Functional Ito calculus and hedging of path-dependent options

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Background

Practitioners often perform sensitivity analysis of derivatives by "bumping"/perturbating a variable, repricing the derivative and taking the difference.
When applied to the "delta" of a path-dependent option, this amounts adding a jump/shift of size ϵ today to the current path ω and recomputing the price F_t in the new path $\omega + \epsilon 1_{[t,T]}$

$$\frac{F_t(\omega + \epsilon 1_{[t,T]}) - F_t(\omega)}{\epsilon}$$

Dupire's functional calculus

Bruno Dupire (2009) formalized this notion and defines, for a functional $F : [0, T] \times D([0, T], \mathbb{R}) \mapsto \mathbb{R}$ defined on cadlag paths,

$$\nabla_x F_t(\omega) = \lim_{\epsilon \to 0} \frac{F_t(\omega + \epsilon 1_{[t,T]}) - F_t(\omega)}{\epsilon}$$

Dupire and argues that this is the correct hedge ratio for path-dependent options: if the option price F is twice differentiable in the functional sense and $F, \nabla F, \nabla^2 F$ are continuous in supremum norm, then

$$F_T = E[F_T] + \int_0^T \nabla_x F_t . dS_t$$

Summary

- Dupire's assumptions apply to integral functionals $F_t(\omega) = \int_0^t g(\omega(t))dt$ but not to stochastic integrals or functionals involving quadratic variation.
- We show that these ideas can be in fact extended, in a mathematically rigorous fashion, to a much larger class of functionals including stochastic integrals.
- We develop a **non-anticipative pathwise calculus** for functionals defined on cadlag paths.
- This leads to a non-anticipative calculus for path-dependent functionals of a semimartingale, which is (in a precise sense) a "non-anticipative" equivalent of the Malliavin calculus.
- In particular we extend Dupire's hedging/martingale representation formula to *all square-integrable martingales*.

References

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Framework

- Consider a $\mathbb{R}^d$-valued Ito process on $(\Omega, \mathcal{B}, \mathcal{B}_t, \mathbb{P})$:

$$X(t) = \int_0^t \mu(u)du + \int_0^t \sigma(u).dW_u$$

  μ integrable, σ square integrable $\mathcal{B}_t$-adapted processes.
- Quadratic variation process
  $[X](t) = \int_0^t {}^t\sigma.\sigma(u)du = \int_0^t A(u)du$
- $D([0,T], \mathbb{R}^d)$ space of cadlag functions.
- $\mathcal{F}_t = \mathcal{F}^X_{t+}$: natural filtration / history of X
- $C_0([0,T], \mathbb{R}^d)$ space of continuous paths.

Functional notation

For a path $x \in D([0,T],\mathbb{R}^d)$, denote by

- $x(t) \in \mathbb{R}^d$ the value of x at t
- $x_t = x_{|\,[0,t]} = (x(u), 0 \le u \le t) \in D([0,t],\mathbb{R}^d)$ the restriction of x to $[0,t]$.

We will also denote x_{t-} the function on $[0,t]$ given by

$$x_{t-}(u) = x(u) \quad u < t \qquad x_{t-}(t) = x(t-)$$

For a process X we shall similarly denote

- $X(t)$ its value and
- $X_t = (X(u), 0 \le u \le t)$ its path on $[0,t]$.

Path dependent functionals

In stochastic analysis, statistics of processes and mathematical finance, one is interested in path-dependent functionals such as

- (weighted) averages along a path $Y(t) = \int_0^t \quad f(X(t))\rho(t)dt$
- Quadratic variation and p-variation:

$$Y(t) = \lim_{n\to\infty} \sum_{k=1}^{t/n-1} \|X(\frac{k}{n}) - X(\frac{k-1}{n})\|^p$$

- Exponential functionals: $Y(t) = \exp(X(t) - [X](t)/2 \quad)$
- Functionals of quadratic variation: e.g. variance swaps and volatility derivatives

$$([X](t) - K)_+, \qquad \int_0^t f(X(t))d[X] \qquad f(t, X(t), [X]_t)$$

Outline

- We define pathwise derivatives for functionals of the type

$$Y(t) = F_t(\{X(u), 0 \leq u \leq t\}, \{A(u), 0 \leq u \leq t\}) = F_t(X_t, A_t)$$

  where $A = {}^t\sigma.\sigma$ and $F_t : D([0,t], \mathbb{R}^d) \times D([0,t], S_d^+) \to \mathbb{R}$ represents the dependence on the path of X and its quadratic variation process.
- Using this pathwise derivative, we derive a functional change of variable formula which extends the Ito formula in two ways: it allows for path-dependence and for dependence with respect to the quadratic variation of X.
- This pathwise derivative admits a closure ∇_X on the space of square integrable stochastic integrals w.r.t. X, which is shown to be a *stochastic derivative* i.e. an *inverse* of the Ito stochastic integral.
- We derive a (constructive) martingale representation formula and an integration by parts formula for stochastic integrals.

Outline

- I: Pathwise calculus for non-anticipative functionals.
- II: An Ito formula for functionals of semimartingales.
- III: Weak derivatives and relation with Malliavin calculus.
- IV: Numerical computation of functional derivatives
- V: Functional Kolmogorov equations. Pricing equations for path-dependent options.

Functional representation of non-anticipative processes

A process Y adapted to $\mathcal{F}_t$ may be represented as a family of functionals

$$Y(t,.) : \Omega = D([0,T],\mathbb{R}^d) \mapsto \mathbb{R}$$

with the property that $Y(t,.)$ only depends on the path stopped at t: $Y(t,\omega) = Y(t,\omega(.\wedge t)\ )$ so

$$\omega_{|[0,t]} = \omega'_{|[0,t]} \Rightarrow Y(t,\omega) = Y(t,\omega')$$

Denoting $\omega_t = \omega_{|[0,t]}$, we can thus represent Y as

$$Y(t,\omega) = F_t(\omega_t) \text{for some} \quad F_t : D([0,t],\mathbb{R}^d) \to \mathbb{R}$$

which is $\mathcal{F}_t$-measurable.

Non-anticipative functionals on the space of cadlag functions

This motivates the following definition:

Definition (Non-anticipative functional)

A *non-anticipative functional* on the (canonical) path space $\Omega = D([0,T],\mathbb{R}^d)$ is a family $F = (F_t)_{t\in[0,T]}$ where

$$F_t : D([0,t],\mathbb{R}^d) \to \mathbb{R}$$

is $\mathcal{F}_t$-measurable.

$F = (F_t)_{t\in[0,T]}$ naturally induces a functional on the vector bundle $\bigcup_{t\in[0,T]} D([0,t],\mathbb{R}^d)$.

Functional representation of predictable processes

An $\mathcal{F}_t$-*predictable* process Y may be represented as a family of functionals

$$Y(t,.) : \Omega = D([0,T],\mathbb{R}^d) \mapsto \mathbb{R}$$

with the property

$$\omega_{|[0,t[} = \omega'_{|[0,t[} \Rightarrow Y(t,\omega) = Y(t,\omega')$$

We can thus represent Y as

$$Y(t,\omega) = F_t(\omega_{t-}) \qquad \text{for some} \quad F_t : D([0,t],\mathbb{R}^d) \to \mathbb{R}$$

where $\omega_{t-}(u) = \omega(u), u < t$ and $\omega_{t-}(t) = \omega(t-)$.
So: an $\mathcal{F}_t$-*predictable* Y can be represented as $Y(t,\omega) = F_t(\omega_{t-})$ for some non-anticipative functional F.
Ex: integral functionals $Y(t,\omega) = \int_0^t g(\omega(u))\rho(u)du$

Functional representation of non-anticipative processes

The previous examples of processes have a non-anticipative dependence in X and a "predictable" dependence on A since they only depend on $[X] = \int_0^{\cdot} A(u)du$.
We will thus consider processes which may be represented as

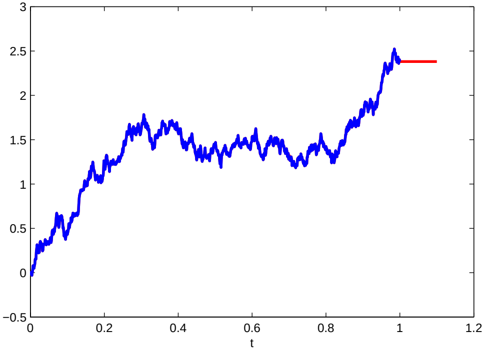
$$Y(t) = F_t(\{X(u), 0 \leq u \leq t\}, \{A(u), 0 \leq u \leq t\}) = F_t(X_t, A_t)$$

where the functional $F_t : D([0,t],\mathbb{R}^d) \times D([0,t], S_d^+) \to \mathbb{R}$ represents the dependence of $Y(t)$ on the path of X and $A = {}^t\sigma.\sigma$ and is "predictable with respect to the 2nd variable":

$$\forall t, \quad \forall (x,v) \in D([0,t],\mathbb{R}^d) \times D([0,t], S_d^+), \quad F_t(x_t, v_t) = F_t(x_t, v_{t-})$$

$F = (F_t)_{t\in[0,T]}$ may then be viewed as a functional on the vector bundle $\Upsilon = \bigcup_{t\in[0,T]} D([0,t],\mathbb{R}^d) \times D([0,t], S_d^+)$.

Horizontal extension of a path



Horizontal extension of a path

Let $x \in D([0,T] \times R^d)$, $x_t \in D([0,T] \times R^d)$ its restriction to $[0,t]$. For $h \geq 0$, the *horizontal* extension $x_{t,h} \in D([0,t+h],\mathbb{R}^d)$ of x_t to $[0,t+h]$ is defined as

$$x_{t,h}(u) = x(u) \qquad u \in [0,t] \ ; \qquad x_{t,h}(u) = x(t) \qquad u \in]t,t+h]$$

d_∞ metric on $\Upsilon = \bigcup_{t\in[0,T]} D([0,t],\mathbb{R}^d) \times D([0,t],S_d^+)$

Extends the supremum norm to paths of different length.
For $T \geq t' = t+h \geq t \geq 0$, $(x,v) \in D([0,t],\mathbb{R}^d) \times S_t^+$ and $(x',v') \in D([0,t+h],\mathbb{R}^d) \times \mathcal{S}_{t+h}^+$

$$d_\infty(\ (x,v),(x',v')\ ) = \sup_{u\in[0,t+h]} |x_{t,h}(u) - x'(u)| + \sup_{u\in[0,t+h]} |v_{t,h}(u) - v'(u)| + \ h$$

Continuity for non-anticipative functionals

A non-anticipative functional $F = (F_t)_{t \in [0,T]}$ is said to be **continuous at fixed times** if for all $t \in [0, T[$,

$$F_t : D([0,t], \mathbb{R}^d) \times \mathcal{S}_t \to \mathbb{R}$$

is continuous w.r.t. the supremum norm.

Definition (Left-continuous functionals)

Define $\mathbb{C}_l^{0,0}$ as the set of non-anticipative functionals $F = (F_t, t \in [0, T[)$ which are continuous at fixed times and

$$\begin{aligned} \forall t \in [0, T[, \quad & \forall \epsilon > 0, \forall (x, v) \in D([0,t], \mathbb{R}^d) \times \mathcal{S}_t, \\ \exists \eta > 0, \forall h \in [0, t], \quad & \forall (x', v') \in \forall (x, v) \in D([0, t-h], \mathbb{R}^d) \times \mathcal{S}_{t-h}, \\ & d_\infty((x, v), (x', v')) < \eta \Rightarrow |F_t(x, v) - F_{t-h}(x', v')| < \epsilon \end{aligned}$$

Boundedness-preserving functionals

We call a functional "boundedness preserving" if it is bounded on each bounded set of paths:

Definition (Boundedness-preserving functionals)

Define $\mathbb{B}([0,T))$ as the set of non-anticipative functionals F on $\Upsilon([0,T])$ such that for every compact subset K of $\mathbb{R}^d$, every $R>0$ and $t_0<T$

$$\exists C_{K,R,t_0}>0,\quad \forall t\le t_0,\quad \forall (x,v)\in D([0,t],K)\times\mathcal{S}_t,$$
$$\sup_{s\in[0,t]}|v(s)|\le R \Rightarrow |F_t(x,v)|\le C_{K,R,t_0}$$

Measurability and continuity

A non-anticipative functional $F = (F_t)$ applied to X generates an $\mathcal{F}_t-$adapted process

$$Y(t) = F_t(\{X(u), 0 \le u \le t\}, \{A(u), 0 \le u \le t\}) = F_t(X_t, A_t)$$

Theorem

Let $(x, v) \in D([0,T], \mathbb{R}^d) \times D([0,T], S_d^+)$. If $F \in \mathbb{C}_l^{0,0}$,

- *the path $t \mapsto F_t(x_{t-}, v_{t-})$ is left-continuous.*
- *$Y(t) = F_t(X_t, A_t)$ defines an optional process.*
- *If A is continuous, $Y(t) = F_t(X_t, A_t)$ is a predictable process.*

Horizontal derivative

Definition (Horizontal derivative)

We will say that the functional $F = (F_t)_{t\in[0,T]}$ on $\Upsilon([0,T])$ is horizontally differentiable at $(x,v) \in D([0,t],\mathbb{R}^d) \times \mathcal{S}_t$ if

$$\mathcal{D}_t F\ (x,v) = \lim_{h\to 0^+} \frac{F_{t+h}(x_{t,h}, v_{t,h}) - F_t(x_t, v_t)}{h} \qquad \text{exists}$$

We will call (1) the horizontal derivative $\mathcal{D}_t F$ of F at (x,v).

$\mathcal{D}F = (\mathcal{D}_t F)_{t\in[0,T]}$ defines a non-anticipative functional.
If $F_t(x,v) = f(t,x(t))$ with $f \in C^{1,1}([0,T]\times\mathbb{R}^d)$ then $\mathcal{D}_t F\ (x,v) = \partial_t f(t,x(t))$.

Vertical perturbation of a path

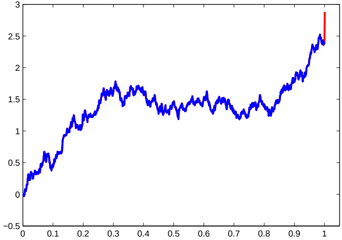


Figure: For $e \in \mathbb{R}^d$, the **vertical** perturbation x_t^e of x_t is the cadlag path obtained by shifting the endpoint:
$x_t^e(u) = x(u)$ for $u < t$ and $x_t^e(t) = x(t) + e$.

Definition (Dupire 2009)

A non-anticipative functional $F = (F_t)_{t\in[0,T[}$ is said to be *vertically differentiable* at $(x, v) \in D([0,t]),\mathbb{R}^d) \times D([0,t], S_d^+)$ if

$$\begin{array}{rcl} \mathbb{R}^d & \mapsto & \mathbb{R} \\ e & \to & F_t(x_t^e, v_t) \end{array}$$

is differentiable at 0. Its gradient at 0 is called the *vertical derivative* of F_t at (x, v)

$$\nabla_x F_t\ (x, v) = (\partial_i F_t(x, v),\ i = 1..d) \qquad \text{where}$$

$$\partial_i F_t(x, v) = \lim_{h\to 0} \frac{F_t(x_t^{he_i}, v) - F_t(x, v)}{h}$$

Vertical derivative of a non-anticipative functional

- $\nabla_x F_t\,(x, v).e$ is simply a directional (Gateaux) derivative in the direction of the indicator function $1_{\{t\}}e$.
- Note that to compute $\nabla_x F_t\,(x, v)$ we need to compute F outside C_0: even if $x \in C_0$, $x_t^h \notin C_0$.
- $\nabla_x F_t\,(x, v)$ is 'local' in the sense that it is computed for t fixed and involves perturbating the endpoint of paths ending at t.

Spaces of differentiable functionals

Definition (Spaces of differentiable functionals)

For $j, k \geq 1$ define $\mathbb{C}_b^{j,k}([0,T])$ as the set of functionals $F \in \mathbb{C}_r^{0,0}$ which are differentiable j times horizontally and k times vertically at all $(x, v) \in D([0,t], \mathbb{R}^d) \times \mathcal{S}_t^+$, $t < T$, with

- horizontal derivatives $\mathcal{D}_t^m F$, $m \leq j$ continuous on $D([0,T]) \times \mathcal{S}_t$ for each $t \in [0, T[$
- left-continuous vertical derivatives: $\forall n \leq k, \nabla_x^n F \in \mathbb{F}_l^\infty$.
- $\mathcal{D}_t^m F, \nabla_x^n F \in \mathbb{B}([0,T])$.

We can have $F \in \mathbb{C}_b^{1,1}([0,T])$ while F_t *not* Fréchet differentiable for any $t \in [0,T]$.

Examples of regular functionals

Example

$Y = \exp(X - [X]/2) = F(X, A)$ where

$$F_t(x_t, v_t) = e^{x(t) - \frac{1}{2}\int_0^t v(u)du}$$

$F \in \mathbb{C}_b^{1,\infty}$ with:

$$\mathcal{D}_t F(x, v) = -\frac{1}{2} v(t) F_t(x, v) \qquad \nabla_x^j F_t(x_t, v_t) = F_t(x_t, v_t)$$

Examples of regular functionals

Example (Cylindrical functionals)

For $g \in C_0(\mathbb{R}^d)$,

$$F_t(x_t, v_t) = [x(t) - x(t_n-)] \quad 1_{t \geq t_n} \qquad g(x(t_1-), x(t_2-) \ldots, x(t_n-))$$

is in $\mathbb{C}_b^{1,2}$

Examples of regular functionals

Example (Integrals w.r.t quadratic variation)

For $g \in C_0(\mathbb{R}^d)$, $Y(t) = \int_0^t g(X(u))d[X](u) = F_t(X_t, A_t)$ where

$$F_t(x_t, v_t) = \int_0^t g(x(u))v(u)du$$

$F \in \mathbb{C}_b^{1,\infty}$, with:

$$\mathcal{D}_t F(x_t, v_t) = g(x(t))v(t) \qquad \nabla_x^j F_t(x_t, v_t) = 0$$

Obstructions to regularity

Example (Jump of x at the current time)

$F_t(x_t, v_t) = x(t) - x(t-)$ has regular pathwise derivatives:

$$\mathcal{D}_t F(x_t, v_t) = 0 \qquad \nabla_x F_t(x_t, v_t) = 1$$

But $F \notin F_r^\infty \cup F_l^\infty$.

Example (Jump of x at a fixed time)

$F_t(x_t, v_t) = 1_{t \geq t_0}(x(t_0) - x(t_0-))$
$F \in \mathbb{F}^\infty$ has horizontal and vertical derivatives at any order, but $\nabla_x F_t(x_t, v_t) = 1_{t=t_0}$ fails to be left (or right) continuous.

Obstructions to regularity

Example (Maximum)

$F_t(x_t, v_t) = \sup_{s \le t} x(s)$
$F \in \mathbb{F}^\infty$ but is not vertically differentiable on

$$\{(x_t, v_t) \in D([0,t], \mathbb{R}^d) \times \mathcal{S}_t, \quad x(t) = \sup_{s \le t} x(s)\}.$$

Non-uniqueness of functional representation

Take $d = 1$.

$$F^0(x_t, v_t) = \int_0^t v(u)du$$

$$F^1(x_t, v_t) = \left(\lim_n \sum_{i=0}^{t2^n} |x(\tfrac{i+1}{2^n}) - x(\tfrac{i}{2^n})|^2\right) \quad \mathbf{1}_{\lim_n \sum_{i \le t2^n} (x(\frac{i+1}{2^n}) - x(\frac{i}{2^n}))^2 < \infty}$$

$$F^2(x_t, v_t) = \left(\underbrace{\lim_n \sum |x(\frac{i+1}{2^n}) - x(\frac{i}{2^n})|^2}_{V_2(x)} - \underbrace{\sum_{s<t} |\Delta x(s)|^2}_{J_2(x)} \right) \mathbf{1}_{V_2(x)<\infty} \mathbf{1}_{J_2(x)<\infty}$$

Then

$$F_t^0(X_t, A_t) = F_t^1(X_t, A_t) = F_t^2(X_t, A_t) = [X](t)$$

Yet $F^0 \in \mathbb{C}_b^{1,2}$ but $F^1, F^2 \notin \mathbb{F}_r^\infty$.

Non-uniqueness of functional representation

$F^1, F^2 \in \mathbb{C}^{1,1}$ coincide on continuous paths

$$\forall t < T, \quad \forall (x, v) \in C_0([0,t], \mathbb{R}^d) \times D([0,t], S_d^+), \qquad F_t^1(x, v) = F_t^2(x, v)$$

then

$$\mathbb{P}(\forall t \in [0,T], F^1(X_t, A_t) = F^2(X_t, A_t)\ ) = 1$$

Yet, $\nabla_x F$ depends on the values of F computed at *discontinuous* paths...

Derivatives of functionals defined on continuous paths

Theorem

If $F^1, F^2 \in \mathbb{C}^{1,1}$ coincide on continuous paths

$$\forall t < T, \quad \forall (x, v) \in C_0([0,t], \mathbb{R}^d) \times D([0,t], S_d^+), \\ F_t^1(x, v) = F_t^2(x, v)$$

then their pathwise derivatives also coincide:

$$\forall t < T, \quad \forall (x, v) \in C_0([0,t], \mathbb{R}^d) \times D([0,t], S_d^+), \\ \nabla_x F_t^1(x, v) = \nabla_x F_t^2(x, v), \qquad \mathcal{D}_t F_t^1(x, v) = \mathcal{D}_t F_t^2(x, v)$$

Quadratic variation for cadlag paths

Föllmer (1979): $f \in D([0,T],\mathbb{R})$ is said to have finite quadratic variation along a subdivision $\pi_n = (t_0^n < ..t_n^{k(n)} = T)$ if the measures:

$$\xi^n = \sum_{i=0}^{k(n)-1} (f(t_{i+1}^n) - f(t_i^n))^2 \delta_{t_i^n}$$

where δ_t is the Dirac measure at t, converge vaguely to a Radon measure ξ on $[0,T]$ such that

$$[f](t) = \xi([0,t]) = [f]^c(t) + \sum_{0<s\leq t} (\Delta f(s))^2$$

where $[f]^c$ is the continuous part of $[f]$. $[f]$ is called the quadratic variation of f along the sequence (π_n).

Change of variable formula for cadlag paths

Let $(x, v) \in D([0, T] \times \mathbb{R}^d) \times D([0, T] \times \mathbb{R}^n)$ where x has finite quadratic variation along (π_n) and

$$\sup_{t \in [0,T] - \pi_n} |x(t) - x(t-)| + |v(t) - v(t-)| \to 0$$

Denote

$$x^n(t) = \sum_{i=0}^{k(n)-1} x(t_{i+1}-) 1_{[t_i, t_{i+1})}(t) + x(T) 1_{\{T\}}(t)$$

$$v^n(t) = \sum_{i=0}^{k(n)-1} v(t_i) 1_{[t_i, t_{i+1})}(t) + v(T) 1_{\{T\}}(t), \qquad h_i^n = t_{i+1}^n - t_i^n$$

A pathwise change of variable formula for functionals

Theorem (C. & Fournié (2009))

For any $F \in \mathbb{C}_b^{1,2}([0,T[)$, *the Föllmer integral, defined as*

$$\int_0^T \nabla_x F_t(x_{t-}, v_{t-}) d^\pi x := \lim_{n\to\infty} \sum_{i=0}^{k(n)-1} \nabla_x F_{t_i^n}(x_{t_i^n-}^{n,\Delta x(t_i^n)}, v_{t_i^n-}^n)(x(t_{i+1}^n) - x(t_i^n))$$

exists and $$F_T(x_T, v_T) - F_0(x_0, v_0) = \int_0^T \mathcal{D}_t F_t(x_{u-}, v_{u-}) du$$

$$+ \int_0^T \frac{1}{2} \mathrm{tr}\left({}^t\nabla_x^2 F_t(x_{u-}, v_{u-}) d[x]^c(u)\right) + \int_0^T \nabla_x F_t(x_{t-}, v_{t-}) d^\pi x$$

$$+ \sum_{u\in]0,T]} [F_u(x_u, v_u) - F_u(x_{u-}, v_{u-}) - \nabla_x F_u(x_{u-}, v_{u-}).\Delta x(u)]$$

Functional Ito formula

This pathwise formula implies a functional change of variable formula for semimartingales:

Theorem (Functional Ito formula)

Let $F \in \mathbb{C}_b^{1,2}([0,T[)$. For any $t \in [0,T[$,

$$F_t(X_t, A_t) - F_0(X_0, A_0) = \int_0^t \mathcal{D}_u F(X_u, A_u)du + \int_0^t \nabla_x F_u(X_u, A_u).dX(u) + \int_0^t \frac{1}{2}\mathrm{tr}\left({}^t\nabla_x^2 F_u(X_u, A_u)\, d[X](u)\right) \quad a.s.$$

In particular, $Y(t) = F_t(X_t, A_t)$ is a semimartingale.

Functional Ito formula

- If $F_t(X_t, A_t) = f(t, X(t))$ where $f \in C^{1,2}([0,T] \times \mathbb{R}^d)$ this reduces to the standard Ito formula.
- $Y = F(X)$ depends on F and its derivatives only via their values on continuous paths: Y can be reconstructed from the second-order jet of F on $\Upsilon_c = \bigcup_{t \in [0,T]} C_0([0,t], \mathbb{R}^d) \times D([0,T], S_d^+) \subset \Upsilon$.

Sketch of proof

Consider first a cadlag piecewise constant process:

$$X(t) = \sum_{k=1}^{n} 1_{[t_k, t_{k+1}[}(t)\phi_k \qquad \phi_k \ \mathcal{F}_{t_k} - \text{measurable}$$

Each path of X is a sequence of horizontal and vertical moves:

$$X_{t_{k+1}} = (X_{t_k,h_k})^{\phi_{k+1}-\phi_k} \qquad h_k = t_{k+1} - t_k$$

$$\begin{aligned}
F_{t_{k+1}}(X_{t_{k+1}}, A_{t_{k+1}}) - F_{t_k}(X_{t_k}, A_{t_k}) &= \\
F_{t_{k+1}}(X_{t_{k+1}}, A_{t_{k+1}}) - F_{t_{k+1}}(X_{t_{k+1}}, A_{t_k,h_k}) &+ \\
F_{t_{k+1}}(X_{t_{k+1}}, A_{t_k,h_k}) - F_{t_{k+1}}(X_{t_k,h_k}, A_{t_k,h_k}) &+ \qquad \text{vertical move} \\
F_{t_{k+1}}(X_{t_k,h_k}, A_{t_k,h_k}) - F_{t_k}(X_{t_k}, A_{t_k}) & \qquad \text{horizontal move}
\end{aligned}$$

Sketch of proof

Horizontal step: fundamental theorem of calculus for $\phi(h) = F_{t_k+h}(X_{t_k,h}, A_{t_k,h})$

$$F_{t_{k+1}}(X_{t_k,h_k}, A_{t_k,h_k}) - F_{t_k}(X_{t_k}, A_{t_k})$$
$$= \phi(h_k) - \phi(0) = \int_{t_k}^{t_{k+1}} \mathcal{D}_t F(X_t, A_t)dt$$

Vertical step: apply Ito formula to $\psi(u) = F_{t_{k+1}}(X^u_{t_k,h_k}, \overbrace{A_{t_k,h_k}}^{\text{frozen}})$

$$F_{t_{k+1}}(X_{t_{k+1}}, A_{t_k,h_k}) - F_{t_{k+1}}(X_{t_k,h_k}, A_{t_k,h_k}) = \psi(X(t_{k+1}) - X(t_k)) - \psi(0)$$
$$= \int_{t_k}^{t_{k+1}} \nabla_x F_t(X_t, A_{t_k,h_k}).dX + \frac{1}{2}\mathrm{tr}(\nabla_x^2 F_t(X_t, A_{t_k,h_k})d[X])$$

Sketch of proof

General case: approximate X by a sequence of simple predictable processes ${}_nX$ with ${}_nX(0) = X(0)$:

$$F_T({}_nX_T) - F_0(X_0) = \int_0^T \mathcal{D}_t F({}_nX_t)dt + \int_0^T \nabla_X F({}_nX_t).dX + \frac{1}{2}\int_0^T \mathrm{tr}[{}^t\nabla_x^2 F({}_nX_t)\ A(t)]\ dt$$

The $\mathbb{C}_b^{1,2}$ assumption on F implies that all derivatives involved in the expression are left continuous in d_∞ metric, which allows to control their convergence as $n \to \infty$ using dominated convergence + the dominated convergence theorem for stochastic integrals.

Definition (Vertical derivative of a process)

Define $\mathcal{C}_b^{1,2}(X)$ the set of processes Y which admit a representation in $\mathbb{C}_b^{1,2}$:

$$\mathcal{C}_b^{1,2}(X) = \{Y, \exists F \in \mathbb{C}_b^{1,2}([0,T]), \quad Y(t) = F_t(X_t, A_t) \text{ a.s.}\}$$

If $\det(A) > 0$ a.s. then for $Y \in \mathcal{C}_b^{1,2}(X)$, the predictable process:

$$\nabla_X Y(t) = \nabla_x F_t(X_t, A_t)$$

is uniquely defined up to an evanescent set, independently of the choice of $F \in \mathbb{C}_b^{1,2}$. We call $\nabla_X Y$ the *vertical derivative* of Y with respect to X.

Vertical derivative for Brownian functionals

In particular when X is a standard Brownian motion, $A = I_d$:

Definition

Let W be a standard d-dimensional Brownian motion. For any $Y \in \mathcal{C}_b^{1,2}(W)$ with representation $Y(t) = F_t(W_t, t)$, the predictable process

$$\nabla_W Y(t) = \nabla_x F_t(W_t, t)$$

is uniquely defined up to an evanescent set, independently of the choice of the representation $F \in \mathbb{C}_b^{1,2}$.

Martingale representation formula

Consider now the case where $X(t) = \int_0^t \sigma(t).dW(t)$ is a Brownian martingale. Consider an $\mathcal{F}_T$-measurable functional $H = H(X(t), t \in [0, T]) = H(X_T)$ with $E[|H|^2] < \infty$ and define the martingale $Y(t) = E[H|\mathcal{F}_t]$.

Theorem

If $Y \in \mathcal{C}_b^{1,2}(X)$ then

$$\begin{aligned} Y(T) &= E[Y(T)] + \int_0^T \nabla_X Y(t) dX(t) \\ &= E[H] + \int_0^T \nabla_X Y(t) \sigma(t) dW(t) \end{aligned}$$

This is a non-anticipative version of Clark's formula (under weaker assumptions).

A hedging formula for path-dependent options

Consider now a (discounted) asset price process $S(t) = \int_0^t \sigma(t).dW(t)$ assumed to be a square-integrable martingale under a pricing measure $\mathbb{Q}$. Let $H = H(S(t), t \in [0,T])$ with $E[|H|^2] < \infty$ be a path-dependent payoff. The *price* at date t is then $Y(t) = E[H|\mathcal{F}_t]$.

Theorem (Hedging formula)

If $Y \in \mathcal{C}_b^{1,2}(S)$ *then*

$$H = E^{\mathbb{Q}}[H] + \int_0^T \nabla_S Y(t) dS(t) \quad \mathbb{Q} - a.s.$$

The hedging strategy for H is given by the vertical derivative of the option price with respect to S:

A hedging formula for path-dependent options

So the hedging strategy for H may be computed **pathwise** as

$$\phi(t) = \nabla_X Y(t, X_t(\omega)) = \lim_{h \to 0} \frac{Y(t, X_t^h(\omega)) - Y(t, X_t(\omega))}{h}$$

where

- $Y(t, X_t(\omega))$ is the option price at date t in the scenario ω.
- $Y(t, X_t^h(\omega))$ is the option price at date t in the scenario obtained from ω by moving up the current price ("bumping" the price) by h.

So, the usual "bump and recompute" sensitivity actually gives.. the hedge ratio!

Pathwise computation of hedge ratios

Consider for example the case where X is a (component of a) multivariate diffusion. Then we can use a numerical scheme (ex: Euler scheme) to simulate X.

Let ${}_nX$ be the solution of a n-step Euler scheme and $\hat{Y}_n$ a Monte Carlo estimator of Y obtained using ${}_nX$.

- Compute the Monte Carlo estimator $\widehat{Y}_n(t, {}_nX_t^h(\omega))$
- Bump the endpoint by h.
- Recompute the Monte Carlo estimator $\widehat{Y}_n(t, {}_nX_t^h(\omega))$ (with the same simulated paths)
- Approximate the hedging strategy by

$$\widehat{\phi_n}(t,\omega) := \frac{\widehat{Y}_n(t, {}_nX_t^h(\omega)) - \widehat{Y}_n(t, {}_nX_t^h(\omega))}{h}$$

Numerical simulation of hedge ratios

$$\widehat{\phi_n}(t,\omega) \simeq \frac{\widehat{Y}_n(t, {}_nX_t^h(\omega)) - \widehat{Y}_n(t, {}_nX_t^h(\omega))}{h}$$

For a general $\mathcal{C}_b^{1,2}(S)$ path-dependent claim, with a few regularity assumptions

$$\forall 1/2 > \epsilon > 0, n^{1/2-\epsilon}|\widehat{\phi}_n(t) - \phi(t)| \to 0 \qquad \mathbb{P} - a.s.$$

This rate is attained for $h = cn^{-1/4+\epsilon/2}$
By exploiting the structure further (Asian options, lookback options,...) one can greatly improve this rate.

A non-anticipative integration by parts formula

$$\mathcal{I}^2(X) = \{\int_0^{\cdot} \phi dX, \quad \phi\ \mathcal{F}_t-\text{adapted},\ E[\int_0^T \|\phi(t)\|^2 d[X](t)] < \infty\}$$

Theorem

Let $Y \in \mathcal{C}_b^{1,2}(X)$ be a $(\mathbb{P}, (\mathcal{F}_t))$-martingale with $Y(0) = 0$ and ϕ an $\mathcal{F}_t-$adapted process with $E[\int_0^T \|\phi(t)\|^2 d[X](t)] < \infty$. Then

$$E\left(Y(T)\int_0^T \phi dX\right) = E\left(\int_0^T \nabla_X Y.\phi d[X]\right)$$

This allows to extend the functional Ito formula to the closure of $\mathcal{C}_b^{1,2}(X) \cap \mathcal{I}^2(X)$ wrt to the norm

$$E\|Y(T)\|^2 = E[\int_0^T \|\nabla_X Y(t)\|^2 d[X](t)\ ]$$

Martingale Sobolev space

Definition (Martingale Sobolev space)

Define $\mathcal{W}^{1,2}(X)$ as the closure in $\mathcal{I}^2(X)$ of $D(X) = \mathcal{C}_b^{1,2}(X) \cap \mathcal{I}^2(X)$.

Lemma

$\{\nabla_X Y, Y \in D(X)\}$ *is dense in* $\mathcal{L}^2(X)$ *and*

$$\mathcal{W}^{1,2}(X) = \{\int_0^{\cdot} \phi dX, \quad E\int_0^T \|\phi\|^2 d[X] < \infty\}.$$

So $\mathcal{W}^{1,2}(X)$=all square-integrable integrals with respect to X.

Weak derivative

Theorem (Weak derivative on $\mathcal{W}^{1,2}(X)$)

The vertical derivative $\nabla_X : D(X) \mapsto \mathcal{L}^2(X)$ is closable on $\mathcal{W}^{1,2}(X)$. Its closure defines a bijective isometry

$$\begin{array}{rccc} \nabla_X : & \mathcal{W}^{1,2}(X) & \mapsto & \mathcal{L}^2(X) \\ & \int_0^T \phi . dX & \mapsto & \phi \end{array}$$

characterized by the following integration by parts formula*: for $Y \in \mathcal{W}^{1,2}(X)$, $\nabla_X Y$ is the unique element of $\mathcal{L}^2(X)$ such that*

$$\forall Z \in D(X), \qquad E[Y(T)Z(T)] = E\left[\int_0^T \nabla_X Y(t) \nabla_X Z(t) d[X](t)\right].$$

Computation of the weak derivative

For $D(X) = \mathcal{C}_b^{1,2}(X) \cap \mathcal{I}^2(X)$, the weak derivative may be computed **pathwise**: for $Y \in \mathcal{W}^{1,2}(X)$, $\nabla_X Y = \lim_n \nabla_X Y^n$ where $Y^n \in D(X)$ is an approximating sequence with

$$E\|Y^n(T) - Y(T)\|^2 \stackrel{n\to\infty}{\to} 0$$

An example of such an approximation is given by a Monte Carlo estimator $\widehat{Y}^n$ (computed for example from an Euler scheme for X).

$$\nabla_X Y(t, X_t(\omega)) = \lim_{n\to\infty} \lim_{h\to 0} \frac{\widehat{Y}^n(t, X_t^h(\omega)) - \widehat{Y}^n(t, X_t(\omega))}{h}$$

In practice one may compute instead

$$\widehat{\nabla_X Y}(t, X_t(\omega)) = \frac{\hat{Y}^n(t, X_t^{h(n)}(\omega)) - \hat{Y}^n(t, X_t(\omega))}{h(n)}$$

where $h(n) \sim cn^{-\alpha}$ is chosen according to the "smoothness" of

Relation with Malliavin derivative

Consider the case where $X = W$. Then for $Y \in \mathcal{W}^{1,2}(W)$

$$Y(T) = E[Y(T)] + \int_0^T \nabla_W Y(t) dW(t)$$

If $H = Y(T)$ is Malliavin-differentiable e.g. $H = Y(T) \in \mathbf{D}^{1,1}$ then the Clark-Haussmann-Ocone formula implies

$$Y(T) = E[Y(T)] + \int_0^T {}^p E[\mathbb{D}_t H | \mathcal{F}_t] dW(t)$$

where $\mathbb{D}$ is the Malliavin derivative.

Relation with Malliavin derivative

Theorem (Intertwining formula)

Let Y be a $(\mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ martingale. If $Y \in \mathcal{C}^{1,2}(W)$ and $Y(T) = H \in \mathbf{D}^{1,2}$ then

$$E[\mathbb{D}_t H | \mathcal{F}_t] = (\nabla_W Y)(t) \qquad dt \times d\mathbb{P} - a.e.$$

i.e. the conditional expectation operator intertwines ∇_W and $\mathbb{D}$:

$$E[\mathbb{D}_t H | \mathcal{F}_t] = \nabla_W \left(E[H | \mathcal{F}_t] \right) \qquad dt \times d\mathbb{P} - a.e.$$

Relation with Malliavin derivative

The following diagram is commutative, in the sense of $dt \times d\mathbb{P}$ almost everywhere equality:

$$\begin{array}{ccc} \mathcal{W}^{1,2}(W) & \stackrel{\nabla_W}{\to} & \mathcal{L}^2(W) \\ \uparrow (E[.|\mathcal{F}_t])_{t\in[0,T]} & & \uparrow (E[.|\mathcal{F}_t])_{t\in[0,T]} \\ \mathbf{D}^{1,2} & \stackrel{\mathbb{D}}{\to} & L^2([0,T]\times\Omega) \end{array}$$

Note however that ∇_X may be constructed for any Ito process X and its construction does not involve Gaussian properties of X.

Functional equation for martingales

Consider now a semimartingale X whose characteristics are left-continuous functionals:

$$dX(t) = b_t(X_t, A_t)dt + \sigma_t(X_t, A_t)dW(t)$$

where b, σ are non-anticipative functionals on Ω with values in $\mathbb{R}^d$-valued (resp. $\mathbb{R}^{d\times n}$) whose coordinates are in $\mathbb{F}_l^\infty$.
Consider the *topological support* of the law of (X, A) in $(C_0([0,T],\mathbb{R}^d) \times \mathcal{S}_T, \|.\|_\infty)$:

$$\text{supp}(X, A) = \{(x, v), \text{any neighborhood } V \text{ of } (x, v), \mathbb{P}((X, A) \in V) > 0$$

A functional Kolmogorov equation for martingales

Theorem

Let $F \in \mathbb{C}_b^{1,2}$. Then $Y(t) = F_t(X_t, A_t)$ is a local martingale if and only if F satisfies

$$\mathcal{D}_t F(x_t, v_t) + b_t(x_t, v_t)\nabla_x F_t(x_t, v_t)$$
$$+\frac{1}{2}\mathrm{tr}[\nabla_x^2 F(x_t, v_t)\sigma_t{}^t\sigma_t(x_t, v_t)] = 0,$$

for $(x, v) \in$supp(X, A).

We call such functionals $X-$*harmonic functionals*.

Structure equation for Brownian martingales

In particular when $X = W$ is a d-dimensional Wiener process, we obtain a characterization of 'regular' Brownian local martingales:

Theorem

Let $F \in \mathbb{C}_b^{1,2}$. Then $Y(t) = F_t(W_t)$ is a local martingale on $[0, T]$ if and only if

$$\forall t \in [0, T], \qquad (x, v) \in C_0([0, T], \mathbb{R}^d),$$
$$\mathcal{D}_t F(x_t) + \frac{1}{2} \operatorname{tr}\left(\nabla_x^2 F(x_t) \right) = 0.$$

Theorem (Uniqueness of solutions)

Let h be a continuous functional on $(C_0([0,T]) \times \mathcal{S}_T, \|.\|_\infty)$. Any solution $F \in \mathbb{C}_b^{1,2}$ of the functional equation (1), verifying $\forall (x,v) \in C_0([0,T]) \times \mathcal{S}_T$,

$$\begin{aligned} D_t F(x_t, v_t) + & \quad b_t(x_t, v_t) \nabla_x F_t(x_t, v_t) \\ + & \quad \frac{1}{2} \mathrm{tr}[\nabla_x^2 F(x_t, v_t) \sigma_t {}^t\sigma_t(x_t, v_t)] = 0 \\ F_T(x,v) = h(x,v), & \quad E[\sup_{t\in[0,T]} |F_t(X_t, A_t)|] < \infty \end{aligned}$$

is uniquely defined on the topological support $supp(X,A)$ of (X,A): if $F^1, F^2 \in \mathbb{C}_b^{1,2}([0,T])$ are two solutions then

$$\forall (x,v) \in \mathrm{supp}(X,A), \quad \forall t \in [0,T], \qquad F_t^1(x_t, v_t) = F_t^2(x_t, v_t).$$

A universal pricing equation

Theorem (Pricing equation for path-dependent options)

Let $\exists F \in \mathbb{C}_b^{1,2}$, $F_t(X_t, A_t) = E[H|\mathcal{F}_t]$ *then* F *is the unique solution of the pricing equation*

$$\mathcal{D}_t F(x_t, v_t) + b_t(x_t, v_t)\nabla_x F_t(x_t, v_t)$$
$$+\frac{1}{2}\mathrm{tr}[\nabla_x^2 F(x_t, v_t)\sigma_t{}^t\sigma_t(x_t, v_t)] = 0,$$

for $(x, v) \in$*supp*(X, A).

This equations implies all known PDEs for path-dependent options: barrier, Asian, lookback,...but also leads to new pricing equations for other examples.

A diffusion example

Consider a scalar diffusion

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t) \qquad X(0) = x_0$$

defined as the solution $\mathbb{P}^{x_0}$ of the martingale problem on $D([0,T], \mathbb{R}^d)$ for

$$L_t f = \frac{1}{2}\sigma^2(t,x)\partial_x^2 f(t,x) + b(t,x)\partial_x f(t,x)$$

where b and $\sigma \geq a > 0$ are continuous and bounded functions.
By the Stroock-Varadhan support theorem, the topological support of (X, A) under $\mathbb{P}^{x_0}$ is

$$\{(x, (\sigma^2(t, x(t)))_{t\in[0,T]}) \quad | x \in C_0(\mathbb{R}^d, [0,T]), x(0) = x_0\}.$$

Weighted variance swap

A weighted variance swap with weight function $g \in C_b([0,T] \times \mathbb{R}^d)$, we are interested in computing

$$Y(t) = E[\int_0^T g(t, X(t)) d[X](t) | \mathcal{F}_t]$$

If $Y = F(X, A)$ with $F \in \mathbb{C}_b^{1,2}([0,T])$ then F is X-harmonic and solves the functional Kolmogorov equation.
Taking conditional expectations and using the Markov property of X:

$$F_t(x_t, v_t) = \int_0^t g(u, x(u)) v(u) du + f(t, x(t))$$

An example

$$F_t(x_t, v_t) = \int_0^t g(u, x(u))v(u)du + f(t, x(t))$$

solves the functional Kolmogorov eq. iff f solves

$$\frac{1}{2}\sigma^2(t,x)\partial_x^2 f(t,x) + b(t,x)\partial_x f(t,x) + \partial_t f(t,x) = -g(t,x)\sigma^2(t,x)$$

$$\text{with terminal condition} \qquad f(T,x) = 0$$

so that $Y(T) = F_T(X_T, A_T)$.
On the other hand, the (unique) $C^{1,2}$ solution of this PDE defines a unique $\mathbb{C}_b^{1,2}([0,T])$ functional on supp(X,A).

Weighted variance swap

Applying now the Ito formula to $f(t, X(t))$ we obtain that the hedging strategy is given by

$$\phi(t) = \frac{\partial f}{\partial x}(t, x)$$

where f solves the PDE with source term:

$$\frac{1}{2}\sigma^2(t,x)\partial_x^2 f(t,x) + b(t,x)\partial_x f(t,x) + \partial_t f(t,x) = -g(t,x)\sigma^2(t,x)$$

$$\text{with terminal condition} \qquad f(T,x) = 0$$

Extensions and applications

- The result can be extended to discontinuous functionals of cadlag processes i.e. Y and X can both have jumps.
- The result can be *localized* using stopping times: important for applying to functionals involving stopped processes/ exit times.
- Pathwise maximum principle for non-Markovian control problems.
- Infinite-dimensional extensions.
- $\theta - \Gamma$ tradeoff for path-dependent derivatives.

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