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On SDEs with State-dependent Jump Measure

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Mark Davis
Department of Mathematics
Imperial College London
London SW7 2AZ
www.ma.ic.ac.uk/∼mdavis

Agenda

- Poisson random measures and point processes
- Jump-diffusion SDEs
- Reduction to SDE with autonomous random measure driving term (Gihman & Skorohod)
- Examples in which finite-activity processes are reduced to infinite-activity processes.
- Solution of SDEs with state-dependent jump compensator
- A quick introduction to Piecewise-Deterministic processes
- The Doléans exponential
- Application to change of measure for PDPs.
- Concluding remarks

Poisson random measures and point processes

$(Y, \mathcal{Y})$ is a measurable space. Let M be the set of $(Z^+ \cup \{+\infty\})$-valued integer-valued measures on $(Y, \mathcal{Y})$ and $\mathcal{M}$ be the smallest σ-field in M such that $\mu \mapsto \mu(B)$ is measurable $\forall B \in \mathcal{Y}$.

An $(M, \mathcal{M})$-valued random variable is a *Poisson random measure* with Lévy measure ν if

(i) For $B \in \mathcal{Y}$, $\mu(B)$ has Poisson distribution with parameter $\nu(B)$, i.e. if $\mu(B) < \infty$ then

$$\mathbb{P}[\mu(B) = n] = e^{-\nu(B)} \frac{(\nu(B))^n}{n!},$$

while $\mu(B) + \infty$ a.s. if $\mu(B) = \infty$.

(ii) $\mu(B_1), \mu(B_2), \ldots$ are independent for disjoint $B_1, B_2, \ldots$.

Construction is simple given any σ-finite measure ν. We take disjoint $B_1, B_2, \ldots$ such that $\nu(B_n) < \infty$ and $\bigcup B_n = Y$ and define for each n on some probability space independent r.v. as follows

(a) A random variable p_n with Poisson($\nu(B_n)$) distribution.

(b) a sequence $\xi_{n,j},\ j = 1, 2, \ldots$ of r.v. B_n-valued with $\mathbb{P}[\xi_{n,j} \in dx] = \nu(dx)/\nu(B_n)$.

We now define

$$\mu(B) = \sum_{n=1}^{\infty} \sum_{j=1}^{p_n} \mathbf{1}_{p_n \geq 1} \mathbf{1}_{B \cap B_n}(\xi_{n,j}).$$

A stationary (or, homogeneous) *Poisson point process* on a measurable space $(Z, \mathcal{Z})$ with Lévy measure ν is simply a Poisson random measure μ on $Y = \mathbb{R}^+ \times Z$ with Lévy measure $dt \times d\nu$). We define

$$N(t, A) = \mu([0, t] \times A), \quad \hat{N}(t, A) = t\nu(A).$$

The for $\nu(A) < \infty$, $t \mapsto N(t, A)$ is a Poisson process with rate $\nu(A)$ and $\tilde{N}(t, A) = N(t, A) - \hat{N}(t, A)$ is the Poisson martingale.

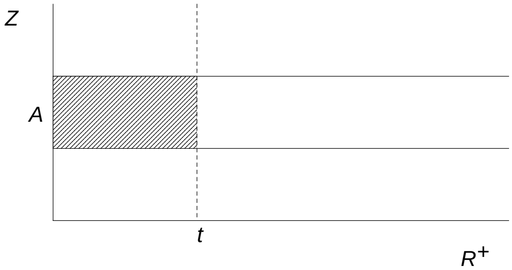

Figure 1: Poisson Random Measure

Jump-diffusion SDEs

Conventionally, the SDE takes the form

$$
\begin{aligned}
X(t) &= X(0) + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s) \qquad (1) \\
&\quad + \int_0^t \int_{Z \setminus Z_0} \gamma(X(s-), z) N(ds, dz) + \int_0^t \int_{Z_0} \gamma(X(s-), z) \tilde{N}(ds, dz)
\end{aligned}
$$

where $Z_0 \in \mathcal{Z}$ is such that $\nu(Z \setminus Z_0) < \infty$. We require

$$
\|\sigma(x)\|^2 + \|b(x)\|^2 + \int_{Z_0} \|\gamma(x,z)\|^2 \nu(dz) \leq K(1 + \|x\|)^2.
$$

To get a strong solution we need the Lipschitz condition

$$\|\sigma(x)-\sigma(y)\|^2+\|b(x)-b(y)\|^2+\int_{Z_0}\|\gamma(x,z)-\gamma(y,z)\|^2\nu(dz)\leq K\|x-y\|)^2.$$

The argument is as follows:

1. The random points (s,p) such that $p\in Z\backslash Z_0$ occur at isolated times $s=\tau_j$ with $0<\tau_1<\tau_2\cdots$. Call these points (τ_j,p_j).

2. Conventional Picard iteration shows (1) has a unique solution $X_1(t)$ on $[0,\tau_1[$.

3. We define $X(\tau_1)=X(\tau_1-)+\gamma(X(\tau_1-),p_1)$.

4. Now restart from $X(\tau_1)$.

The basic question that concerns us is

> *Is it sufficiently general to consider an SDE driven by an autonomous Poisson point process rather than a point process with solution-dependent compensator?*

Various people have answered "yes" to this question ...

The Gihman and Skorohod ("SDEs", 1972) argument

Take a small time interval $]t, t+h]$. Given $X_t = x$ an Euler step for the Brownian motion is $\sigma(x)(W(t+h) - W(t)) \sim N(0, \sigma^2(t,x)h)$. Correspondingly the Poisson martingale increment is

$$I \equiv \int_{Z_0} \gamma(t,x,z)\tilde{N}(h,dz).$$

Suppressing (t,x) dependence, suppose

$$\int_{Z_0} |\gamma(z) - \gamma^{(n)}(z)|^2 \nu(dz) \to 0$$

where $\gamma^{(n)}$ are piecewise constant, $\gamma^{(n)} = \gamma_k^n$ for $z \in A_k^n$. Then $I = \lim_{L_2} I_n$ where $I_n = \sum_k \gamma_k^n \tilde{N}(h, A_k^n)$. The ch. fn. of I_n is

$$\begin{aligned}
\phi_n(u) &= \mathbb{E}[\exp(iu\sum_k \gamma_k^n \tilde{N}(h, A_k^n))] \\
&= \prod \mathbb{E}[\exp(iu\gamma_k^n \tilde{N}(h, A_k^n))] \\
&= \prod \exp\left(e^{iu\gamma_k^n} - 1 - iu\gamma_k^n)h\nu(A_k^n)\right) \\
&= \exp\left(h\int (e^{iu\gamma^{(n)}} - 1 - iu\gamma^{(n)})\nu(dz)\right)
\end{aligned}$$

Hence the ch.fn. of I is

$$\log \phi_{t,x}(u) = h \int (e^{iu\gamma(t,x,z)} - 1 - iu\gamma(t,x,z))\nu(dz)$$

where γ satisfies

$$\int_{Z_0} |\gamma(t,x,z)|^2 \nu(dz) < \infty.$$

Changing variables to $\zeta = \gamma(t,x,z)$ gives

$$\log \phi_{t,x}(u) = h \int (e^{iu\zeta} - 1 - iu\zeta)\nu \circ \gamma_{t,x}^{-1}(d\zeta).$$

Claim: If $Z = \mathbb{R}^d$ and $\hat{\nu}(t,x,d\zeta)$ is an 'arbitrary' family of σ-finite measures then there exists a measurable function $\gamma : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ such that $\lambda \circ \gamma_{t,x}^{-1}(d\zeta) = \hat{\nu}(t,x,d\zeta)$, where λ is Lebesgue measure.

Proof (not given by G&S) – standard fact in measure theory + measurable selection.

Question: what about other measurable spaces?

Example: Non-homogeneous Poisson process

Consider a Poisson process $X(t)$ with arrival rate $\eta(t)$. This is a Poisson random measure with $Z = \mathbb{R}^+$, $\nu(dz) = \delta_{\{1\}}(dz)$ and compensator $\hat{N}(t, A) = \mathbf{1}_{1 \in A} \int_0^t \eta(s) ds$.

To realize it in terms of a homogeneous Poisson random measure, we take $Z = \mathbb{R}^+$ and $\nu =$ Lebesgue measure. Then

$$X(t) = \int_0^t \int_{\mathbb{R}^+} \gamma(t, z) N(ds, dz)$$

where $\gamma = \mathbf{1}_{[0,\eta(t)]}(z)$. (The Poisson rate is the shaded area in the figure.)

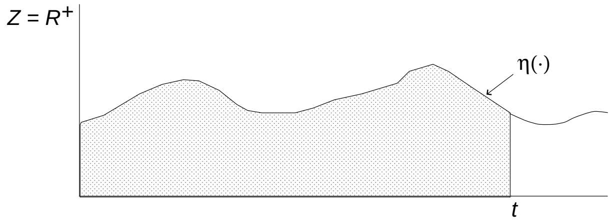

Figure 2: Realization of non-homogeneous Poisson process

Modelling prices of defaultable assets

Many people have studied reduced-form default models in which default times have hazard rate $h(X(t))$ depending on a factor process $X(t)$ which could be autonomous or could be part of the solution of an SDE (the 'self-exciting' case). This leads us to consider non-homogeneous point processes as above but with compensator $\eta(t) = h(X(t))$.

Suppose $X(t)$ has continuous paths and $h(\cdot)$ is not a bounded function (it could be, for example, affine). Then the resulting point process certainly has finite activity.

However, if we wish to realize it in terms of an autonomous driving Poisson point process, this will have to have infinite activity (since there is no uniform upper bound to $\eta(\cdot)$.)

SDEs with state-dependent jump measure

There are two approaches:

(i) *Strong solutions.* The Picard iteration approach seems well suited: at each stage we define $X^n(t)$ in terms of the already defined $X^{n-1}(\cdot)$ so the point process will simply have a 'random compensator'.

(ii) *Weak solutions* We have the freedom to change the compensator by absolutely continuous change of measure.

We examine the latter in the context of Piecewise-deterministic Markov processes (=jump-diffusions without the diffusion!)

Piecewise-deterministic Markov processes

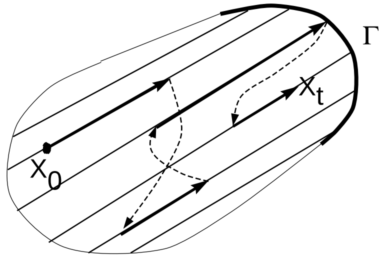

Figure 3: PDP sample function

A PDP (X_t) is a random motion in a state space $E \subset \mathbb{R}^d$ consisting of possibly disconnected components in $\mathbb{R}^d$. The process is specified by four 'local characteristics':

- Vector field $\mathcal{X}$ in E,
- jump rate $\lambda : E \to \mathbb{R}^+$,
- transition measures $Q : E \to \mathcal{P}(E)$, $R : \Gamma \to \mathcal{P}(E)$

$\mathcal{P}(E)$ is the set of probability measures on E and Γ is a subset of the boundary ∂E, defined below.

Start with an open set $E^0 \subset \mathbb{R}^d$, let $\mathcal{X}$ be a C^1 vector field on E^0 and let $\zeta(t,x)$ be the integral curve of $\mathcal{X}$, i.e. $\zeta(t,x)$ is the solution of the ordinary differential equation

$$\frac{d}{dt} f(\zeta(t,x)) = \mathcal{X} f(\zeta(t,x)), \quad \zeta(0,x) = x, \quad f \in C^1(E^0).$$

The 'active boundary' of E^0 is the set of points in ∂E^0 which are hit by some integral curve, i.e.,

$$\Gamma = \{z \in \partial E^0 : z = \lim \zeta(t_n, x) \text{ for some } x, t \in E^0 \times \mathbb{R}^+ \text{ and sequence } t_n \uparrow t\}.$$

We now define $E = \overline{E^0} \backslash \Gamma$, and $t^*(x) = \inf\{t : \zeta(t,x) \in \Gamma\}$ (with $\inf \emptyset = +\infty$).

Construction: Starting at $x \in E$, $X_t = \zeta(t,x)$ for $t \in [0, T_1)$ where the first jump time T_1 is a random variable whose distribution function is

$$F(t,x) = \mathbb{P}(T_1 \le t) = 1 - \mathbf{1}_{(t<t^*(x))} e^{-\int_0^t \lambda(\zeta(s,x))ds}.$$

Thus T_1 has hazard rate $\lambda(X_s)$ on $[0, t^*(x))$, with a mandatory jump at $t^*(x)$ if T_1 has not occurred by then.

The sample path is right-continuous, so $X_{T_1-} = \zeta(T_1, x)$.

If $X_{T_1-} \in E$ then $X_{T_1} \in E$ is a random variable whose conditional distribution given T_1 is given by the transition measure Q:

$$\mathbb{P}[X_{T_1} \in A | T_1] = Q(A, \zeta(T_1, x)), \quad A \in \mathcal{B}(E).$$

If $X_{T_1-} \in \Gamma$ then the recipe is the same, but using the transition measure R.

Having determined X_{T_1} we restart the process from $x' = X_{T_1}$, so that $X_t = \zeta(t - T_1, x')$ for $t \in [T_1, T_2)$, where the 'gap' $T_2 - T_1$ is determined by the same recipe as used above to determine T_1.

Continuing in this way we obtain an increasing sequence of random time T_n and, for any n, $X_t = \zeta(t - T_n, X_{T_n})$ for $t \in [T_n, T_{n+1})$. It is assumed that $\lim_n T_n = \infty$ a.s., a condition that is generally easily checked in applications.

The extended generator

The main general result of PDP theory is that the process just described is a homogeneous strong Markov process whose extended generator is the operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ given by

$$\mathcal{A}f(x) = \mathcal{X}f(x) + \lambda(x)\int_E (f(y) - f(x))Q(dy, x), \quad f \in \mathcal{D}(\mathcal{A}).$$

By definition, the extended generator has the property that for $f \in \mathcal{D}(\mathcal{A})$ the process

$$C_t^f = f(X_t) - f(x) - \int_0^t \mathcal{A}f(X_s)ds \tag{2}$$

is a local martingale. The domain $\mathcal{D}(\mathcal{A})$ can be precisely characterized. Sufficient conditions under which $f \in \mathcal{D}(\mathcal{A})$ are

$$\text{The function } t \mapsto f(\zeta(t, x)) \text{ is continuously differentiable} \tag{3}$$

$$\mathbb{E}_x \sum_i |f(X_{T_i \wedge t}) - f(X_{T_i \wedge t-})| < \infty \quad \text{for}(t, x) \in \mathbb{R}^+ \times E \tag{4}$$

$$f(x) = \int_E f(y)R(dy, x), \quad x \in \Gamma. \tag{5}$$

The key point here is that $f \in \mathcal{D}(\mathcal{A})$ only if the *boundary condition* (5) is satisfied for all $x \in \Gamma$.

Stochastic integrals Let $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ be the natural filtration of a PDP, complete with all null sets of $\mathcal{F}_\infty$. For $A \in \mathcal{B}(E)$, define counting processes p, p^* as follows

$$\begin{aligned} p(t, A) &= \sum_{j=1}^{\infty} \mathbf{1}_{(t \geq T_j)} \mathbf{1}_{(X_{T_j} \in A)}, \\ p^*(t) &= \sum_{j=1}^{\infty} \mathbf{1}_{(t \geq T_j)} \mathbf{1}_{(X_{T_j -} \in \Gamma)} \end{aligned}$$

These count the number of times the process jumps into a given set A, and the number of times the boundary is hit, respectively. p^* is a predictable process (i.e. measurable with respect to the predictable σ-field in $\Omega \times \mathbb{R}^+$). Hence the process $\tilde{p}$ defined as follows is also predictable.

$$\tilde{p}(t, A) = \int_{(0,t]} Q(A, X_s)\lambda(X_s)ds + \int_{(0,t]} R(A, X_{s-})dp^*(s).$$

For each fixed A, $\tilde{p}$ is the *predictable compensator* of p, i.e. the process

$$q(t, A) = p(t, A) - \tilde{p}(t, A)$$

is a local martingale. In fact, these local martingales span the filtration $\mathcal{F}_t$, in that there is a 1-1 correspondence between $\mathcal{F}_t$-local martingales and stochastic integrals with respect to this family of fundamental local martingales. The appropriate class of integrands is the set $L^1_{\mathrm{loc}}(p)$ of functions $g : E \times \mathbb{R}^+ \times \Omega \to \mathbb{R}$ such that

$$\begin{aligned}
&g \text{ is a measurable function on the product space;}\\
&\text{for each } y \in E, \text{ the function } (t, \omega) \mapsto g(y, t, \omega) \text{ is predictable}\\
&\text{for } k = 1, 2 \ldots \text{ we have } \mathbb{E}\left[\sum_{j=1}^{k} |g(X_{T_j}, T_j, \omega)|\right] < \infty.
\end{aligned}$$

For $g \in L^1_{\mathrm{loc}}(p)$ and $t > 0$ we define the stochastic integral M^g pathwise by

$$\begin{aligned} M^g &= \int_0^t g(y,t)q(dy,dt) \\ &= \int_0^t g(y,t)p(dy,dt) - \int_0^t g(y,t)\tilde{p}(dy,dt) \\ &= \sum_{T_i \le t} g(X_{T_i}, T_i) - \int_0^t Qg(X_{s-}, s)\lambda(X_{s-})ds - \int_0^t Rg(X_{s-}, s)dp^*(s). \quad (6) \end{aligned}$$

We use the notation $Qg(x,t,\omega) = \int_E g(y,t,\omega)Q(dy,x)$ and similarly for Rg. With this definition, M^g is a local martingale for each $g \in L^1_{\mathrm{loc}}(p)$.

Proposition 1 *For the PDP* (X_t) *with extended generator* $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ *we have*
(i) For $f \in \mathcal{D}(\mathcal{A})$ *the local martingale* C^f *of* (2) *is given by* $C^f = M^{\mathcal{B}f}$, *where*

$$\mathcal{B}f(y,t,\omega) = f(y) - f(X_{t-}(\omega)). \tag{7}$$

(ii) (PDP 'Ito formula') If a function $f : E \rightarrow \mathbb{R}$ *satisfies conditions* (3) *and* (4) *then*

$$f(X_t) - f(X_0) = \int_0^t \mathcal{A}f(X_s)ds + \int_0^t \int_E \mathcal{B}f(y, X_{s-})q(dy, ds) + \int_0^t \mathcal{C}f(X_{s-})dp_s^*, \tag{8}$$

where $\mathcal{C}f(x) = Rf(x) - f(x)$.
(iii) (Martingale representation) If M *is an* $\mathcal{F}_t$*-local martingale then* $M = M^g$ *for some* $g \in L^1_{\text{loc}}(p)$.

Note: The multiplicity of $(\mathcal{F}_t)$ is determined by the support of $Q(\cdot, x)$ and $R(\cdot, x)$.

The Doléans theorem for semimartingales (Section II.8 of Protter's book)

We are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

Theorem 1 *Let M be an $\mathcal{F}_t$-semimartingale with $M_0 = 0$. Then there exists a unique semimartingale Z, denoted $Z = \mathcal{E}(M)$, satisfying the equation*

$$Z_t = 1 + \int_0^t Z_{s-} dM_t. \tag{9}$$

Z is given explicitly by

$$Z_t = e^{M_t - \frac{1}{2}[M,M]_t^c} \prod_{0<s\leq t} (1 + \Delta M_s) e^{-\Delta M_s}, \tag{10}$$

where the infinite product converges.

In (10), $[M, M]_t^c$ denotes the quadratic variation of the continuous martingale part M^c of M. *In this talk, $M^c = 0$ and the product in* (10) *contains only a finite number of terms.*

When M is a local martingale, Z is a positive local martingale and hence a supermartingale, so that $\mathbb{E}[Z_T] \leq 1$. By standard arguments, it is a martingale on any finite time interval $[0,T]$ provided $\mathbb{E}[Z_T] = 1$. We may then define a measure $\mathbb{Q}$ on $\mathcal{F}_T$ by its Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(M)_T. \tag{11}$$

Theorem 2 *Let M, N be local martingales. Define $Z = \mathcal{E}(M)$, assume $\mathbb{E}[Z_T] = 1$ and define $\mathbb{Q}$ by (11). Let A be a predictable process and define $X_t = N_t - A_t$. Then X is a $\mathbb{Q}$-local martingale iff A is the predictable compensator of $[M,N]$. Here, $[M,N]$ is the cross-variation process defined by*

$$[M,N] = \frac{1}{4}([M+N, M+N] - [M-N, M-N]).$$

PROOF: It is standard that X is a $\mathbb{Q}$-local martingale iff XZ is a $\mathbb{P}$-local martingale. By the Ito product formula

$$d(XZ) = X_- dZ + Z_- dN - Z_- dA + d[Z,N],$$

and from (9)

$$[Z,N] = [Z \cdot M, N] = Z \cdot [M,N].$$

Thus

$$d(XZ) = X_- dZ + Z_- dN + Z_-(d[M,N] - dA),$$

and XZ is a local martingale iff $[M,N] - A$ is a local martingale. $\square$

When is the Doléans exponential a martingale?

The major work here is by Jean Mémin around 1980. The multiplicative decomposition of local martingales described below is covered in Protter's book.

Mémin's Additive Decomposition of Local Martingales

Let $M(t)$ be a local martingale. We define an additive decomposition $M(t) = M_1(t) + M_2(t)$. Indeed

$$M_1(t) = L(t) - \tilde{L}(t)$$

where

$$L(t) = \sum_{0<s\leq t} \Delta M_s \mathit{1}_{\{|\Delta M_s| \geq \frac{1}{2}\}}$$

and $\tilde{L}(t)$ is the compensator of $L(t)$. Then $M_2(t) = M(t) - M_1(t)$

Proposition Let $M(t)$ be a local martingale with the additive decomposition above and such that $M_0 = 0$. Then

1. $\mathcal{E}(M)$ has the decomposition

$$\mathcal{E}(M) = \mathcal{E}(M_2)\mathcal{E}(\tilde{M}_1)$$

   where

$$\tilde{M}_1(t) = M_1(t) - \sum_{0<s\leq t} \frac{\Delta M_1(s)\Delta M_2(s)}{1 + \Delta M_2(s)}, \qquad t < \infty$$

2. $\mathcal{E}(M_2)\tilde{M}_1$ is a local martingale.

3. If $\Delta M(s) > -1$ then $\Delta \tilde{M}_1(s) > -1$ for all finite s.

Corollary

Let N be a local martingale such that $\Delta N(s) > -1$ for all finite s, and such that $\mathcal{E}(N(\infty)$ is uniformly integrable. Let $\mathbb{P}'$ be the probability defined as

$$\frac{d\mathbb{P}'}{d\mathbb{P}} = \mathcal{E}(N)(\infty)$$

Let N_1 be a local martingale with locally integrable variations and denote by $\tilde{N}_1$ the $\mathbb{P}$-semimartingale defined as

$$\tilde{N}_1(t) = N_1(t) - \sum_{0<s\leq t} \frac{\Delta N_1(s)\Delta N(s)}{1+\Delta N(s)}, \qquad t < \infty$$

then $\tilde{N}_1$ is a $\mathbb{P}'$ local martingale, with locally integrable variations. Moreover, the $\mathbb{P}'$ predictable compensator of $\sum_{0<s\leq t} |\Delta \tilde{N}_1(s)|$ is equal to the $\mathbb{P}$ predictable compensator of $\sum_{0<s\leq t} |\Delta N_1(s)|$.

Theorem Let $M(t)$ be a local martingale with additive decomposition as above. If the predictable compensator of the process

$$Y(t) = [M^c, M^c]_t + \sum_{0<s\leq t} |\Delta M_1(s)| + \sum_{0<s\leq t} (\Delta M_2(s))^2 \tag{12}$$

is bounded, then $\mathcal{E}(M)(t)$ is uniformly integrable.

Change of measure for PDPs

We start by calculating the Doléans exponential when M is a local martingale in the natural filtration of a PDP.

Lemma 1 *For a PDP* (X_t)*, let* M^g *be the stochastic integral defined by* (6) *for some* $g \in L^1_{\mathrm{loc}}(p)$*. Then*

$$\begin{aligned} \mathcal{E}(M^g)_t \;&=\; \left(\prod_{\substack{T_i \le t \\ X_{T_i-} \notin \Gamma}} (1 + g(X_{T_i}, T_i)) \right) \\ &\quad \times \left(\prod_{\substack{T_i \le t \\ X_{T_i-} \in \Gamma}} (1 + g(X_{T_i}, T_i) - Rg(X_{T_i-}, T_i) \right) \\ &\quad \times \exp\left(- \int_0^t Qg(X_s, s)\lambda(X_s) ds \right). \end{aligned} \tag{13}$$

PROOF: Writing $M^g = M$, we have from (10)

$$\mathcal{E}(M)_t = \exp\left(M_t - \sum_{s\le t}\Delta M_s\right)\prod_{s\le t}(1+\Delta M_s).$$

Now

$$M_t = \sum_{\substack{T_i\le t\\ T_i\notin\Gamma}} g(X_{T_i},T_i) + \sum_{\substack{T_i\le t\\ T_i\in\Gamma}}(g(X_{T_i},T_i) - Rg(X_{T_i-},T_i)) - \int_0^t Qg(X_s,s)\lambda(X_s)ds,$$

so

$$M_t - \sum_{s\le t}\Delta M_s = -\int_0^t Qg(X_s,s)\lambda(X_s)ds,$$

and

$$\begin{aligned}(1+\Delta M_{T_i}) &= 1 + g(X_{T_i}) \quad \text{if } X_{T_i-}\notin\Gamma,\\ (1+\Delta M_{T_i}) &= 1 + g(X_{T_i},T_i) - Rg(X_{T_i-},T_i) \quad \text{if } X_{T_i-}\in\Gamma.\end{aligned}$$

The result follows. $\square$

We now investigate what happens to the PDP when we replace the original measure $\mathbb{P}$ by a new measure $d\mathbb{Q} = \mathcal{E}(M^g)d\mathbb{P}$.

- In general, X_t will no longer be a PDP.
- We identify the class of integrands g for which X_t is a $\mathbb{Q}$-PDP and identify the new local characteristics.
- Under any absolutely continuous change of measure the vector field $\mathcal{X}$ must remain the same.

In the notation of Theorem 2, take $M = M^g$ and $N = M^{\mathcal{B}f}$, where $\mathcal{B}f$ is defined by (7). From the 'Ito formula' (8) we have

$$\begin{aligned} N_t \;=\; & f(X_t) - f(X_0) - \int_0^t \mathcal{X}f(X_s)ds - \int_0^t (Qf(X_s) - f(X_s))\lambda(X_s)ds \qquad (14) \\ & - \int_0^t (f(X_{s-}) - Rf(X_{s-}))dp_s^*. \end{aligned}$$

From (6) and (14) we see that when $t = T_i$ and $X_{T_i-} \notin \Gamma$ then $\Delta M_t = g(X_t, t)$ and $\Delta N_t = \Delta f_t = f(X_t) - f(X_{t-})$, while if $X_{T_i-} \notin \Gamma$ then $\Delta M_t = g(X_t, t) - Rg(X_{t-}, t)$ and $\Delta N_t = \Delta f_t = f(X_t) - Rf(X_{t-})$. Hence the predictable compensator of $[M, N]$ is

$$\begin{aligned} A_t &= \int_0^t \int_E (f(y) - f(X_{s-})g(y,t))Q(dy, X_s)\lambda(X_s)ds \\ &\quad - \int_0^t \int_E (g(y) - Rg(X_{s-}))(f(y) - Rf(X_{s-}))R(dy, X_{s-})dp^*_s.) \end{aligned} \tag{15}$$

From Theorem 2, $N - A$ is $\mathbb{Q}$-local martingale. From (14) and (15), the integrand of dp^* is (in compressed notation)

$$-f + Rf + R((g - Rg)(f - Rf)) = -f + R((1 + g - Rg)f.$$

Thus

$$\begin{aligned} N_t - A_t &= f(X_t) - f(X_0) - \int_0^t \mathcal{X} f(X_s) ds \qquad (16) \\ &- \int_0^t \int_E (f(y) - f(X_{s-}))(1 + g(y,t)) Q(dy, X_s) \lambda(X_s) ds \\ &- \int_0^t (f - R(1 + g - Rg) f)(X_{s-}) dp_s^*. \end{aligned}$$

On the other hand X_t is a PDP under $\mathbb{Q}$ if and only if the last two terms above are given by

$$- \int_0^t (f(y) - f(X_{s-})) \tilde{Q}(dy, X_s) \tilde{\lambda}(X_s) ds - \int_0^t (f(X_{s-}) - \tilde{R} f(X_{s-})) dp_s^*. \qquad (17)$$

where $(\mathcal{X}, \tilde{\lambda}, \tilde{Q}, \tilde{R})$ are the $\mathbb{Q}$-local characteristics of the process. We therefore have the following result.

Theorem 3 *PDPs with local characteristics* $(\mathcal{X}, \lambda, Q)$ *and* $(\mathcal{X}, \tilde{\lambda}, \tilde{Q}, \tilde{R})$ *have mutually absolutely continuous probability laws if and only if there is a strictly positive function* $\beta : E \to \mathbb{R}^+$ *and a measurable function* $\gamma : E \times E \to \mathbb{R}^+$*, satisfying* $\int_E \gamma(y,x) Q(dy,x) = 1$ *for all* x*, such that*

$$\begin{aligned}
\tilde{\lambda}(x) &= \beta(x)\lambda(x), \\
\tilde{Q}(A,x) &= \int_A \gamma(y,x) Q(dy,x), \\
\tilde{R}(A,x) &= \int_A \eta(y,x) R(dy,x).
\end{aligned}$$

The Radon-Nikodym derivative is

$$\begin{aligned}
\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} &= \prod_{\substack{T_i \le T \\ X_{T_i -} \notin \Gamma}} \gamma(X_{T_i}, X_{T_i -}) \beta(X_{T_i -}) \prod_{\substack{T_i \le T \\ X_{T_i -} \in \Gamma}} \eta(X_{T_i}, X_{T_i -}) \qquad (18) \\
&\quad \times \exp\left(- \int_0^T \int_E \gamma(y, X_s) Q(dy, X_s) \beta(X_s) \lambda(X_s) ds \right).
\end{aligned}$$

PROOF: (Outline) Comparing the expressions in (16) and (17) we see that for X_t to be a PDP under $\mathbb{Q}$ with local characteristics $(\mathcal{X}, \tilde{\lambda}, \tilde{Q}, \tilde{R})$, the integrand g must be given in for $X_{t-} \in E$ by

$$g(y,t,\omega) = \gamma(y, X_{t-}(\omega))\beta(X_{t-}(\omega)) - 1$$

where γ, β have the stated properties.

Turning to the boundary term, the only way in which the dp^* integrand in (16) can be expressed in the form $f(x) - \tilde{R}f(x)$ is to suppose that

$$g(y,t,\omega) = \xi(y, X_{t-}(\omega)) \quad \text{for } X_{t-} \in \Gamma$$

for some function $\xi : E \times \Gamma \to \mathbb{R}$ such that

$$\int_E \eta(y,x)R(dy,x) = 0 \quad \forall x \in \Gamma.$$

Then taking $\eta(y,x) = 1+\xi(y,x)$ we have $\int_\Gamma \eta(y,x)R(dy,x) \equiv 1$ and $\eta = d\tilde{R}/dR$.

Concluding Remarks

It seems that there is a case for including state-dependent jump measures explicitly in SDEs intended for applications in, specifically, financial modelling.

In that case there's quite a lot to do.

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