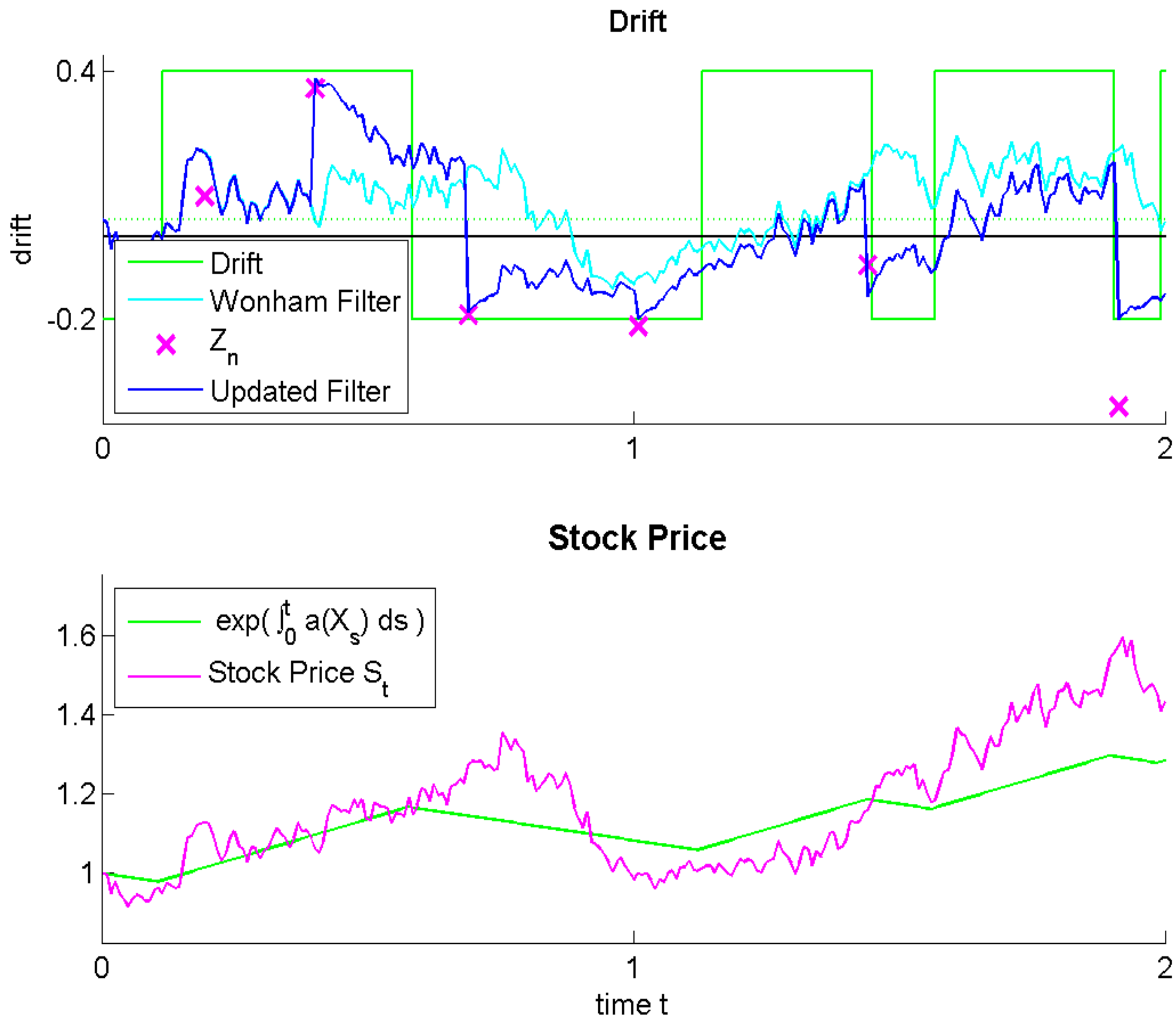
Extra information has no impact on filter p_t between 'information dates', but there is Bayesian updating at $t = T_n$. Recall that $f(X_{T_n}, z)$ is density of Z_n given X_{T_n}. Hence

$$p^k_{T_n} \propto p^k_{T_n-} f(x_k, Z_n) \;\text{ with normalizer }\; \sum_{j=1}^{K} p^j_{T_n-} f(x_j, Z_n).$$

Denote by $\mu(dt \times dz) := I(dt \times dz) - \lambda dt \sum_{k=1}^{K} p^k_{t-} f(x_k, z)\, dz$ the $\mathbb{F}$-compensated random measure of I. Then

$$\begin{aligned} dp^k_t &= \sum_{j=1}^{K} Q^{jk} p^j_t dt + p^k_t \Big(a(x_k) - \sum_{j=1}^{K} p^j_t a(x_j) \Big)^{\top} (\sigma^{\top})^{-1} d\widetilde{W}_t \\ &+ p^k_t \int_{\mathcal{Z}} \Big(\frac{f(x_k, z)}{\sum_{j=1}^{K} p^j_{t-} f(x_j, z)} - 1 \Big) \mu(dt \times dz) \end{aligned}$$

Filter: Example



Reduction to an OP Under Full Information

Consider $K+1$-dimensional state process (V_t, p_t)

Wealth $dV_t^h = V_t^h \; h_t^\top \sum_{k=1}^K p_t^k a(x_k)\, dt + \sigma d\widetilde{W}_t), \quad V_0^h = v_0$

Filter
$$dp_t^k = \sum_{j=1}^K Q^{jk} p_t^j dt + p_t^k \Big(a_k - \sum_{j=1}^K p_t^j a_j \Big)^\top (\sigma^\top)^{-1} d\widetilde{W}_t$$
$$+ p_{t-}^k \int_{\mathcal{Z}} \Big(\frac{f(x_k, z)}{\overline{f}(p_{t-}, z)} - 1 \Big) \mu(dt \times dz), \qquad p_0^k = \pi^k$$

Value $J^h(t,v,p) = E_{t,v,p}[\; U(V_T^h) \;]$ for $h \in \mathcal{H}(t)$

$$J(t,v,p) \;=\; \sup_{h \in \mathcal{H}(t)} \; J^h(t,v,p)$$

Problem Find $h^* \in \mathcal{H}(0)$ such that $J(0, v_0, \pi) = J^{h^*}(0, v_0, \pi)$

4. Logarithmic Utility

For $U(x) = \log x$ one has

$$U(V_T^h) = \log v_0 + \int_0^T \Big(h_s^\top \widehat{a(X_s)} - \frac{1}{2} h_s^\top \sigma\sigma^\top h_s \Big) ds + \int_0^T h_s^\top \sigma d\widetilde{W}_s$$

$$E[U(V_T^h)] = \log v_0 + E\Big[\int_0^T \Big(h_s^\top \widehat{a(X_s)} - \frac{1}{2} h_s^\top \sigma\sigma^\top h_s \Big) ds \Big] + 0$$

Optimal Strategy $h_t^* = (\sigma\sigma^\top)^{-1} \widehat{a(X_t)}$.

Certainty equivalence principle. h^* is obtained by replacing in the optimal strategy under full information $h_t^{\text{full}} = (\sigma\sigma^\top)^{-1} a(X_t)$ the unknown drift $a(X_t)$ by its filter $\widehat{a(X_t)}$

5. Solution for Power Utility

Risk-sensitive control problem (Nagai & Runggaldier (2008))

$$
\begin{aligned}
U(V_T^h) &= \frac{v_0^\theta}{\theta} Z_T^h \exp\Big\{ -\int_0^T b^{(\theta)}(p_s, h_s)ds \Big\} \\
\text{where} \quad Z_T^h &:= \exp\Big\{ \theta \int_0^T h_s^\top \sigma d\widetilde{W}_s - \frac{\theta^2}{2}\int_0^T h_s^\top \sigma\sigma^\top h_s ds \Big\} \\
\text{and} \quad b^{(\theta)}(p,h) &:= -\theta\big(h^\top Ap - \frac{1-\theta}{2} h^\top \sigma\sigma^\top h\big)
\end{aligned}
$$

Change of measure: assume $E[Z_T^h] = 1$ and define $P^h(A) = E[Z_T^h 1_A]$ for $A \in \mathcal{F}_T$. Then $W_t^h = \widetilde{W}_t - \theta \int_0^t \sigma^\top h_s ds$ is an $\mathbb{F}$-Brownian motion under P^h

Expected utility $E[U(V_T^h)] = \frac{v_0^\theta}{\theta} E^h\Big[\exp\Big\{ -\int_0^T b^{(\theta)}(p_s, h_s)ds \Big\}\Big]$

Expectation depends only on filter p and strategy h

Power Utility cont.

Filter dynamics under P^h.

$$dp_t^k = \sum_{j=1}^{K} Q^{jk} p_t^j dt + d_k^\top(p_t)\, dW_t^h + \theta d_k^\top(p_t)\sigma^\top h_t dt$$
$$+ p_{t-}^k \int_{\mathcal{Z}} \Big(\frac{f(x_k, z)}{\overline{f}(p_{t-}, z)} - 1 \Big) \mu(dt \times dz)$$

Admissible strategies. $\mathcal{A}(t) = \mathcal{H}(t) \cap \{h \colon E(Z_T^h) = 1\}$.

Value functions.

$$V(t,p) = E_{t,p}\Big(\exp\Big\{ - \int_t^T b^{(\theta)}(p_s^h, h_s) ds \Big\}\Big), \quad h \in \mathcal{A}(t)$$
$$J(t,p) = \sup\{J^h(t,p) \colon h \in \mathcal{A}(t)\}$$

New Problem. Find $h^* \in \mathcal{A}(0)$ such that $J(0,\pi) = J^{h^*}(0,\pi)$.

HJB-Equation and Optimal Strategy

Formal HJB equation

$$\begin{aligned} J_t(t,p) + \sup_{h \in \mathbb{R}^n} \left\{ \mathcal{L}^h J(t,p) - b^{(\theta)}(p,h) J(t,p) \right\} &= 0, \\ J(T,p) &= 1 \end{aligned}$$

where $\mathcal{L}^h$ generator of the Markov process p_t^h

Optimal Strategy

$$h^* = h^*(t,p) \;=\; \underbrace{\frac{1}{(1-\theta)} (\sigma\sigma^\top)^{-1} \Big\{ Ap}_{\text{myopic strategy}} + \underbrace{\frac{1}{J(t,p)} \sigma \sum_{k=1}^{K} d_k(p)\, J_{p^k}(t,p)}_{\text{correction}} \Big\}$$

Certainty equivalence principle does not hold !

HJB-Equation (cont.)

Plugging in h^* into the HJB equation and substituting $J = G^{1-\theta}$ we derive a transformed HJB-Equation for $G = G(t, p)$

$$G_t + \frac{1}{2} tr[\alpha^\top(p)\alpha(p)\, D^2 G] + \Phi^\top(p)\,\nabla G + \Psi(p)\, G$$
$$+ \frac{\lambda}{1-\theta} \int_{\mathcal{Z}} \frac{G^{1-\theta}(t, p + \Delta(p,z)) - G^{1-\theta}(t,p)}{G^{-\theta}(t,p)} \overline{f}(p,z) dz = 0,$$

The functions α, ϕ, ψ, Δ are defined in the paper. Note that the equation has a linear diffusion part but nonlinear integral term.

6. Approximative computation of optimal strategy

- Numerical solution of HJB equation via finite differences (only for K small)
- Policy improvement (possibly via Monte Carlo)
- Linearized HJB-equation (possibly via Monte Carlo)

Policy Improvement

Start from myopic strategy $h_t^{(0)} = \frac{1}{1-\theta}(\sigma\sigma^\top)^{-1}Ap_t$ with value function

$$J^{(0)}(t,p) := J^{h^{(0)}}(t,p) = E_{t,p}\Big[\exp\Big(-\int_t^T b^{(\theta)}(p_s^{h^{(0)}}, h_s^{(0)})ds\Big)\Big].$$

Motivated by fixed point interpretation of HJB, consider the problem $\max_h \left\{\mathcal{L}^h J^{(0)}(t,p) - b^{(\theta)}(p,h)J^{(0)}(t,p)\right\}$ with maximizer

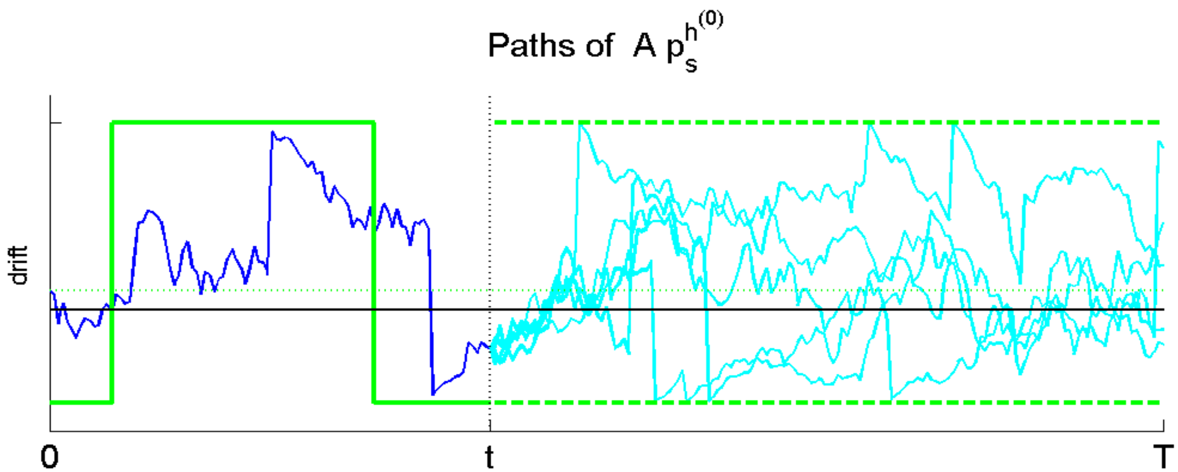
$$h^{(1)}(t,p) = h^{(0)}(t,p) + \frac{1}{(1-\theta)J^{(0)}(t,p)}(\sigma^\top)^{-1}\sum_{k=1}^{K} d_k(p)\, J^{(0)}_{p^k}(t,p).$$

For the corresponding value function $J^{(1)}(t,p) := J^{h^{(1)}}(t,p)$ one has

Lemma. $h^{(1)}$ is an improvement of $h^{(0)}$ i.e. $J^{(1)}(t,p) \geq J^{(0)}(t,p)$.

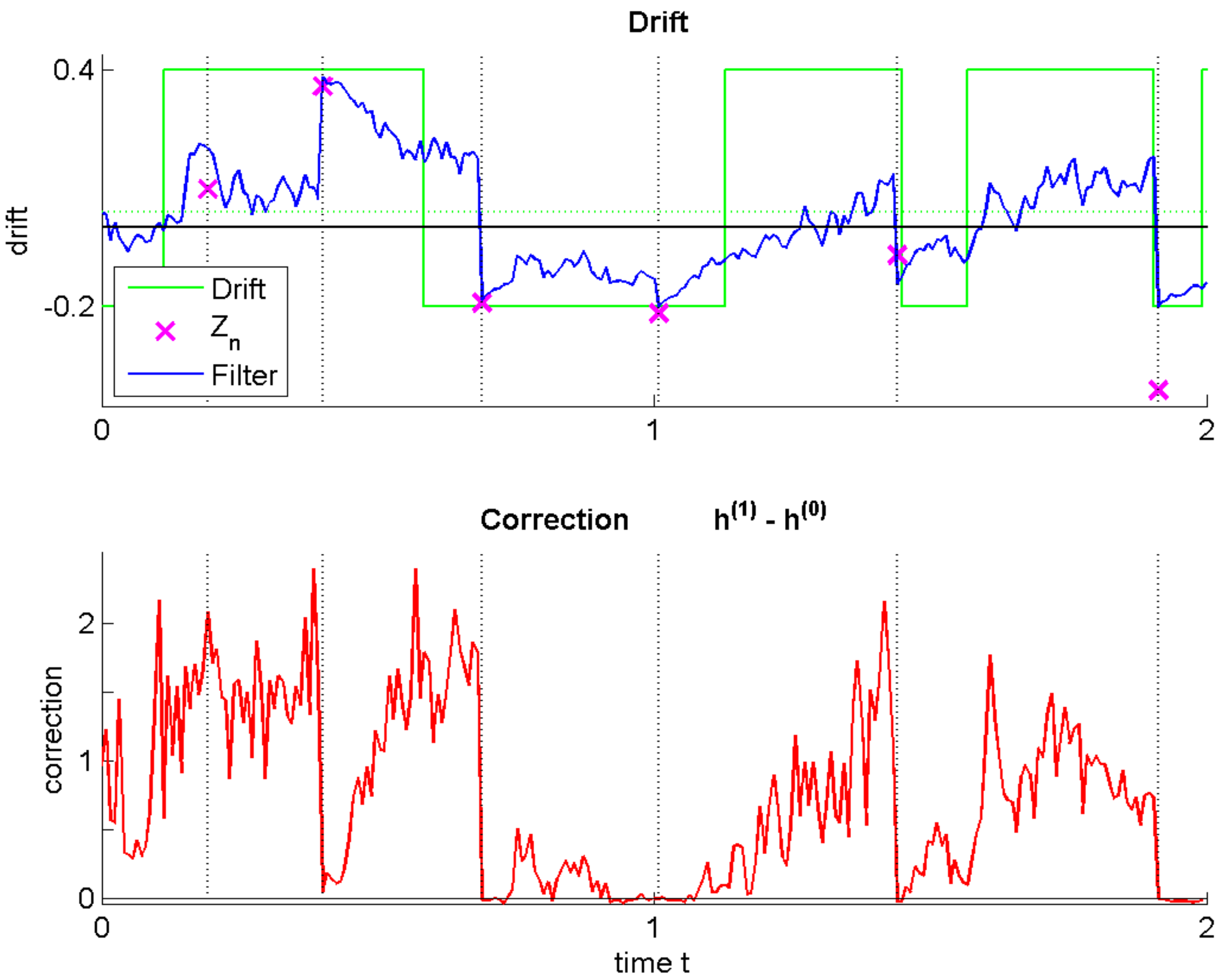
Policy Improvement (cont.)

Policy improvement requires Monte-Carlo approximation of value function $J^{h^{(0)}}(t,p)$.



- Generate N paths of $p_s^{h^{(0)}}$ starting at time t with $p = p_t$
- Estimate expectation $E_{t,p}(\cdot)$
- Approximate partial derivatives $J_{p^k}^{(0)}(t,p)$ by finite differences
- Compute first iterate $h^{(1)}$

Numerical Results



For $t = T_n$ nearly full information $\Longrightarrow$ correction ≈ 0

Outlook

- Framework that allows for a consistent integration of views/expert opinions into a dynamic portfolio optimization framework ('dynamic Black-Litterman')
- Feasible approximations to optimal strategy via dynamic programming
- Outlook/open points
  - ⋆ further testing of our approximations
  - ⋆ numerical experiments: value of information
  - ⋆ verification theorems or viscosity characterization
  - ⋆ Similar analysis for linear Gaussian case and Kalman filtering

References

[1] Lakner, P. (1998): Optimal trading strategy for an investor: the case of partial information. *Stochastic Processes and their Applications* 76, 77–97.

[2] 2003Litterman (2003)Litterman, R. (2003) *Modern Investment Management: An Equilibrium Approach*, Wiley, New Jersey.

[3] Nagai, H. and Runggaldier, W.J. (2008): PDE approach to utility maximization for market models with hidden Markov factors. In: *Seminar on Stochastic Analysis, Random Fields*

and Applications V (R.C.Dalang, M.Dozzi, F.Russo, eds.). Progress in Probability, Vol.59, Birkhäuser Verlag, 493–506.

[4] Rieder, U. and Bäuerle, N. (2005): Portfolio optimization with unobservable Markov-modulated drift process. *Journal of Applied Probability* 43, 362–378

[5] Sass, J. and Haussmann, U.G (2004): Optimizing the terminal wealth under partial information: The drift process as a continuous time Markov chain. *Finance and Stochastics* 8, 553–577.

[6] Sass, J. and Wunderlich, R. (2010): Optimal Portfolio Policies Under Bounded Expected Loss and Partial Information. *Mathematical Methods of Operations Research*, to appear.