Absolutely Continuous Compensators

Workshop on Computational Methods in Finance

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Based on work with Svante Janson and Sokhna M'Baye

Structural Versus Reduced Form Models in Credit Risk (Merton, 1973)

- We begin with a filtered space $(\Omega, \mathcal{H}, P, \mathbb{H})$ where $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$
- Let X be a Markov process on $(\Omega, \mathcal{H}, P, \mathbb{H})$ given by

$$dX_t = 1 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t \mu(s, X_s) ds$$

- In a structural model we assume we observe $\mathbb{G} = (\sigma(X_s; 0 \leq s \leq t))_{t \geq 0}$ and so $\mathbb{G} \subset \mathbb{H}$
- Default occurs when the firm's value X crosses below a given threshold level process $L = (L_t)_{t \geq 0}$
- If L is constant, then the default time is $\tau = \inf\{t > 0 : X_t \leq L\}$, and τ is a predictable time for $\mathbb{G}$ and $\mathbb{H}$

Two objections to the Structural Model Approach

- It is assumed that the coefficients σ and μ in the diffusion equation are knowable
- It is also assumed the level crossing that leads to default is knowable
- The default time is a predictable stopping time

The Reduced Form Approach (Jarrow, Turnbull, Duffie, Lando, Jeanblanc...)

- We assume that a stopping time τ is given, which is a default time
- We assume that τ is a totally inaccessible time
- This means that $M_t = 1_{\{t \geq \tau\}} - A_t =$ a martingale
- A is adapted, continuous, and non decreasing
- Usually it is **implicitly assumed** that A is of the form

$$A_t = \int_0^t \lambda_s ds,$$

where λ is the instantaneous likelihood of the arrival of τ

The Hybrid Approach (Giesecke, Goldberg, ...)

- We assume the structural approach, but instead of a level crossing time as a default time, we replace it with a random curve
- This can make the stopping time totally inaccessible, and of the form found in the reduced form approach
- Giesecke has also pointed out that the increasing process A need no longer have absolutely continuous paths

The Filtration Shrinkage Approach (Çetin, Jarrow, Protter, Yildirim)

- τ can be the time of default for the structural approach
- One does not know the structural approach, so one models this by shrinking the filtration to the presumed level of observable events
- The result is that τ becomes totally inaccessible, and one recovers the reduced form approach
- **Advantage:** This relates the structural and reduced form approaches which facilitate empirical methods to estimate τ
- Motivates studying compensators of stopping times and their behavior under filtration shrinkage

When does the compensator A have absolutely continuous paths?

- **Ethier-Kurtz Criterion:** $A_0 = 0$ and suppose for $s \leq t$

$$E\{A_t - A_s | \mathcal{G}_s\} \leq K(t-s)$$

then A is of the form $A_t = \int_0^t \lambda_s ds$

- **Yan Zeng, PhD Thesis, Cornell, 2006:** There exists an increasing process D_t with $dD_t \ll dt$ a.s. and

$$E\{A_t - A_s | \mathcal{G}_s\} \leq E\{D_t - D_s | \mathcal{G}_t\},$$

then A is of the form $A_t = \int_0^t \lambda_s ds$

Shrinkage Result; M. Jacobsen, 2005; New proof

- Suppose $1_{\{t \geq \tau\}} - \int_0^t \lambda_s ds$ is a martingale in $\mathbb{H}$
- Suppose also τ is a stopping time in $\mathbb{G}$ where $\mathbb{G} \subset \mathbb{H}$. Then

$$1_{\{t \geq \tau\}} - \int_0^t {}^o\lambda_s ds \text{ is a martingale in } \mathbb{G}$$

where ${}^o\lambda$ denotes the optional projection of the process λ onto the filtration $\mathbb{G}$

Is there a general condition such that all stopping times have absolutely continuous compensators?

- Let X be a strong Markov process; suppose it also a Hunt process
- **(Çinlar and Jacod, 1981)** On a space $(\Omega, \mathcal{F}, \mathbb{F}, P^x)$, up to a change of time and space, if X is a semimartingale we have the representation

$$\begin{aligned}
X_t = X_0 \quad &+ \quad \int_0^t b(X_s)ds + \int_0^t c(X_s)dW_s \\
&+ \quad \int_0^t \int_{\mathbb{R}} k(X_{s-}, z)1_{\{|k(X_{s-},z)|\leq 1\}}[n(ds, dz) - ds\nu(dz)] \\
&+ \quad \int_0^t \int_{\mathbb{R}} k(X_{s-}, z)1_{\{|k(X_{s-},z)|>1\}}n(ds, dz)
\end{aligned}$$

Lévy system of a Hunt process

- For a Hunt process semimartingale X with measure P^μ a **Lévy system** (K, H) where K is a kernel on $\mathbb{R}$ and H is a continuous additive functional of X, satisfies the following relationship:

$$
\begin{aligned}
& E^\mu\left(\sum_{0<s\leq t} f(X_{s-}, X_s) 1_{\{X_{s-}\neq X_s\}}\right) \\
= \quad & E^\mu\left(\int_0^t dH_s \int_{\mathbb{R}} K(X_{s-}, dy) f(X_s, y)\right)
\end{aligned}
$$

- For X a strong Markov process as in the Çinlar-Jacod theorem, we can take the continuous additive functional H to be $H_t = t$

In a "natural" Markovian space, all compensators of stopping times have absolutely continuous paths

Theorem: Let $\mathbb{F}$ be the natural (completed) filtration of a Hunt process X on a space $(\Omega, \mathcal{F}, P^\mu)$ and let (K, H) be a Lévy system for X. If $dH_t \ll dt$ then for any totally inaccessible stopping time τ the compensator of τ has absolutely continuous paths a.s. That is, there exists an adapted process λ such that

$$1_{\{t \geq \tau\}} - \int_0^t \lambda_s ds \text{ is an } \mathbb{F} \text{ martingale.} \tag{1}$$

Moreover if dH_t is not equivalent to dt, then there exists a stopping time ν such that (1) does not hold.

Jumping Filtrations

- Jacod and Skorohod define a **jumping filtration** $\mathbb{F}$ to be a filtration such that there exists a sequence of stopping times $(T_n)_{n=0,1,\dots}$ increasing to ∞ a.s. with $T_0 = 0$ and such that for all $n \in \mathbb{N}, t > 0$, the σ-fields $\mathcal{F}_t$ and $\mathcal{F}_{T_n}$ coincide on $\{T_n \leq t < T_{n+1}\}$

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- **Theorem:** Let $N = (N_t)_{t \geq 0}$ be a point process without explosions that generates a quasi-left continuous jumping filtration, and suppose there exists a process $(\lambda_s)_{s \geq 0}$ such that

$$N_t - \int_0^t \lambda_s ds = \text{ a martingale.} \tag{2}$$

  Let $\mathbb{D} = (\mathcal{D}_t)_{t \geq 0}$ be the (automatically right continuous) filtration generated by N and completed in the usual way. Then for any $\mathbb{D}$ totally inaccessible stopping time R we have that the compensator of $1_{\{t \geq R\}}$ has absolutely continuous paths, a.s.

Increasing Processes

- **Theorem:** *Z is an increasing process; suppose there exists λ such that*

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- *Let R be a stopping time such that $P(\Delta Z_R > 0 \cap \{R < \infty\}) = P(R < \infty)$; then R too has an absolutely continuous compensator; that is, there exists a process μ such that*

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- **Consequence:** If N is a **Poisson process** with parameter λ, and R is a totally inaccessible stopping time on the minimal space generated by N, then the compensator of R has absolutely continuous paths.

Filtration Shrinkage and Compensators

- **Dellacherie's Theorem:** Let R be a nonnegative random variable with $P(R=0)=0, P(R>t)>0$ for each $t>0$. Let $\mathcal{F}_t=\sigma(t\wedge R)$. Let F denote the law of R. Then the compensator $A=(A_t)_{t\geq 0}$ of the process $1_{\{R\geq t\}}$ is given by

$$A_t=\int_0^t \frac{1}{1-F(u-)}dF(u).$$

If F is continuous, then A is continuous, R is totally inaccessible, and $A_t=-\ln(1-F(R\wedge t))$.

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  If F is continuous, then A is continuous, R is totally inaccessible, and $A_t = -\ln(1 - F(R \wedge t))$.

- We know by Jacobsen's theorem, that once a compensator is absolutely continuous, it still is in any smaller filtration

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- **Conjecture:** If a stopping time R has an absolutely continuous law, then it has an absolutely continuous compensator in any filtration rendering it totally inaccessible.
- **This conjecture is false.** A stopping time can be constructed with Brownian local time at zero as its compensator. In its minimal filtration, the compensator is absolutely continuous with respect to $t \mapsto E(L_t)$, which is absolutely continuous with respect to dt.

Equivalent Probabilities

- Let τ be a stopping time on a space $(\Omega, \mathcal{F}, P\mathbb{F})$ and suppose it has an absolutely continuous compensator; that is,

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- Then τ has an absolutely continuous compensator, given by the relation

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- **Note:** Since $[M, M]_t = 1_{\{t \geq \tau\}}$ we have that $\langle M, M\rangle_t = \int_0^t \lambda_s ds$, and the result folows by the Kunita-Watanabe inequality.

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- Let $Q_t(\omega, dx)$ be the conditional distribution of L given $\mathcal{F}_t$, and suppose further that $Q_t(\omega, ds) \ll \eta(dx)$ and we write $Q_t(\omega, dx) = q_t^x \eta_t(dx)$

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- We write

$$\langle q^x, M \rangle_t = \int_0^t k_s^x q_{s-}^x d\langle M, M \rangle_s$$

- The compensator of τ under the enlarged filtration $\mathbb{G}$ given by $\mathcal{G}_t = \mathcal{F}_t \wedge \sigma(t \wedge T)$ is

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- Again, note that $\langle M, M\rangle_t = \int_0^t \lambda_s ds$, so that the compensator is absolutely continuous

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- Then τ has an absolutely continuous compensator in $\mathbb{G}$ as well.

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- **Theorem: Suppose that $(\Omega, \mathcal{G}, P, \mathbb{G}, X)$ is a given system, and that there exists a probability $Q^{\star}$ equivalent to P such that $Q^{\star}$ has Property AC. Then if $\mathcal{Q}$ is the set of all probability measure equivalent to P, we have that Property AC holds under any $Q \in \mathcal{Q}$.**

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- This last theorem is especially useful for applications in Finance

- **Theorem:** Under initial expansion, we have an analogous result. Expand $\mathbb{G}$ by adding a random variable L initially to obtain $\mathbb{H}$. If there exists $Q^\star \in \mathcal{Q}$ with Property AC under $\mathbb{G}$, then $Q^\star$ has Property AC in $\mathbb{H}$, and so all $Q \in \mathcal{Q}$.

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- **Theorem:** Let L be a positive random variable and progressively expand $\mathbb{G}$ with L to get a filtration $\mathcal{J}$. If $Q^{\star} \in \mathcal{Q}$ has Property AC for $\mathbb{G}$, then it also does for $\mathcal{J}$. Moreover so does any $Q \in \mathcal{Q}$.

Thank you