Hedging of counterparty risk

T.R. Bielecki[†], S. Crépey[*] and M. Jeanblanc[*]

[*]Université d'Évry Val-d'Essonne and CRIS Consortium
[†]Illinois Institute of Technology

Workshop on Financial Derivatives and Risk Management, Fields Institute, 24-28 May 2010

Outline

Where do we come from?

- Assefa, S., Bielecki. T.R, Crepey, S. and Jeanblanc, M. (2009)
  CVA computation for counterparty risk assessment in credit portfolios
- Crepey, S., Jeanblanc, M. and Zargari, B. (2010)
  Counterparty Risk on a CDS in a Model with Joint Defaults and Stochastic Spreads
- Bielecki. T.R, Crepey, S., Cousin, A. and Hendersson, A. (20..)
  Pricing and Hedging Portfolio Credit Derivatives in a Bottom-up Model with Simultaneous Defaults
- Cesari, G., Aquilina, J., Charpillon, N., Filipović, Z., Lee, G., Manda, I. (2010)
  Modeling, Pricing and Hedging Counterparty Credit Exposure, Springer-Finance
- Gregory, J. (2010)
  Counterparty Credit Risk: The New Challenge for Global Financial Markets Overview, Features, and Description, Wiley Finance

Outline

Basic Concept

Risk that some value is lost by a party in OTC derivatives contracts due to the default of the other party

- Early termination of a contract with positive value at time of default of the other party
  - Cum-dividend value, including promised payment not paid at default time

The primary form of (credit) risk – vulnerability
Very significant during the crisis
An important dynamic modeling issue/challenge, particularly in connection with credit derivatives

- Pricing at any future time
- Defaults dependence modeling
  - Wrong way risk

General Set-Up: I

$(\Omega, \bar{\mathbb{G}}, \mathbb{P})$, $\bar{\mathbb{G}} = (\bar{\mathcal{G}}_t)_{t\in[0,T]}$, $(\Omega, \mathbb{G}, \mathbb{P})$, $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ risk-neutral pricing models, where $\bar{\mathbb{G}} \subset \mathbb{G}$ (with $r = 0$ for notational simplicity)

$\bar{\mathbb{E}}_t$, $\mathbb{E}_t$ Conditional expectation under $\mathbb{P}$ given $\bar{\mathcal{G}}_t$ or given $\mathcal{G}_t$, respectively.

Note that in both cases we use the same pricing measure; in essence, we only consider models such that pricing of counterparty risk free claims and pricing of counterparty risky claims is done using the same measure. Models of this type are for example models where so called immersion property is satisfied between filtrations $\bar{\mathbb{G}}$ and $\mathbb{G}$; another example is provided by Markov copula models.

General Set-Up: II

τ_{-1} and τ_0 Default times of the two parties, referred to henceforth as the investor, labeled -1, and its counterparty, labeled 0

- $[0, +\infty]$-valued $\mathbb{F}$-stopping times
- Bilateral counterparty risk $\leftrightarrow$ counterparty risk on both sides is considered $\leftrightarrow \tau_{-1} < +\infty,\ \tau_0 < +\infty$
- Unilateral counterparty risk$\leftrightarrow \tau_{-1} = +\infty$

R_{-1} and R_0 Recovery rates, given as $\mathcal{G}_{\tau_{-1}}$- and $\mathcal{G}_{\tau_0}$- measurable $[0, 1]$-valued random variables

$\tau =$ $\tau_{-1} \wedge \tau_0$, with related default and non-default indicator processes denoted by H and J, so $H_t = \mathbb{1}_{\tau \leq t}$ and $J = 1 - H$.

- No actual cash flow after τ

All cash flows and prices considered from the perspective of the investor

Cash Flows

General case reduced to that of a

Fully netted and collateralized portfolio

$\mathcal{D}$ Counterparty risky cumulative cash flows

D Counterparty clean cumulative cash flows

$\Longrightarrow$

$$\mathcal{D} = JD + HD_{\tau-} + H\Gamma_\tau$$
$$+ (R_0\chi^+ - \chi^-)[H, H^0] - (R_{-1}\chi^- - \chi^+)[H, H^{-1}] - \chi[[H, H^0], H^{-1}]$$

Γ_τ Value of the collateral (or margin account) at time τ

$\chi = P_{(\tau)} + (D_\tau - D_{\tau-}) - \Gamma_\tau$ Algebraic 'debt' of the counterparty to the investor at time τ

$P_{(\tau)}$ 'Fair (ex-dividend) value' of the portfolio at τ

$D_\tau - D_{\tau-}$ Promised cash flow at τ

Valuation

$\Pi_t := \mathbb{E}_t\left[\mathcal{D}_T - \mathcal{D}_t\right]$ Counterparty Risky Ex-dividend Value of the portfolio

$P_t := \bar{\mathbb{E}}_t\left[D_T - D_t\right]$ Counterparty Clean Ex-dividend Value of the portfolio

$\widehat{\Pi}_t := \mathbb{E}_t\left[\mathcal{D}_T\right]$ Counterparty Risky Cumulative-dividend Value of the portfolio

$\widehat{P}_t := \bar{\mathbb{E}}_t\left[D_T\right]$ Counterparty Clean Cumulative-dividend Value of the portfolio

Projection Property: For any $t \in [0, T]$ it holds that

$$P_t = \mathbb{E}_t\left[D_T - D_t\right], \quad \widehat{P}_t = \mathbb{E}_t\left[D_T\right].$$

CVA: Definition

CVA (Credit Valuation Adjustment)

CVA of the first kind (null after τ)

$$CVA_t := J_t(\widehat{P}_t - \widehat{\Pi}_t)$$

CVA of the second kind (constant after τ)

$$\widehat{CVA}_t := \widehat{P}_{t\wedge\tau} - \widehat{\Pi}_{t\wedge\tau}$$

Below, market 'Legal Value' standard

$$P_{(\tau)} = P_\tau$$

assumed for simplicity

CVA: Properties

CVA Properties

- $CVA_t = J_t \widehat{CVA}_t = J_t(P_t - \Pi_t), \quad t \in [0.T],$
- Process $\widehat{CVA}$ is a $\mathbb{G}$ martingale under $\mathbb{P}$, and it satisfies

$$\widehat{CVA}_t = \mathbb{E}_t\left[\xi\right],$$

  where the $\mathcal{G}_\tau$-measurable Potential Future Exposure at Default (PFED) ξ is given by

$$\xi = (1 - R_0)\mathbb{1}_{\tau \geq \tau_0}\chi^+_{(\tau)} - (1 - R_{-1})\mathbb{1}_{\tau \geq \tau_{-1}}\chi^-_{(\tau)} .$$

- Random variable ξ is equal to the value at time τ of process

$$Z_t = (1 - R_0)\mathbb{1}_{t \geq \tau_0}\chi^+_t - (1 - R_{-1})\mathbb{1}_{t \geq \tau_{-1}}\chi^-_t$$

  where $\chi_t := P_t + \Delta D_t - \Gamma_t$.

CVA: Dynamics I

Lemma

For any $t \in (0, T]$ we have

$$\begin{aligned} d\widehat{CVA}_t &= (1 - H_{t-})(d\widehat{P}_t - d\widehat{\Pi}_t) \\ &= (1 - H_t)(dP_t - d\Pi_t) + (\Delta\widehat{P}_\tau - \Delta\widehat{\Pi}_\tau)dH_t \\ &= (1 - H_t)(dP_t - d\Pi_t) + (Z_\tau - \widehat{CVA}_{\tau-})dH_t \\ &= (1 - H_t)(dP_t - d\Pi_t) + (Z_t - \widehat{CVA}_{t-})dH_t. \end{aligned}$$

$$dCVA_t = (1 - H_t)(dP_t - d\Pi_t) - CVA_{t-}dH_t.$$

Proof. The first equality comes from

$$\widehat{CVA}_t = (1 - H_t)(\widehat{P}_t - \widehat{\Pi}_t) + H_t(\widehat{P}_\tau - \widehat{\Pi}_\tau).$$

The remaining ones follow easily.

CVA: Dynamics II

- The above lemma is the key to hedging of counterparty risk. The dynamics of $\widehat{CVA}$ splits into the "pre-counter-party-default" part $(1-H_t)(dP_t - d\Pi_t)$, and the "at-counter-party-default" part $(Z_t - \widehat{CVA}_{t-})dH_t$.
- Specification of the above dynamics, that is specification of all martingale terms, is not easy in general. We shall thus proceed with specification of the dynamics in a stylized Markovian model.

Outline

CCR in CDS: I

We now specify the general setup presented above to the situation of the counterparty credit risk in a CDS contract.

Towards this end we postulate that the CDS contract between the investor and his/her counterparty references a credit name denoted as 1. Next, we denote by τ_1 the default time of the reference name, and by H^1 we denote the corresponding indicator process.

Here we consider the following filtration structure:

- $\bar{\mathbb{G}} = \bar{\mathbb{F}} \vee \mathbb{H}^1$, where $\bar{\mathbb{F}}$ is a reference (Brownian) filtration, and $\mathbb{H}^1$ is the filtration of H^1.
- $\mathbb{G} = \bar{\mathbb{G}} \vee \mathbb{F} \vee \mathbb{H}^{-1} \vee \mathbb{H}^0$, where $\mathbb{F}$ is another reference (Brownian) filtration.

CCR in CDS: II

- We consider a payer CDS on name 1 (CDS protection on name 1 bought by the investor, or credit name -1, from its counterparty, represented by the credit name 0).
- Denoting by T the maturity, κ the contractual spread and $R_1 \in [0,1]$ the recovery, this corresponds to the special case of the dividend process D given as

$$D = \int_{[0,\cdot]} J^1_u \, dC_u + \int_{[0,\cdot]} \delta_u \, dH^1_u = J^1 C + H^1 \left(C_{\tau_1 -} + \delta^1_{\tau_1} \right) ,$$

where

$$C_t = -\kappa (t \wedge T) , \;\; \delta_t = (1 - R_1) .$$

Unilateral CCR in CDS

- Here we have that $\tau_{-1} = +\infty$ and $\tau = \tau_0$ is the default time of the investor's (bank's) counterparty, with recovery R_0 simply denoted as R
- Here we have $D_\tau = \mathbb{1}_{\tau_1 \le \tau \le T}(1 - R_1)$, so that $\chi_{(\tau)} = P_\tau + \mathbb{1}_{\tau_1 = \tau \le T}(1 - R_1) - \Gamma_\tau$ if $\tau \le T$, and zero otherwise,

Proposition

One has,

$$\xi = (1 - R)\Big(P_\tau + \mathbb{1}_{\tau_1 \le \tau \le T}(1 - R_1) - \Gamma_\tau\Big)^+ ,$$

so that

$$Z_t = (1 - R)\Big(P_t + \mathbb{1}_{\tau_1 \le t \le T}(1 - R_1) - \Gamma_t\Big)^+ .$$

Outline

Markovian Copula Model: I

- Let $\mathrm{H} = (H^1, H^0)$ denote the pair of the default indicator processes, for the CDS reference name (H^1) and for the counterparty (H^0)
- $\mathrm{X} = (X_1, X_0)$ is a diffusion factor process
- Here, $\bar{\mathbb{G}} = \mathbb{H}^1 \vee \mathbb{F}^{X^1}$ and $\mathbb{G} = \bar{\mathbb{G}} \vee \mathbb{H}^0 \vee \mathbb{F}^{X^0}$.
- (X, H) is a Markov process w.r.t. $\mathbb{G}$
- (X^1, H^1) is a Markov process w.r.t. $\bar{\mathbb{G}}$

Markovian Copula Model: II

Generator of (X, H) given by, for $u = u(t, \chi, \varepsilon)$ with $t \in \mathbb{R}_+, \chi = (x_1, x_0) \in \mathbb{R}^2, \varepsilon = (K_1, K_0) \in \{0,1\}^2$:

$$\mathcal{A}u(t,\chi,\varepsilon) = \partial_t u(t,\chi,\varepsilon) + \sum_{0\le i\le 1} l_i(t,\chi_i)\left(u(t,\chi,\varepsilon^i) - u(t,\chi,\varepsilon)\right)$$

$$+ l_3(t)\left(u(t,\chi,1,1) - u(t,\chi,\varepsilon)\right) + \sum_{0\le i\le 1}\left(b_i(t,x_i)\partial_{x_i}u(t,\chi,\varepsilon) + \frac{1}{2}\sigma_i^2(t,x_i)\partial^2_{x_i^2}u(t,\chi,\varepsilon)\right),$$

where, for $i = 1, 2$:

- ε^i denotes the vectors obtained from ε by replacing the component i, by number one,
- b_i and σ_i^2 denote factor drift and variance functions, and l_i is an individual default intensity function,
- $l_3(t)$ stands for a joint defaults intensity function.

Markovian Copula Model: III

The intensity-matrix function of H is given as

$$A(t,x) = \begin{bmatrix} -l(t,x) & l_1(t,x_1) & l_0(t,x_0) & l_3(t) \\ 0 & -q_0(t,x_0) & 0 & q_0(t,x_0) \\ 0 & 0 & -q_1(t,x_1) & q_1(t,x_1) \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

with, for every $i = 1,2,$

$$q_i(t,x_i) = l_i(t,x_i) + l_3(t)$$

and $l(t,x) = l_0(t,x_1) + l_1(t,x_2) + l_3(t)$.

Markovian Copula Model: IV

The intensity-matrix function of H^i is given by

$$A_i(t, x_i) = \begin{bmatrix} -q_i(t, x_i) & q_i(t, x_i) \\ 0 & 0 \end{bmatrix}$$

for $i = 0, 1$.

Remark

The above model for (X, H) is a Markovian copula model with **given Markovian** marginals (X^i, H^i), for $i = 0, 1$. This property of the marginals is key to pricing risk-free CDSs on the firm or on the counterparty, as intensively done at the stage of model calibration.

Markovian Copula Model: Pricing CDS without CCR

Proposition

The price of the risk-free CDS admits the representation:

$$P_t = (1 - H_t^1) v_1(t, X_t^1) ,$$

for a pre-default pricing function $v_1(t, x_1)$ *solving*

$$\begin{cases} v_1(T, x_1) = 0 \,, \quad x_1 \in \mathbb{R} \\ \left(\partial_t + b_1(t, x_1)\partial_{x_1} + \frac{1}{2}\sigma_1^2(t, x_1)\partial^2_{x_1^2}\right) v_1(t, x_1) - q_1(t, x_1) v_1(t, x_1) + p(t, x_1) = 0 \,, \\ \qquad t \in [0, T),\ x_1 \in \mathbb{R} \,, \end{cases}$$

where

$$p(t, x_1) = (1 - R_1) q_1(t, x_1) - \kappa$$

Markovian Copula Model: Pricing CDS with CCR

- We assume for simplicity that the collateral process is null ($\Gamma = 0$).

Proposition

The price of the counterparty risky CDS admits the representation:

$$\Pi_t = (1 - H_t^1)(1 - H_t^0) v_0(t, \mathrm{X}_t) \ ,$$

for a pre-default pricing function $v_0(t,x)$ *solving*

$$\begin{cases} v_0(T,x) = 0 \,, \quad x \in \mathbb{R}^2 \\ \\ \Big(\partial_t + \sum_{0 \le i \le 1} \big(b_i(t,x_i)\partial_{x_i} + \frac{1}{2}\sigma_i^2(t,x_i)\partial^2_{x_i^2}\big)\Big) v_0(t,x) \\ \qquad -l(t,x)v_0(t,x) + \pi(t,x) = 0 \,, \quad t \in [0,T), \ x = (x_1, x_0) \in \mathbb{R}^2 \ , \end{cases}$$

where

$$\pi(t,x) = (1-R_1)\big[l_1(t,x_1) + Rl_3(t)\big] + l_2(t,x_2)\big[Rv_1^+(t,x_1) - v_1^-(t,x_1)\big] - \kappa$$

Outline

Dynamics of CVA on CDS: I

We define $\mathbb{G}$ stopping times

$$\tau^{\{1\}} = \begin{cases} \tau_1 & \text{if } \tau_1 \neq \tau_0, \\ \infty & \text{otherwise} \end{cases}, \quad \tau^{\{0\}} = \begin{cases} \tau_0 & \text{if } \tau_1 \neq \tau_0, \\ \infty & \text{otherwise} \end{cases}, \quad \tau^{\{0,1\}} = \begin{cases} \tau_1 & \text{if } \tau_1 = \tau_0 \\ \infty & \text{otherwise} \end{cases},$$

and we define the default indicator processes accordingly,

$$H_t^{\{1\}} = \mathbb{1}_{\tau^{\{1\}} \leq t}, \; H_t^{\{0\}} = \mathbb{1}_{\tau^{\{0\}} \leq t}, \; H_t^{\{0,1\}} = \mathbb{1}_{\tau^{\{0,1\}} \leq t}.$$

We see that

$$H^0 = H^{\{0\}} + H^{\{0,1\}}, \quad H^1 = H^{\{1\}} + H^{\{0,1\}}.$$

Dynamics of CVA on CDS: II

Lemma

For every $\iota \in I = \{\{0\}, \{1\}, \{0,1\}\}$, the process M^ι defined by,

$$M_t^\iota = H_t^\iota - \int_0^t q_\iota(s, \mathrm{X}_s, \mathrm{H}_s) ds \ ,$$

is a $\mathbb{G}$-martingale, where the intensity processes $q_\iota(t, X_t, H_t)$s are given by

$$q_{\{1\}}(t, X_t, H_t) = (1 - H_t^1) \left((1 - H_t^0) l_1(t, X_t^1) + H_t^0 q_1(t, X_t^1) \right)$$
$$q_{\{0\}}(t, X_t, H_t) = (1 - H_t^0) \left((1 - H_t^1) l_0(t, X_t^0) + H_t^1 q_0(t, X_t^0) \right)$$
$$q_{\{0,1\}}(t, X_t, H_t) = (1 - H_t^1)(1 - H_t^0) l_3(t) \ .$$

Dynamics of CVA on CDS: III

Proposition

In our Markovian copula model the dynamics of the CVA on CDS are given as

$$\begin{aligned} d\,\widehat{CVA}_t &= -(1-H^0_{t-})(1-H^1_{t-})(v_1-v_0)dM^{\{1\}}_t \\ &\quad +(1-H^0_{t-})(1-H^1_{t-})\Big((1-R)(1-R_1)-(v_1-v_0)\Big)dM^{\{0,1\}}_t \\ &\quad +(1-H^0_{t-})(1-H^1_{t-})\Big((1-R)v_1^+-(v_1-v_0)\Big)dM^{\{0\}}_t \\ &\quad +(1-H^0_t)(1-H^1_t)((\partial_{x_1}v_1-\partial_{x_1}v_0)\sigma_1 dW^1_t-(\partial_{x_0}v_0)\sigma_0 dW^0_t) \\ &=: \chi^{\{1\}}_t dM^{\{1\}}_t+\chi^{\{0,1\}}_t dM^{\{0,1\}}_t+\chi^{\{0\}}_t dM^{\{0\}}_t \\ &\qquad +\gamma^1_t dW^1_t+\gamma^0_t dW^0_t \end{aligned}$$

Remark

Recall that if $R=1$ then $v_1=v_0$ so that $d\,\widehat{CVA}_t=0$ for all t.

Outline

Rolling CDS: I (Bielecki, T.R., Jeanblanc, M. and Rutkowski, M. (2008) *Pricing and trading credit default swaps in a hazard process model*)

- We consider a rolling CDS contract written on the counterparty name 0. For simplicity we assume that recovery is R.

Definition

A rolling CDS contract, initiated at $t = 0$ and maturing at $t = T$, is equivalent to a self-financing trading strategy that at any given time t enters into a CDS contract of maturity T and then unwinds the contract at time $t + dt$.

- Similarly as with self-financing strategies in futures contracts, the value of a rolling CDS is null at any point in time, yet due to the trading gains of the strategy the related cumulative value process is not zero.

Rolling CDS: II

Lemma

The cumulative value process, say $\widehat{CDS}$, of the rolling CDS, is a $\mathbb{G}$ martingale and its dynamics are given as

$$\begin{aligned} d\widehat{CDS}_t &= (1-H_t^0)\left((1-R)\partial g(t,X_t^0) - S(t,X_t^0)\partial f(t,X_t^0)\right)\sigma_0(t,X_t^0)dW_t^0 \\ &\quad + (1-R)(dM_t^{\{0\}} + dM_t^{\{0,1\}}) \\ &=: (1-H_t^0)\psi_t^0 dW_t^0 + (1-R)(dM_t^{\{0\}} + dM_t^{\{0,1\}}) \end{aligned}$$

where g and f denote the pre-default pricing functions of the unit protection and fee legs of the ordinary CDS contract initiated at time t, so

$$f(t,X_t^0) = \mathbb{E}\left(\int_t^T e^{-\int_t^u q_0(v,X_v^0)dv} du \,|\, X_t^0\right)$$

$$g(t,X_t^0) = \mathbb{E}\left(\int_t^T e^{-\int_t^u q_0(v,X_v^0)dv} q_0(u,X_u^0) du \,|\, X_t^0\right) .$$

Here $S = g/f$ is the corresponding CDS fair spread function.

Mean-variance Hedge of CVA: I

- Let ζ be an $\mathbb{R}$-valued process, representing the number of units held in the rolling CDS which is used in a self-financing hedging strategy of CVA.
- The tracking error (e_t) of the hedged portfolio satisfies $e_0 = 0$ and, for $t \in [0, T]$,

$$
\begin{aligned}
de_t &= d\widehat{CVA}_t - (1 - H_t^1)\zeta_t d\widehat{CDS}_t \\
&= \chi_t^{\{1\}} dM_t^{\{1\}} + (\chi_t^{\{0,1\}} - (1-R)\zeta_t) dM_t^{\{0,1\}} + (\chi_t^{\{0\}} - (1-R)(1-H_t^1)\zeta_t) dM_t^{\{0\}} \\
&\quad + \gamma_t^1 dW_t^1 + (\gamma_t^0 - (1-R)\zeta_t \psi_t) dW_t^0 \ ,
\end{aligned}
$$

Mean-variance Hedge of CVA: II

Proposition

The self-financing strategy that minimizes the risk-neutral variance of the tracking error is given, on the set $\{t \le \tau_1\}$, as

$$\zeta_t^{va} = \frac{1}{1-R}\,\frac{\chi_t^{\{0,1\}}\, d\langle M^{\{0,1\}}\rangle_t + \chi_t^{\{0\}}\, d\langle M^{\{0\}}\rangle_t + \gamma_t^0 \psi_t\, dt}{d\langle M^{\{0,1\}}\rangle_t + d\langle M^{\{0\}}\rangle_t + \psi_t^2\, dt}. \qquad (2)$$

$$= \frac{1}{1-R}\,\frac{\chi_t^{\{0,1\}} q_{\{0,1\}}(t, X_t, H_t)dt + \chi_t^{\{0\}} q_{\{0\}}(t, X_t, H_t)dt + \gamma_t^0 \psi_t\, dt}{q_{\{0,1\}}(t, X_t, H_t)dt + q_{\{0\}}(t, X_t, H_t)dt + \psi_t^2\, dt}.$$

Mean-variance Hedge of CVA: III

Remark

The self-financing strategy that minimizes the risk-neutral variance of the jump–to–counterparty–default risk is given, on the set $\{t \le \tau_1\}$, as

$$\zeta_t^{jd} = ((1-R)(1-R_1) - CVA_{t-})\mathbb{P}(H_t^{\{0,1\}} = 1 | \mathcal{G}_{t-})$$

$$+((1-R)CDS_{t-}^{+} - CVA_{t-})\mathbb{P}(H_t^{\{0\}} = 1 | \mathcal{G}_{t-}).$$

The ζ^{jd} hedging strategy changes the counterparty jump-to-default exposure from ξ to $\mathbb{E}(\xi \,|\, \mathcal{G}_{\tau-})$, the 'best guess' of ξ available right before τ.