Rational Term Structure Models with Geometric Lévy Martingales

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Talk based on:

- Brody, D. C., Hughston, L. P. & Mackie, E. (2010) "Rational term structure models with geometric Lévy martingales" Imperial College Working Paper.

Related work:

- Flesaker, B. & Hughston, L. P. 1996 Positive interest. *Risk Magazine* **9**, 46–49; reprinted in *Vasicek and Beyond*, L.P. Hughston (ed), London: Risk Publications (1996).
- Brody, D. C. & Hughston, L. P. 2001 Interest rates and information geometry. *Proc. Roy. Soc. London* A**457**, 1343–1364.
- Brody, D. C. & Hughston, L. P. 2002 Entropy and information in the interest rate term structure. *Quantitative Finance* **2**, 70-80.

Term structure density approach

In 1996 Flesaker and Hughston made the observation that for a positive nominal interest-rate system, the price $\{P_{tT}\}_{0\leq t\leq T}$ of a T-maturity discount bond admits the rational representation

$$P_{tT} = \frac{\int_T^\infty (-\partial_u P_{0u}) M_{tu}\, \mathrm{d}u}{\int_t^\infty (-\partial_u P_{0u}) M_{tu}\, \mathrm{d}u}, \tag{1}$$

where $\{M_{tu}\}_{0\leq t\leq u}$ is a family of positive unit-initialised martingales.

To model the interest rate system we thus need to specify the initial term structure together with a family of positive martingales.

The expression appearing in the integrand of (1), namely,

$$\rho_0(T) \equiv -\partial_T P_{0T}, \tag{2}$$

defines a probability density function over $\mathbb{R}_+$ associated with an abstract random variable Z associated with the bond maturity.

More generally, let us switch to the Musiela parameterisation and introduce the tenor variable $z = T - t$.

Then the term-structure density process is defined according to the prescription

$$\rho_t(z) = -\partial_z P_{t,t+z}. \tag{3}$$

Put the matter differently, the term-structure density approach is based on the observation that there exists an abstract positive random variable Z whose conditional density process is given by (3).

The bond price can then be expressed in the form

$$P_{tT} = \frac{\mathbb{E}_t^{\rho}[\mathbb{1}\{Z > T\}]}{\mathbb{E}_t^{\rho}[\mathbb{1}\{Z > t\}]}, \tag{4}$$

where $\mathbb{E}_t^{\rho}[-]$ denotes expectation with respect to the conditional density (3).

Suppose that we assume that the market filtration $\{\mathcal{F}_t\}$ is generated by a family of $\mathbb{P}$-Brownian motions $\{W_t\}$.

Then the arbitrage-free dynamical equation satisfied by the term-structure density process is given by the so-called Brody-Hughston stochastic partial differential equation:

$$\mathrm{d}\rho_t(z) = \big(r_t\rho_t(z) + \partial_x\rho_t(z)\big)\mathrm{d}t + \rho_t(z)(\nu_t(z) - \bar{\nu}_t)(\mathrm{d}W_t - \bar{\nu}_t\mathrm{d}t), \tag{5}$$

where $r_t = \rho_t(0)$ is the short rate, and $\bar{\nu}_t = \mathbb{E}_t^{\rho}[\nu_t(z)]$, or, equivalently,

$$\bar{\nu}_t = \int_0^{\infty} \rho_t(z)\nu_t(z)\mathrm{d}z. \tag{6}$$

The volatility structure is thus specified exogenously via $\{\nu_t(z)\}$, whereas the initial yield curve can be calibrated by the specification of $\rho_0(z) = -\partial_z P_{0z}$.

The market risk premium is given by $\lambda_t = -\bar{\nu}_t$, thus making

$$W_t^* = W_t - \int_0^t \bar{\nu}_s \mathrm{d}s \tag{7}$$

a $\mathbb{Q}$-Brownian motion.

Writing $V_{tT} = \nu_t(T-t)$ to convert back to the maturity variable, we find that the bond price admits the representation

$$P_{tT} = \frac{\int_T^{\infty} \rho_0(u) \exp\left(\int_{s=0}^t V_{su}\mathrm{d}W_s - \frac{1}{2}\int_{s=0}^t V_{su}^2\mathrm{d}s\right)\mathrm{d}u}{\int_t^{\infty} \rho_0(u) \exp\left(\int_{s=0}^t V_{su}\mathrm{d}W_s - \frac{1}{2}\int_{s=0}^t V_{su}^2\mathrm{d}s\right)\mathrm{d}u}. \tag{8}$$

One of the advantages of the term-structure density approach over the more traditional HJM or market approaches is that the positivity of nominal interest rate, or equivalently the arbitrage freeness, is automatically ensured.

On the other hand, for interest rate positivity the HJM forward rate volatility process $\{\sigma_{tT}\}$ has to be of the following form:

$$\sigma_{tT} = f_{tT}\left(V_{tT} - \frac{\mathbb{E}^{\rho}_{t}[V_{tZ}\mathbb{1}\{Z>T\}]}{\mathbb{E}^{\rho}_{t}[\mathbb{1}\{Z>T\}]}\right). \quad (9)$$

In other words, in the HJM or market models, after having chosen $\{V_{tT}\}$ freely, one has to work out the term structure density process first in order to deduce the arbitrage-free form of the forward-rate volatility via (9).

Another advantage of the term-structure density approach has been pointed out more recently by Filipović *et al.* (2009).

Filipović *et al.* showed that it is "less delicate" to add jumps to the Brody-Hughston equation (5) than to the familiar HJM framework in the Musiela representation.

In this spirit we shall consider a range of geometric Lévy martingales $\{M_{tu}\}$ in the rational representation (1).

Geometric Lévy martingales

Our goal now is to construct a class of interest rate models based on various Lévy processes.

Let $\{L_t\}_{t\geq 0}$ be a Lévy process with $L_0 = 0$.

For a suitable function suitable $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ we define a martingale family $\{M_{tx}\}_{0\leq t\leq x}$ by setting

$$M_{tx} = \frac{\mathrm{e}^{\phi(x)L_t}}{\mathbb{E}\left[\mathrm{e}^{\phi(x)L_t}\right]}. \tag{10}$$

Note that $\{M_{tx}\}$ satisfies $M_{tx} > 0$ and $M_{0x} = 1$.

Then by taking various choices for the underlying Lévy process we are able to generate a variety of interest rate models, each with some functional freedom.

Geometric Brownian motion family

For a standard Brownian motion $\{B_t\}_{t\geq 0}$, we obtain a bond price of the form

$$P_{tT} = \frac{\int_T^\infty \rho(x)\mathrm{e}^{\phi(x)B_t - \frac{1}{2}\phi(x)^2 t}\mathrm{d}x}{\int_t^\infty \rho(x)\mathrm{e}^{\phi(x)B_t - \frac{1}{2}\phi(x)^2 t}\mathrm{d}x}, \tag{11}$$

and a corresponding short rate of the form

$$r_t = \frac{\rho(t)\mathrm{e}^{\phi(t)B_t - \frac{1}{2}\phi(t)^2 t}\mathrm{d}x}{\int_t^\infty \rho(x)\mathrm{e}^{\phi(x)B_t - \frac{1}{2}\phi(x)^2 t}\mathrm{d}x}. \tag{12}$$

Here $\rho(t)$ denotes the initial term structure density

$$\rho(t) = -\partial_t P_{0t}. \tag{13}$$

Using Ito's lemma we deduce the dynamics of the bond price system is given by

$$\mathrm{d}P_{tT} = (r_t P_{tT} + \Phi_{tt}(\Phi_{tt} - \Phi_{tT}))\mathrm{d}t + (\Phi_{tT} - \Phi_{tt})\mathrm{d}B_t, \tag{14}$$

where

$$\Phi_{tT} = \frac{\int_T^\infty \phi(x)\rho(x)\mathrm{e}^{\phi(x)B_t - \frac{1}{2}\phi(x)^2 t}\mathrm{d}x}{\int_T^\infty \rho(x)\mathrm{e}^{\phi(x)B_t - \frac{1}{2}\phi(x)^2 t}\mathrm{d}x}, \tag{15}$$

and

$$\Phi_{tt} = \frac{\int_t^\infty \phi(x)\rho(x)\mathrm{e}^{\phi(x)B_t - \frac{1}{2}\phi(x)^2 t}\mathrm{d}x}{\int_t^\infty \rho(x)\mathrm{e}^{\phi(x)B_t - \frac{1}{2}\phi(x)^2 t}\mathrm{d}x}. \tag{16}$$

Positive risk premium implies that $|\phi(x)|$ is decreasing in x.

The price today of a call option expiring at time t with strike price K, on a discount bond maturing at time T is given by

$$C_{0t} = \mathbb{E}^{\mathbb{Q}}[P_{0t}(P_{tT} - K)^+]. \tag{17}$$

The option price in the geometric Brownian motion example turns out to be

$$\begin{aligned} C_{0t} \;=\; & \int_T^\infty \rho(x)\, N\left(\pm\frac{\xi^*}{\sqrt{t}} \mp \phi(x)\sqrt{t}\right)\mathrm{d}x \\ & -K\int_t^\infty \rho(x)\, N\left(\pm\frac{\xi^*}{\sqrt{t}} \mp \phi(x)\sqrt{t}\right)\mathrm{d}x, \end{aligned} \tag{18}$$

where ξ^* is a critical value on the boundary of positive payoffs.

Here the $(\pm, \mp)$ signs corresponds to the combination $(+, -)$ if $\phi(x)$ is increasing in x, and $(-, +)$ if $\phi(x)$ is decreasing in x.

Geometric Brownian motion family: bond price

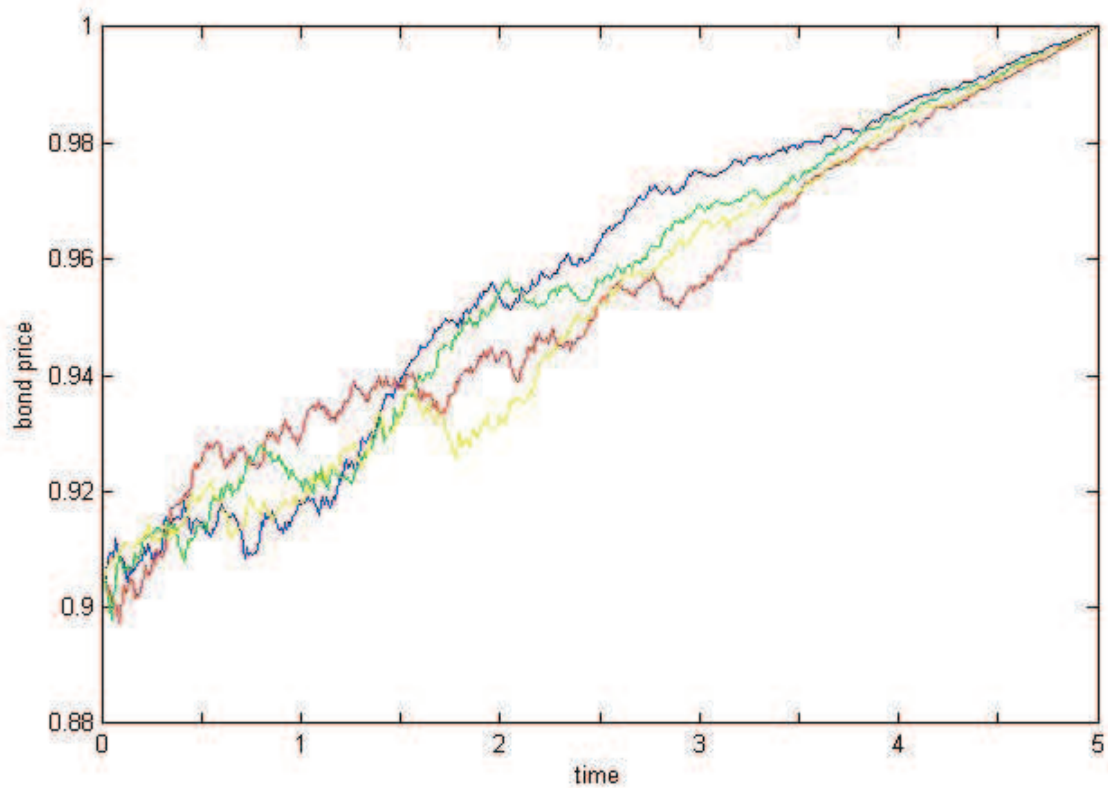


Figure 1: Simulation of the bond price in a term-structure density model with a parametric martingale family based on a geometric Brownian motion. The parameters are $\rho(x) = \mathrm{e}^{-0.02x}$, $\phi(x) = \mathrm{e}^{-0.025x}$.

Geometric Brownian motion family: short rate

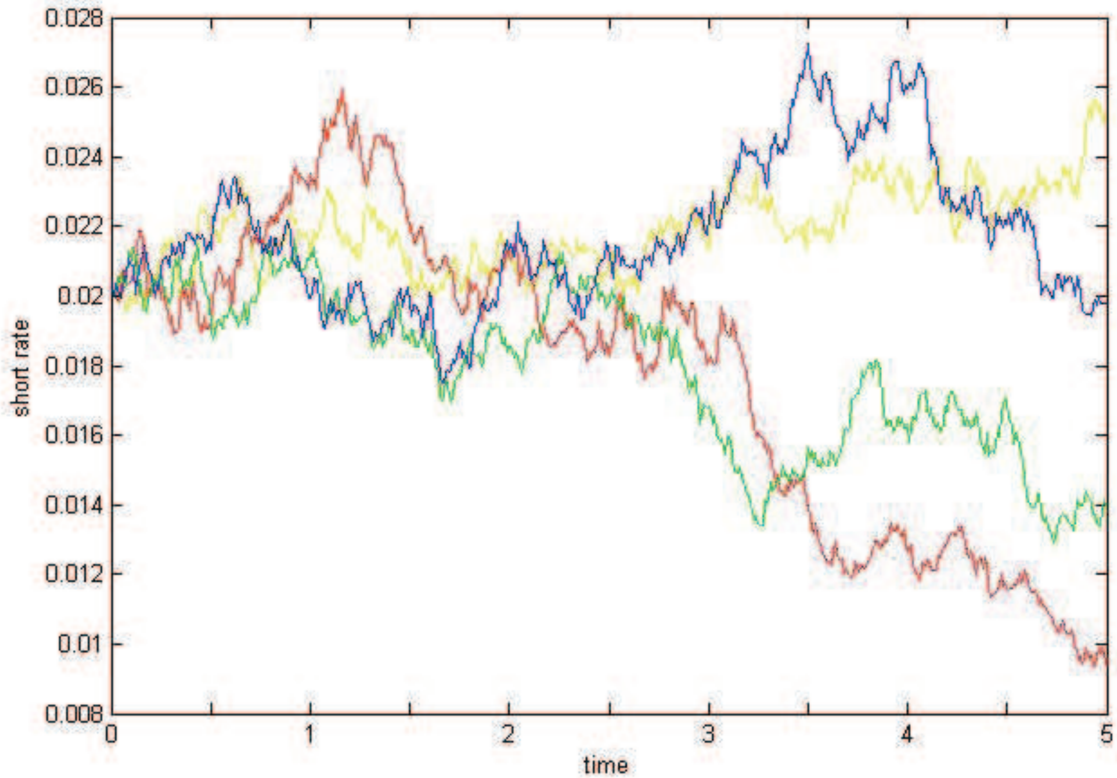


Figure 2: Simulation of the short rate in a term-structure density model with a parametric martingale family based on a geometric Brownian motion. The parameters are $\rho(x) = \mathrm{e}^{-0.02x}$, $\phi(x) = \mathrm{e}^{-0.025x}$.

Geometric Brownian motion family: option price

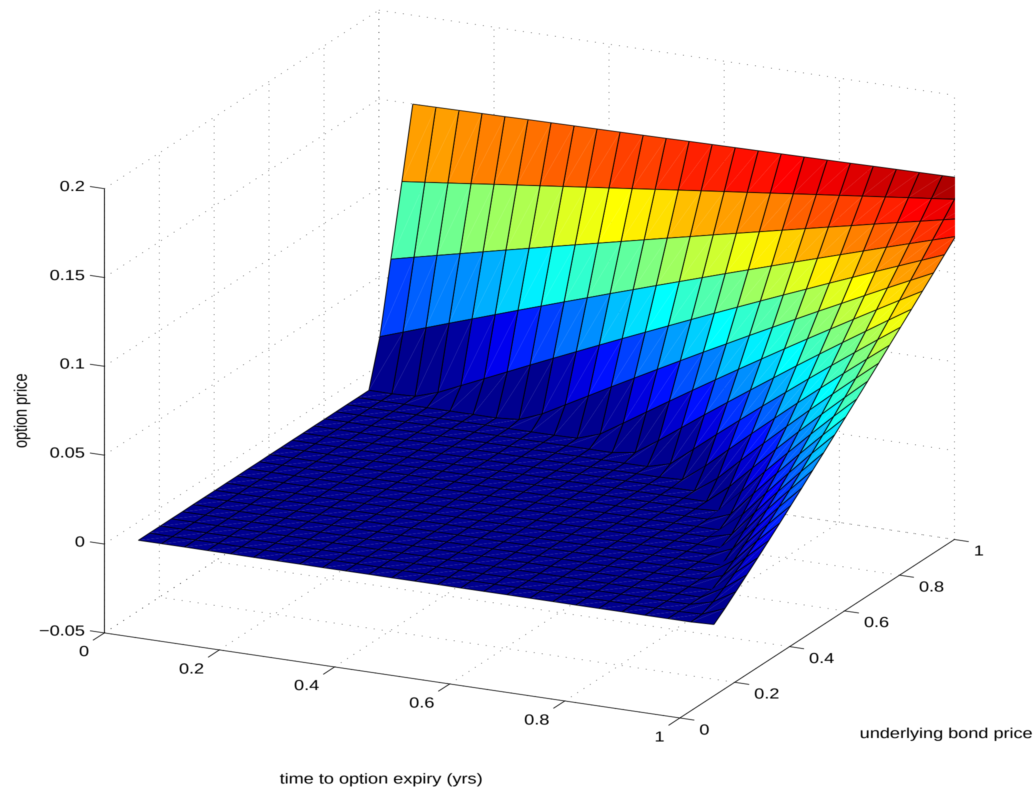


Figure 3: Option price as a function of P_{0T} and option maturity $0 \leq t \leq T$, where the bond maturity is $T = 1$.

Geometric gamma family

Let $\{\gamma_t\}_{t\geq 0}$ be a gamma process whose increments $\gamma_t - \gamma_s$, for $0 \leq s \leq t < \infty$, have a probability density $f(x)$ such that

$$f(x) = g_\gamma(x; (t-s)m, \kappa) = \frac{x^{mt-1}\mathrm{e}^{-x/\kappa}}{\Gamma(mt)\kappa^{mt}}. \tag{19}$$

Here, m is the rate parameter and κ is the scale parameter of $\{\gamma_t\}_{t\geq 0}$.

We are able to show that the bond price is of the form

$$P_{tT} = \frac{\int_T^\infty \rho(x)(1-\phi(x)\kappa)^{-mt}\mathrm{e}^{\phi(x)\gamma_t}\mathrm{d}x}{\int_t^\infty \rho(x)(1-\phi(x)\kappa)^{-mt}\mathrm{e}^{\phi(x)\gamma_t}\mathrm{d}x}, \tag{20}$$

and that the associated short rate is given by

$$r_t = \frac{\rho(t)(1-\phi(t)\kappa)^{-mt}\mathrm{e}^{\phi(t)\gamma_t}\mathrm{d}x}{\int_t^\infty \rho(x)(1-\phi(x)\kappa)^{-mt}\mathrm{e}^{\phi(x)\gamma_t}\mathrm{d}x}. \tag{21}$$

For increasing $\phi(x)$ we deduce that the required option price in the example of the geometric gamma process is of the following form:

$$C_{0t} = \int_T^\infty \rho(x)\Gamma\left(mt, \gamma^*\left(\frac{1}{\kappa} - \phi(x)\right)\right)\mathrm{d}x$$
$$-K\int_t^\infty \rho(x)\Gamma\left(mt, \gamma^*\left(\frac{1}{\kappa} - \phi(x)\right)\right)\mathrm{d}x, \tag{22}$$

where γ^* is a critical value on the boundary of positive payoffs.

Here

$$\Gamma(a, x) = \int_x^\infty \frac{t^{a-1}\mathrm{e}^{-t}}{\Gamma(a)}dt \tag{23}$$

is the "upper" incomplete gamma function.

Geometric gamma process family: bond price

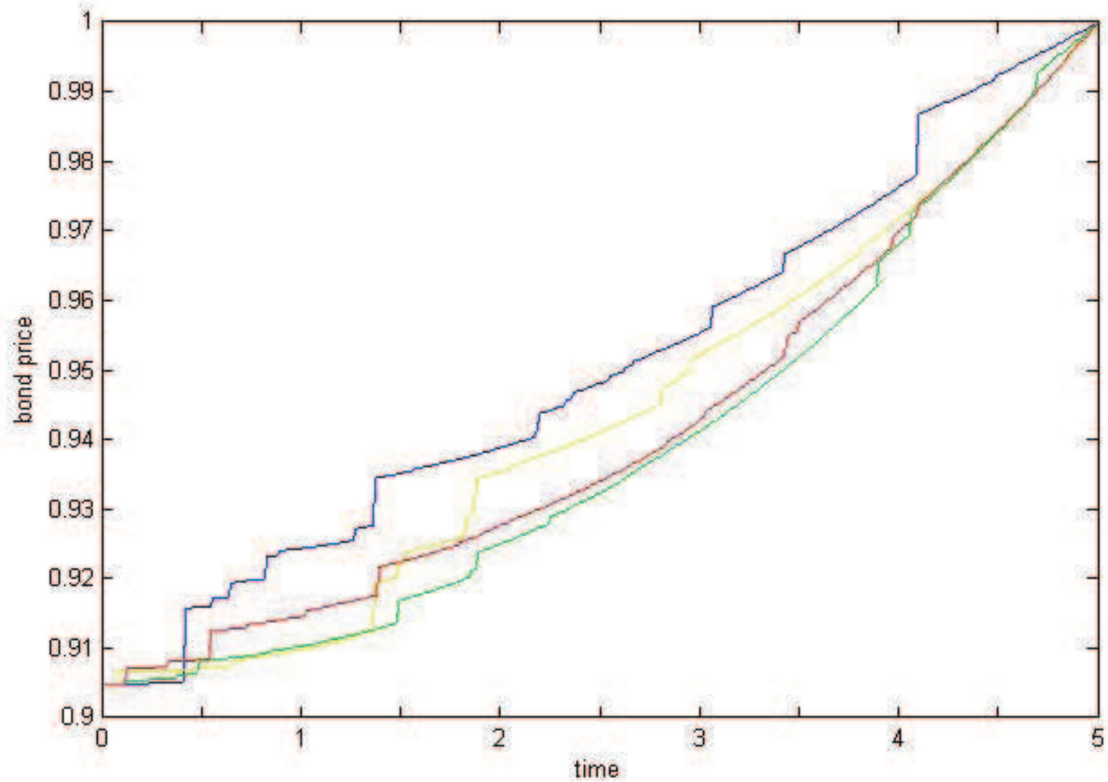


Figure 4: Simulation of the bond price in a term-structure density model with a parametric martingale family based on a geometric gamma martingale. The parameters are $\rho(x) = \mathrm{e}^{-0.02x}$, $\phi(x) = \mathrm{e}^{-0.025x}$, $m = 1$, $\kappa = 0.5$.

Geometric gamma process family: short rate

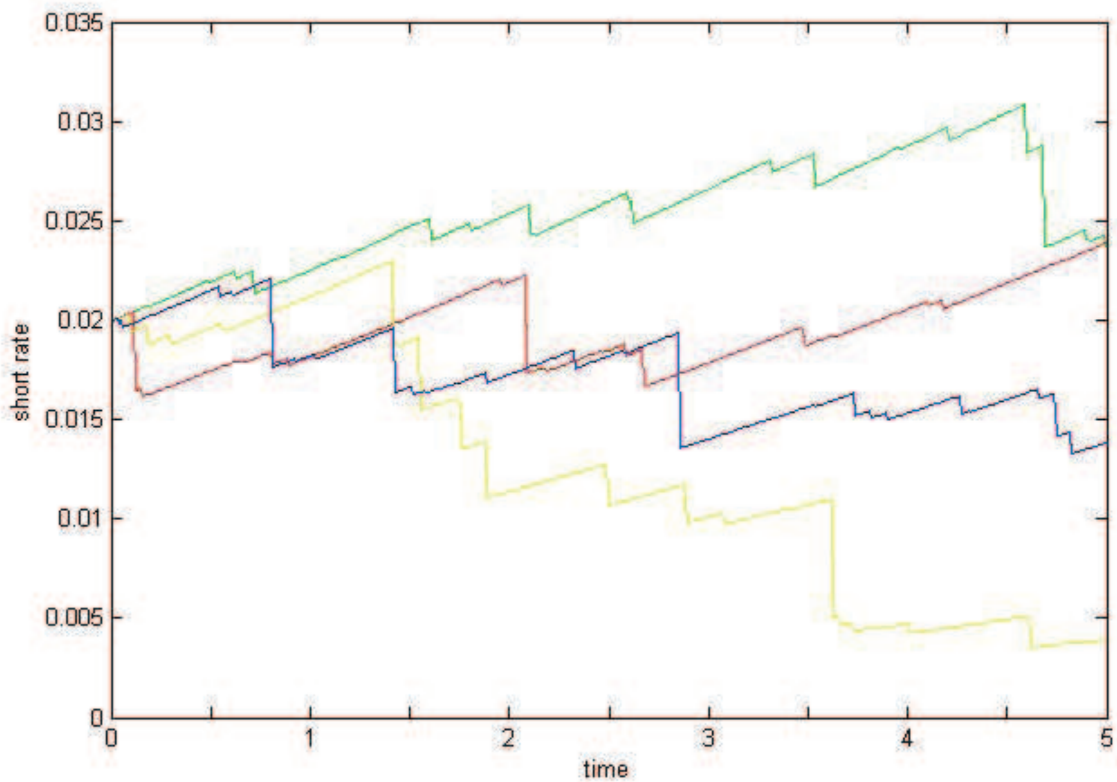


Figure 5: Simulation of the short rate in a term-structure density model with a parametric martingale family based on a geometric gamma martingale. The parameters are $\rho(x) = \mathrm{e}^{-0.02x}$, $\phi(x) = \mathrm{e}^{-0.025x}$, $m = 1$, $\kappa = 0.5$.

Geometric variance-gamma family

Let $\{V_t\}_{t\geq 0}$ be a variance-gamma process.

The increments $V_t - V_s$ of $\{V_t\}$ thus have the same distribution as a Brownian motion with volatility parameter σ and drift μ, time-changed by a gamma subordinator, whose increments have distribution $g(x)$ such that

$$g(x) = g_\gamma(x; (t-s)/\nu, \nu) \tag{24}$$

for $0 \leq s \leq t < \infty$.

The parameters μ, σ and ν control the properties of the variance-gamma distribution.

We are able to show that bond price is of the form

$$P_{tT} = \frac{\int_T^\infty \rho(x)(1-\nu\mu\phi(x) - \frac{1}{2}\nu\sigma^2\phi(x)^2)^{-\frac{t}{\nu}}\mathrm{e}^{\phi(x)V_t}\mathrm{d}x}{\int_t^\infty \rho(x)(1-\nu\mu\phi(x) - \frac{1}{2}\nu\sigma^2\phi(x)^2)^{-\frac{t}{\nu}}\mathrm{e}^{\phi(x)V_t}\mathrm{d}x}, \tag{25}$$

and that the associated short rate is given by

$$r_t = \frac{\rho(t)(1-\nu\mu\phi(t) - \frac{1}{2}\nu\sigma^2\phi(t)^2)^{-\frac{t}{\nu}}\mathrm{e}^{\phi(t)V_t}}{\int_T^\infty \rho(x)(1-\nu\mu\phi(x) - \frac{1}{2}\nu\sigma^2\phi(x)^2)^{-\frac{t}{\nu}}\mathrm{e}^{\phi(x)V_t}\mathrm{d}x}. \tag{26}$$

Using a methodology similar to that of Madan et al. [1998], we derive the option price in the example of the geometric variance-gamma process.

We first condition on the gamma time-change and numerically calculate ξ^*, which is a critical value of the Brownian motion on the boundary of positive payoffs, and then we integrate over the uncertainty of the gamma time-change.

The required option price is of the form

$$\begin{aligned} C_{0t} \;=\; & \int_T^\infty \rho(x)\Psi\left(\pm\xi^*\Phi(x), \mp\frac{\sigma\phi(x)}{\Phi(x)}, \frac{t}{\nu}\right)\mathrm{d}x \\ & -K\int_t^\infty \rho(x)\Psi\left(\pm\xi^*\Phi(x), \mp\frac{\sigma\phi(x)}{\Phi(x)}, \frac{t}{\nu}\right)\mathrm{d}x. \end{aligned} \tag{27}$$

Here

$$\Phi(x) = \sqrt{\frac{1-\nu\mu\phi(x)-\frac{1}{2}\nu\sigma^2\phi(x)^2}{\nu}}, \tag{28}$$

and

$$\Psi(a,b,c) = \int_0^\infty \mathcal{N}\left(\frac{a}{\sqrt{u}}+b\sqrt{u}\right)\frac{u^{c-1}\mathrm{e}^{-u}}{\Gamma(c)}\mathrm{d}u. \tag{29}$$

Geometric variance-gamma process family: bond price

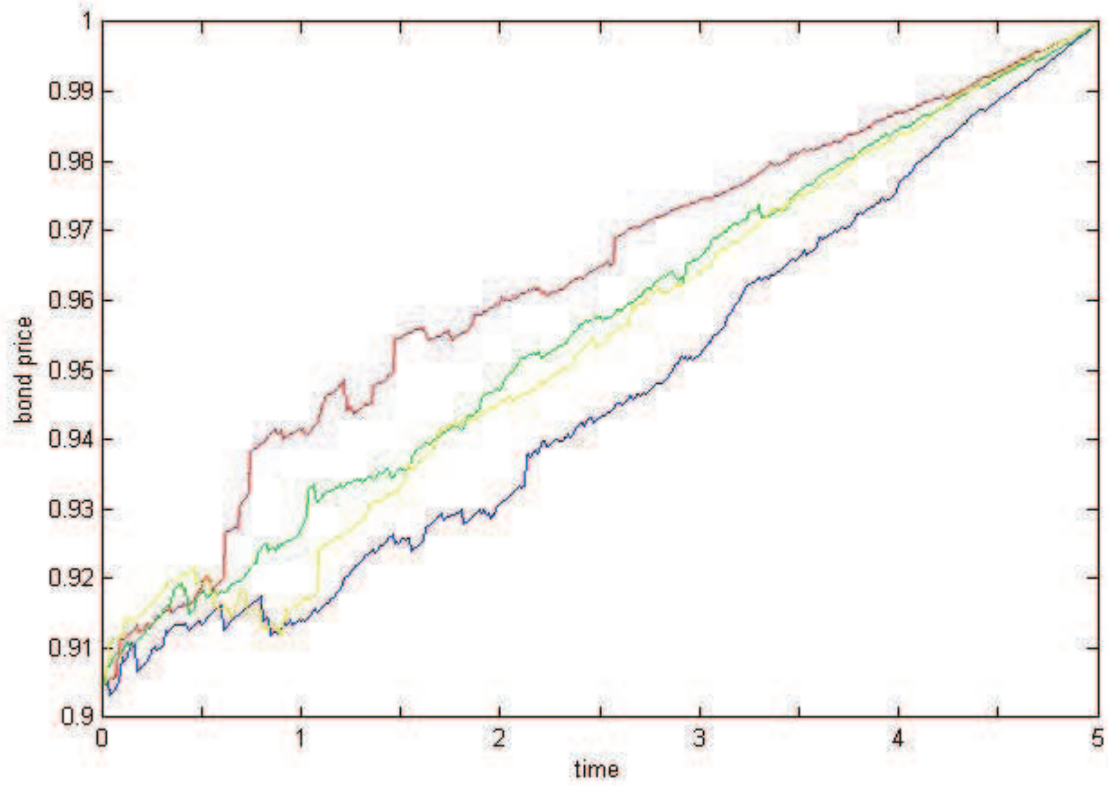


Figure 6: Simulation of the bond price based on a geometric variance-gamma martingale. The parameters are $\rho(x) = \mathrm{e}^{-0.02x}$, $\phi(x) = \mathrm{e}^{-0.025x}$, $\mu = 0.2$, $\sigma = 0.2$ $\nu = 0.1$.

Geometric variance-gamma process family: short rate

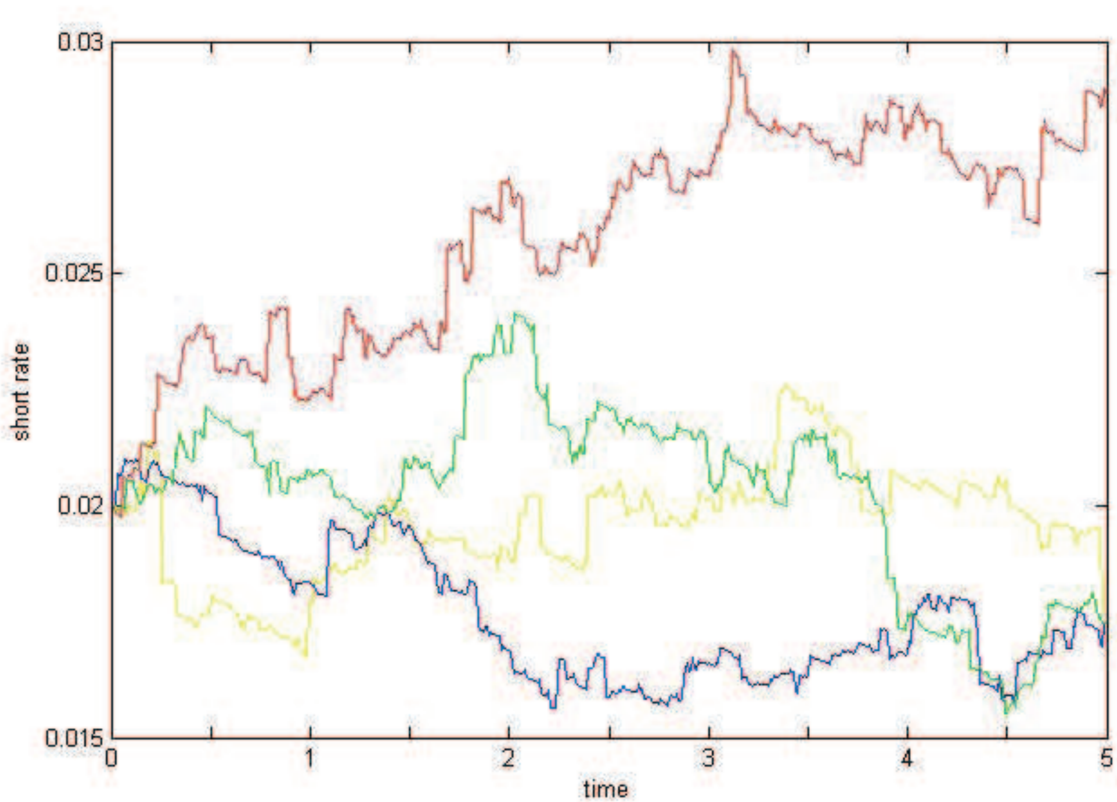


Figure 7: Simulation of the short rate based on a geometric variance-gamma martingale. The parameters are $\rho(x) = \mathrm{e}^{-0.02x}$, $\phi(x) = \mathrm{e}^{-0.025x}$, $\mu = 0.2$, $\sigma = 0.2$ $\nu = 0.1$.

Geometric variance-gamma process family: option price

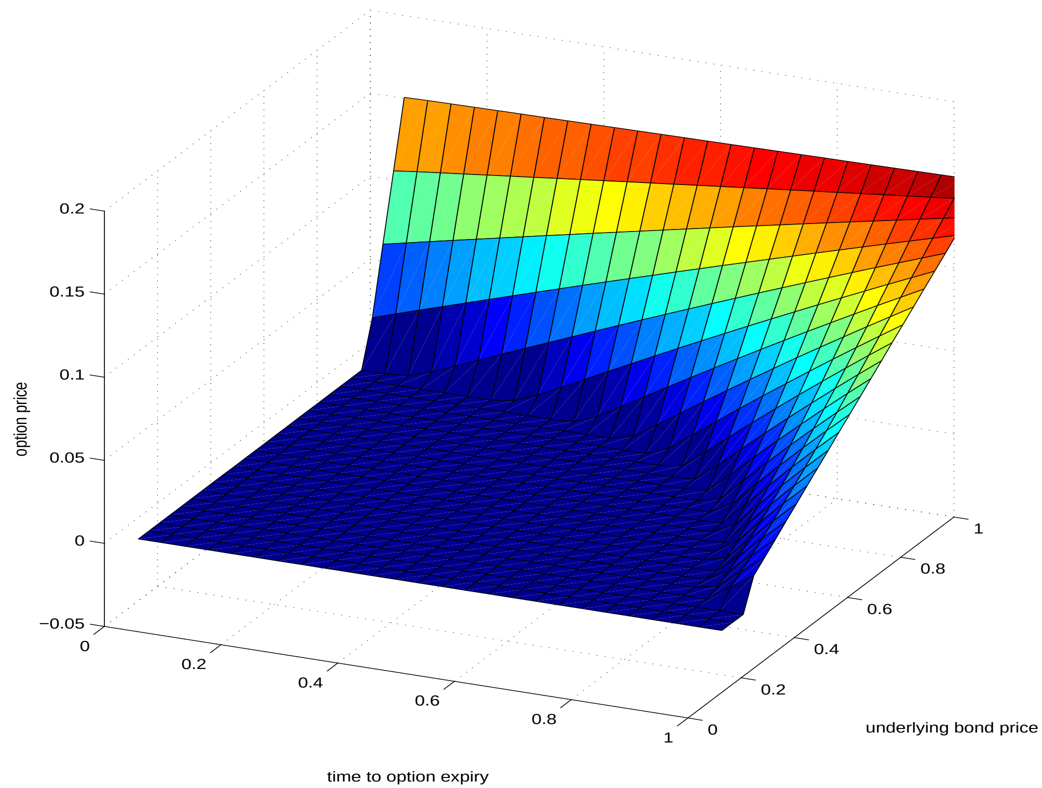


Figure 8: Option price as a function of P_{0T} and option maturity $0 \leq t \leq T$, where the bond maturity is $T = 1$.

Hedging strategy—option delta

How does one determine the sensitivity of the option price C_{0t} on the initial price P_{0T} of the underlying?

Recall that the initial bond price is given by

$$P_{0T} = \int_T^\infty \rho(x)\mathrm{d}x. \tag{30}$$

In the case of interest-rate term structure, the initial price P_{0T} of the bond is a *functional* of the term structure density $\rho(x)$.

Likewise, the option price C_{0t} is a functional of $\rho(x)$.

Therefore, to determine the option sensitivity, we are required to employ the method of *functional derivative*.

This is because there are infinitely many ways of perturbing

$$\rho(x) \longrightarrow \rho(x) + \epsilon\eta(x) \tag{31}$$

to obtain the perturbation

$$P_{0T} \longrightarrow P_{0T} + \delta P_{0T}. \tag{32}$$

We thus proceed as follows.

Given the call price $C_{0t}[\rho]$ as a functional of $\rho(x)$ we consider the functional derivative

$$\frac{\delta C_{0t}}{\delta \rho} = \lim_{\varepsilon \to \infty} \frac{1}{\varepsilon} \left(C_{0t}[\rho + \varepsilon\eta] - C_{0t}[\rho] \right), \tag{33}$$

where the perturbation $\eta(x)$ has to satisfy the condition

$$\int_0^\infty \eta(x)\mathrm{d}x = 0 \tag{34}$$

so that $\rho + \varepsilon\eta$ is a density.

Recall that the option price is given by

$$C_{0t} = \mathbb{E}^{\mathbb{Q}} \left[P_{0t} \left(\frac{\int_T^\infty \rho(x) M_{tx} \mathrm{d}x}{\int_t^\infty \rho(x) M_{tx} \mathrm{d}x} - K \right)^+ \right]. \tag{35}$$

Performing the functional differentiation, and letting $\Theta(\cdot)$ denotes the Heaviside function, we deduce that

$$\frac{\delta C_{0t}}{\delta \rho} = \mathbb{E}\left[\Theta\left(\int_T^\infty \rho(x) M_{tx} \mathrm{d}x - K \int_t^\infty \rho(x) M_{tx} \mathrm{d}x\right) \times \left(\int_T^\infty \eta(x) M_{tx} \mathrm{d}x - K \int_t^\infty \eta(x) M_{tx} \mathrm{d}x\right)\right]. \tag{36}$$

How do we choose the perturbation $\eta(x)$?

In the case of a single factor model, it suffices to consider a single perturbation.

To this end, we take the perturbation to be a constant shift

$$R_{0T} \to R_{0T} + \varepsilon \tag{37}$$

of the initial yield curve R_{0T}.

Under this perturbation, we have

$$\rho(x) \to \rho(x) + \varepsilon(P_{0x} - x\rho(x)). \tag{38}$$

In other words, the perturbation that results in a constant shift of the initial yield curve is given by

$$\eta(x) = P_{0x} + x \frac{\mathrm{d}P_{0x}}{\mathrm{d}x}. \tag{39}$$

Note that the functional derivative of the initial bond price under this perturbation is given by

$$\frac{\delta P_{0T}}{\delta \rho} = -T P_{0T}. \tag{40}$$

Hence, by use of the chain rule we have

$$\frac{\delta C_{0t}}{\delta P_{0T}} = \frac{\delta \rho}{\delta P_{0T}} \frac{\delta C_{0t}}{\delta \rho} = -\frac{1}{T P_{0T}} \frac{\delta C_{0t}}{\delta \rho}. \tag{41}$$

Putting these together, and after a short calculation we find that

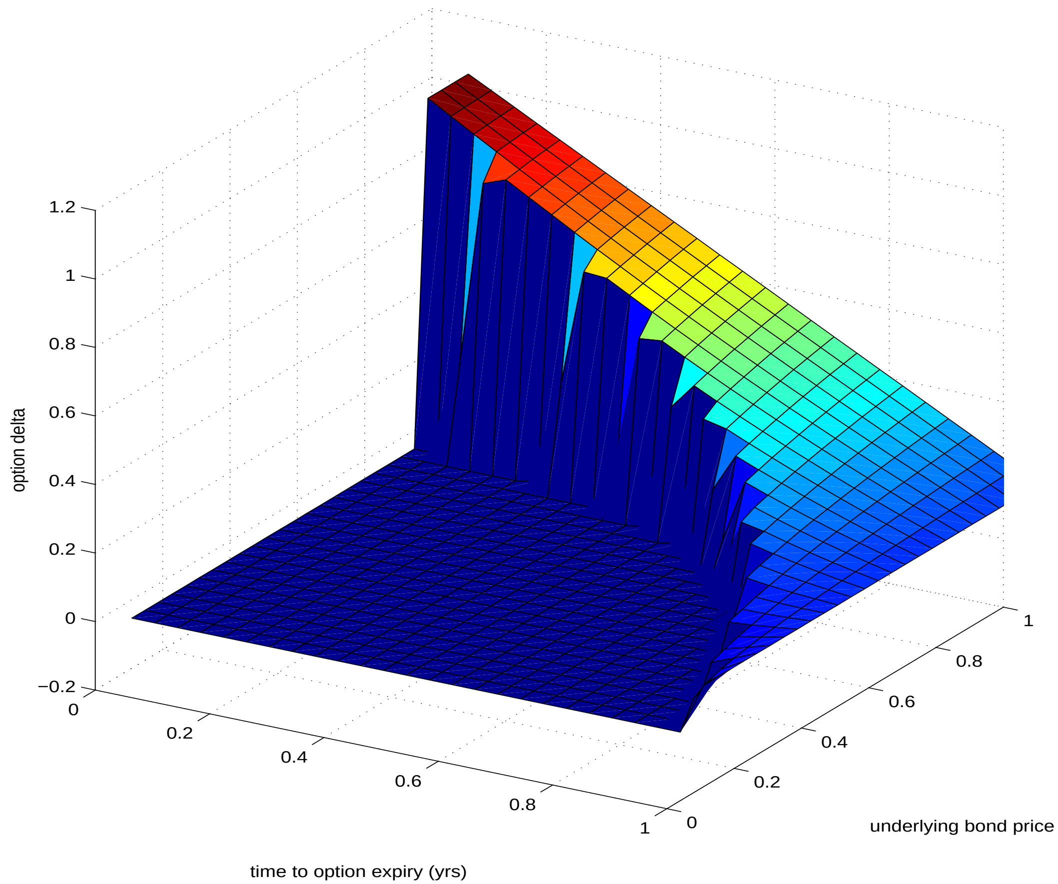
$$\Delta = \int_T^\infty \frac{(x\rho(x) - P_{0x})}{T P_{0T}} m_t(x)\mathrm{d}x - K \int_t^\infty \frac{(x\rho(x) - P_{0x})}{T P_{0T}} m_t(x)\mathrm{d}x, \tag{42}$$

where

$$m_t(x) = \mathbb{E}\left[\Theta\left(\int_T^\infty \rho(x) M_{tx}\mathrm{d}x - K\int_t^\infty \rho(x) M_{tx}\mathrm{d}x\right) M_{tx}\right]. \tag{43}$$

In this way we are able to compute the option delta for a range of Lévy martingales $\{M_{tx}\}$.

Geometric Brownian motion family: option delta



Geometric variance-gamma process family: option delta