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Recent Advances in Option Pricing via Binomial Trees

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OUTLINE

1. ***Binomial Trees in Option Pricing – Basics***

2. ***Binomial Trees in Option Pricing – Where are the problems ?***

3. ***Advanced Single-Asset Trees and the Optimal Drift Method***

4. ***Standard multi-dimensional trees***

5. ***A simple universal tree obtained by decoupling***

1. *Binomial Trees in Option Pricing – The Start*

Rubinstein/Sharpe (1975) conference in Ein Borek, Israel

With nothing to do during the breaks (except to take a dip in the sea), ..., we wondered how it was that the then two-year-old Black-Scholes approach to valuing options could recreate a riskless payoff using only the option and its underlying asset. It was then that Sharpe said:

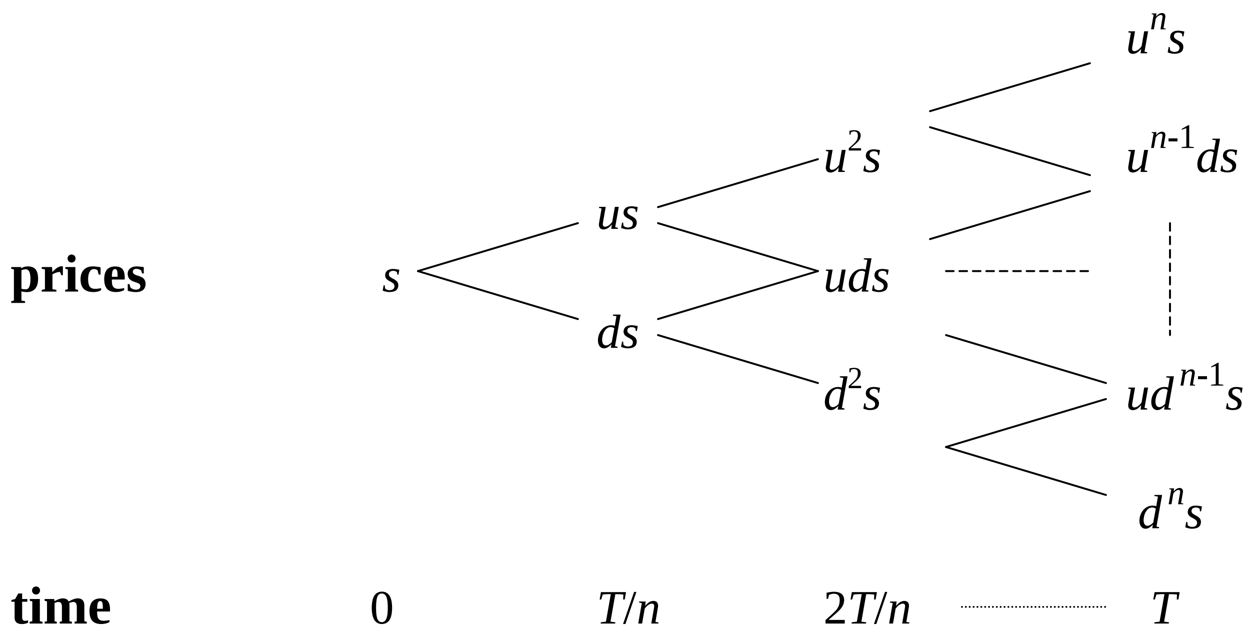
"I wonder if it's really that there are only two states of the world, but three securities, so that any one of the securities can be replicated by the other two"

$\Rightarrow$

Birth of binomial trees !

1. *Binomial Trees in Option Pricing – What is a binomial tree ?*

An ***n-period*** (one-dimensional) ***binomial tree*** is a model for a discrete time stock price process with all possible prices $S^{(n)}(i)$, i=0,1,..., n, being represented by the following tree:



i.e. the price always increases by either a factor u (with prob. p) or d (with prob. (1-p))

Option pricing in a binomial model: ***Risk-neutral valuation and replication***

Assume that in the binomial model we have $d < e^{r\Delta t} < u,\ \Delta t := T/n.$

a) ***Each*** *final payment B in an n-period binomial model can be replicated by an investment strategy in stock and bond.* (*Completeness property*)

b) *The* *initial costs* *of this strategy determine the* *option price* *and both equal*

$$p_B = E_Q\left(e^{-rT}B\right)$$

where the measure Q is the product measure of the Q_i *which are determined by*

$$Q_i\left(S^{(n)}(i+1)/S^{(n)}(i) = u\right) = q = \frac{\exp(r\Delta t) - d}{u - d}$$

and for which we have

$$S^{(n)}(i) = E_Q\left(e^{-r(j-i)T} S^{(n)}(j) \middle| F_i\right),\ 0 \le i \le j \le n.$$

(*Equivalent martingale measure property*)

Why should we consider binomial trees?

- The binomial model is **easy to understand**
- The binomial model contains the aspects of **risk-neutrality and replication**
- The binomial model allows for easily calculatable option prices (see later)

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- **Is the binomial model in any way related to a continuous-time stock price model** (such as the geometric Brownian motion model, the Heston model, …) **?**
- **Does it help us to calculate (an approximation for) the price of an option**

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$$E_Q\left(e^{-rT}B\right)$$ **in a continuous-time setting?**

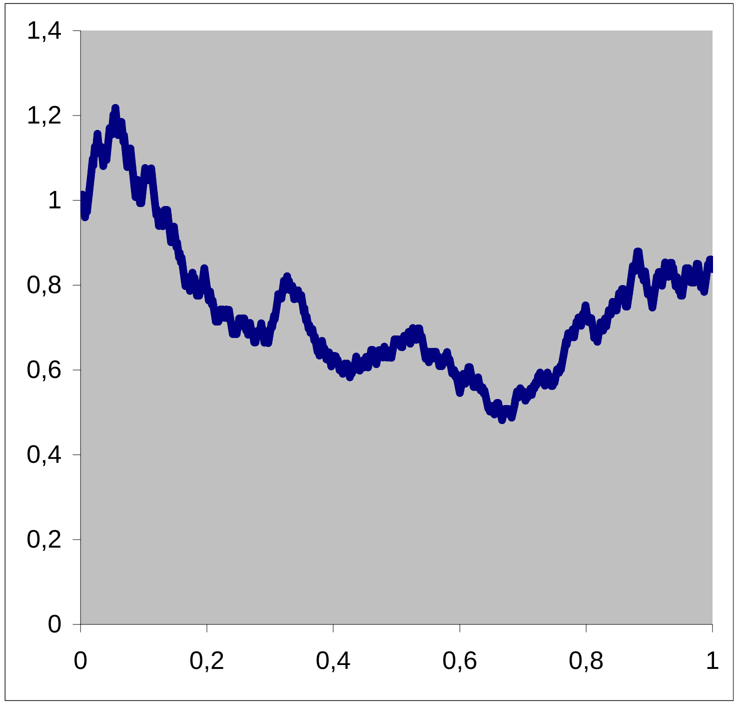
$\Rightarrow$

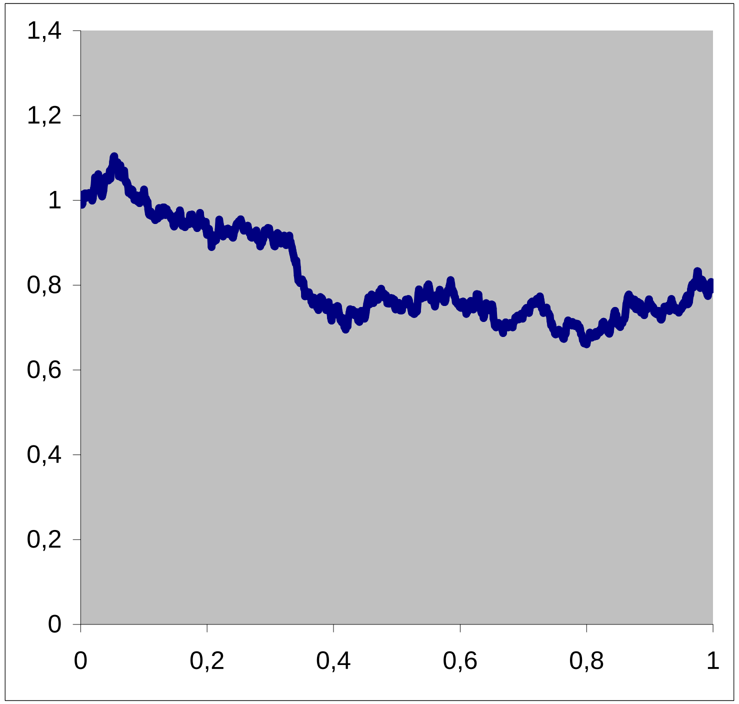
- Look at the **path behaviour** of the corresponding two models
- Find **convergence criteria** for

$$\boxed{\text{“Binomial model} \longrightarrow \text{continuous-time model”}}$$

Approximating binomial trees

Which one is the binomial price process ?





How to find the correct relations between a (sufficiently fine) binomial tree and a geometric Browian motion ?

Convergence of binomial trees towards Geometric Brownian Motion

Donsker's Theorem (special case)

For **given stock price parameters *r*** (drift) and **σ** (volatility) the price process of the binomial tree converges (in distribution) towards the price process in the Black Scholes model if the first two moments of the relative log-returns of both models coincide, i.e. if we have

$$E\left(\ln\left(\frac{S(\Delta t)}{S(0)}\right)\right)=E^{(n)}\left(\ln\left(\frac{S^{(n)}(1)}{S^{(n)}(0)}\right)\right),\quad E\left(\ln\left(\frac{S(\Delta t)}{S(0)}\right)^2\right)=E^{(n)}\left(\ln\left(\frac{S^{(n)}(1)}{S^{(n)}(0)}\right)^2\right)$$

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Remark:

Note that above we only have two (non-linear) equations for three unknowns

$$\left(r-\tfrac{1}{2}\sigma^2\right)\Delta t \;=\; \ln(u)\cdot p+\ln(d)\cdot(1-p),$$

$$\left(r-\tfrac{1}{2}\sigma^2\right)^2(\Delta t)^2+\sigma^2\Delta t=\ln(u)^2\cdot p+\ln(d)^2\cdot(1-p)$$

$\Rightarrow$ (Possibly) one degree of freedom.

Popular choices for u, d, p:

i) **Rendleman and Bartter Tree** (1979)

$$p = \tfrac{1}{2} \quad \Rightarrow \quad u = e^{\left(r - \frac{1}{2}\sigma^2\right)\Delta t + |\sigma|\sqrt{\Delta t}} \;,\; d = e^{\left(r - \frac{1}{2}\sigma^2\right)\Delta t - |\sigma|\sqrt{\Delta t}}$$

No arbitrage condition (not (!) necessary for approximation!): $n > \dfrac{T \cdot \sigma^2}{4}$

ii) **Cox, Ross, Rubinstein Tree** (1979)

$$d = 1/u \quad \Rightarrow \quad u = e^{\sigma\sqrt{\Delta t}} \;,\; p = \tfrac{1}{2}\left(1 + \left(r - \tfrac{1}{2}\sigma^2\right)\frac{1}{\sigma}\sqrt{\Delta t}\right) \quad \text{(only approx. solution)}$$

Necessary condition for $0 < p < 1$: $n \geq \dfrac{T \cdot \sigma^2}{\left(r - \frac{1}{2}\sigma^2\right)^2}$

Consequence:

The expected discounted final payment $E^{(n)}(e^{-rT}B_n)$ in the binomial model (with B_n the final payment in the binomial tree) is an approximation for $E_Q(e^{-rT}B)$ (+techn. cond)

Next question: How to compute $E^{(n)}(e^{-rT}B_n)$? (=> **By backward induction**)

i) $\boxed{p = \frac{1}{2}}$:

$$V^{(n)}\left(T, S^{(n)}(n)\right) = f\left(S^{(n)}(n)\right),$$

For i= n–1, …, 0:

$$V^{(n)}\left(i \cdot \tfrac{T}{n}, S^{(n)}(i)\right) = \tfrac{1}{2}\left[V^{(n)}\left((i+1)\tfrac{T}{n}, uS^{(n)}(i)\right) + V^{(n)}\left((i+1)\tfrac{T}{n}, dS^{(n)}(i)\right)\right] \cdot e^{-\frac{rT}{n}}$$

$$\Rightarrow E^{(n)}\left(e^{-rT}B_n\right) = V^{(n)}(0, s).$$

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For i= n–1, …, 0:

$$V^{(n)}\left(i \cdot \tfrac{T}{n}, S^{(n)}(i)\right) = \left[pV^{(n)}\left((i+1)\tfrac{T}{n}, uS^{(n)}(i)\right) + (1-p)V^{(n)}\left((i+1)\tfrac{T}{n}, \tfrac{1}{u}S^{(n)}(i)\right)\right] \cdot e^{-\frac{rT}{n}}$$

$$\Rightarrow E^{(n)}\left(e^{-rT}B_n\right) = V^{(n)}(0, s)$$

Modifications for American option pricing:

Example: American put, $p = \frac{1}{2}$:

At each time in the calculation of the American put price compare the

intrinsic value of the option $\left(K - S^{(n)}(i)\right)^{+}$

with its ***continuation value***

$$\tilde{V}^{(n)}\left(i \cdot \tfrac{T}{n}, S^{(n)}(i)\right) = \tfrac{1}{2}\left[V^{(n)}\left((i+1)\tfrac{T}{n}, uS^{(n)}(i)\right) + V^{(n)}\left((i+1)\tfrac{T}{n}, dS^{(n)}(i)\right)\right] \cdot e^{-\frac{rT}{n}}$$

and obtain the (approximate) ***value*** of the American put in $S^{(n)}(i)$ as

$$V^{(n)}\left(i \cdot \tfrac{T}{n}, S^{(n)}(i)\right) = \max\left\{\left(K - S^{(n)}(i)\right)^{+},\ \tilde{V}^{(n)}\left(i \cdot \tfrac{T}{n}, S^{(n)}(i)\right)\right\},$$

$$V^{(n)}\left(T, S^{(n)}(n)\right) = f\left(S^{(n)}(n)\right)$$

$\Rightarrow$ $V^{(n)}(0,s)$ is an approximation for $\sup_{\tau} E\left(e^{-r\tau}\left(K - S(\tau)\right)^{+}\right)$, where τ runs through are all possible stopping times with values in [0, T].

2. *Binomial Trees in Option Pricing – Where are the problems ?*

Problem 1: Multi-asset generalizations – **Complications by Correlations**

$$dS_i(t) = S_i(t)\left(rdt + \sigma_i dW_i\right),\; Corr\left(\frac{dS_i}{S_i}, \frac{dS_j}{S_j}\right) = \rho_{ij}dt,\; i, j = 1, \ldots, m$$

How to approximate the multi-asset process by a (correlated) multi-dimensional tree?

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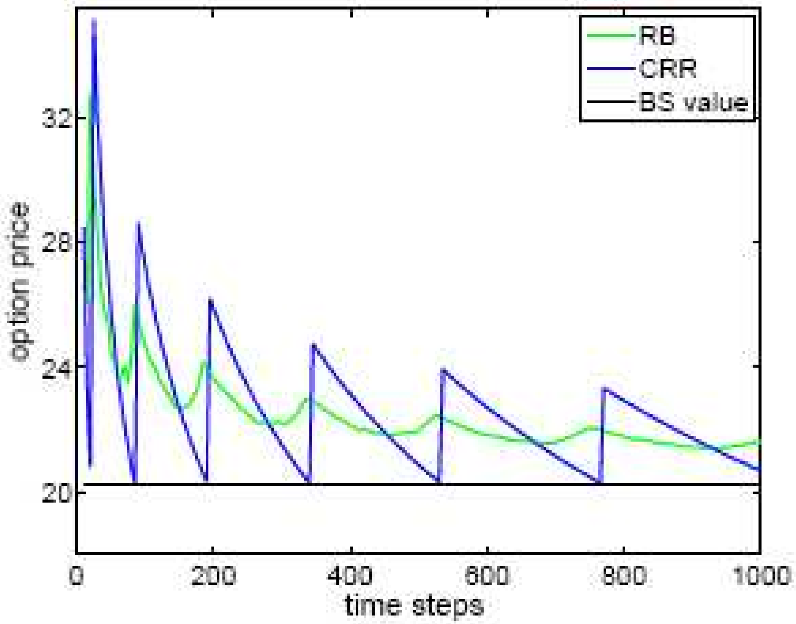
Problem 2: Convergence patterns – **The sawtooth effect**



Convergence pattern for a down-and-out digital call option (Boyle and Lau (1994))

3. *Advanced Single-Asset Trees and the Optimal Drift Method*

A more convenient representation of the binomial tree

$$S^{(n)}(k+1) = S^{(n)}(k)e^{\alpha(n)\Delta t+\beta\sqrt{\Delta t}Z^{(n)}(k+1)},\ S^{(n)}(0) = s$$

with $\beta > 0,\ \alpha(n)$ bounded in n and $Z^{(n)}(k)$ i.i.d. distributed as

$$Z^{(n)}(k) = \begin{cases} 1 \ with \ \ prob. \ p(n) \\ -1 \ with \ \ prob.\, 1-p(n) \end{cases}$$

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Moment conditions revisited

Donsker's theorem is valid if $\beta > 0, \alpha(n), p(n)$ satisfy the approx. moment cond.:

$$\text{(M1)} \quad \mu(n) := \frac{1}{\Delta t} E_{P_n}\left(ln\left(S^{(n)}(1) \big/ S^{(n)}(0) \right)\right) \xrightarrow{\Delta t \to 0} r - \tfrac{1}{2}\sigma^2$$

$$\text{(M2)} \quad \sigma^2(n) := \frac{1}{\Delta t} Var_{P_n}\left(ln\left(S^{(n)}(1) \big/ S^{(n)}(0) \right)\right) \xrightarrow{\Delta t \to 0} \sigma^2$$

Consequences:

a) Cox, Ross, Rubinstein : $\alpha(n)=0, \beta=\sigma, p(n)=\frac{1}{2}+\frac{1}{2}\left(\frac{r-\frac{1}{2}\sigma}{\sigma}\right)\sqrt{\Delta t}$, i.e.

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b) Theorem (Müller (2009)) ***Drift-invariance***
With the choice of the risk-neutral probability

(RNP) $$p(n)=\frac{e^{r\Delta t}-d}{u-d}$$

and $u=e^{\alpha(n)\Delta t+\beta\sqrt{\Delta t}}, d=e^{\alpha(n)\Delta t-\beta\sqrt{\Delta t}}$ the moment conditions of Donsker's theorem are automatically (approximately) satisfied if and only if we choose

(*) $$\beta=\sigma.$$

Note:
This is a key observation as it leaves us the **free choice of the drift process** $\alpha(n)$.

How to use the freedom of choosing the drift ?

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$\Rightarrow$ Second key ingredient:

Theorem (Müller (2009)) ***Asymptotic approximation of the martingale distribution***

Let Q be the unique equivalent martingale measure in the Black-Scholes model, let $\beta = \sigma$ and

(PC) $$p(n) = \tfrac{1}{2} + \tfrac{1}{2\sigma}\left(r - \alpha(n) - \tfrac{1}{2}\sigma^2\right)\sqrt{\Delta t} + c(n)(\Delta t)^{3/2} + o\left(N^{-3/2}\right)$$

where $c(n)$ is a bounded function of $\alpha(n)$.

Then we obtain

$$P^{(n)}\left(S^{(n)}(n) \geq x\right) = Q\left(S(T) \geq x\right) + \frac{e^{-\frac{1}{2}d_2^2(x)}}{\sqrt{2\pi}} b(n)\frac{1}{\sqrt{n}} + h\left(x, \alpha(n), b(n), c(n)\right)\frac{1}{n} + o\left(\frac{1}{n}\right)$$

for some function $b(.)$ with $\left|b(x)\right| \leq 1$. Further, with k = # up-moves, we have

$$b(n) = 1 - 2\frac{ln\left(S^{(n)}(n;k)/x\right)}{\left(S^{(n)}(n;k)/S^{(n)}(n;k-1)\right)} \quad \text{for} \quad S^{(n)}(n;k-1) < x \leq S^{(n)}(n;k).$$

How to use these results ?

For a given x (such as the strike of a call!)**, obtain a good (?) value for** $b(n)$ **by choosing an appropriate drift process** $\alpha(n)$**.**

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Suggestions for good values of $b(n)$:
By starting with the CRR-tree (i.e. $\alpha(n)=0$) obtain

a) **Tian (1994):**

$b(n) = 1$ (i.e. monotone convergence to leading order!) via suitably choosing $\alpha(n)$.

$\Rightarrow$ No sawtooth effect and convergence can be speeded up by extrapolation !

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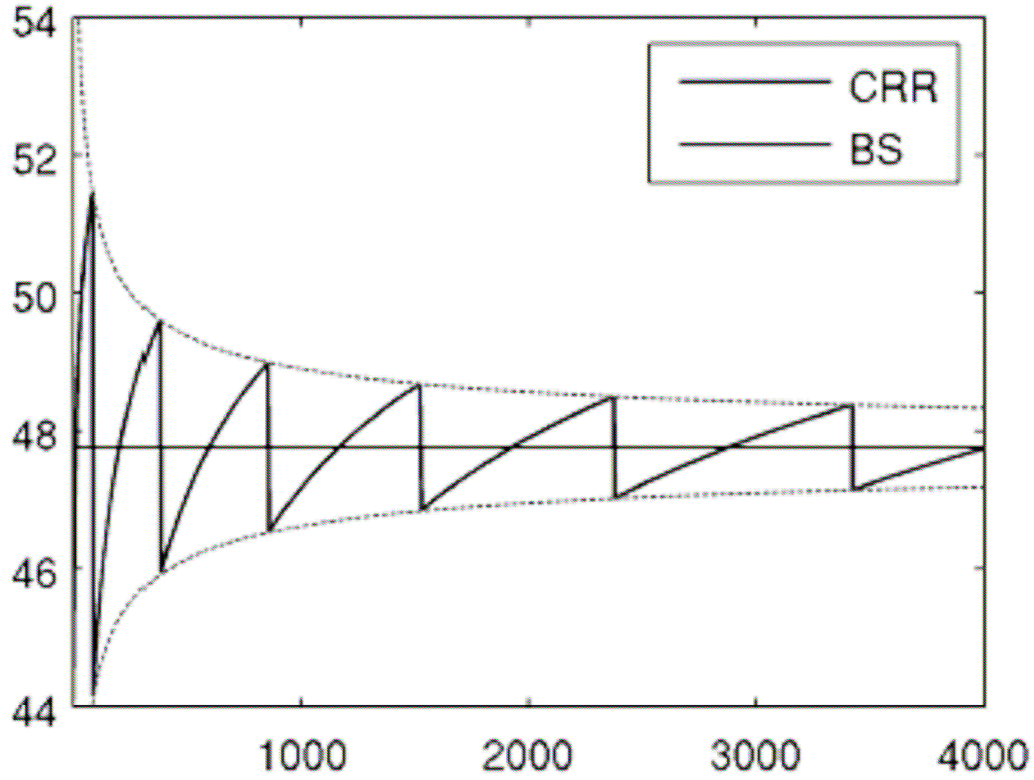
b) **Chang-Palmer (2007):**

$b(n) = 0$ (i.e. increase the order of convergence !) via suitably choosing $\alpha(n)$.

$\Rightarrow$ Faster (but not montone) convergence !

Convergence behaviour for a European digital call with

s = 95**,** r = 0.1**,** σ = 0.25**,** T = 1, K = x =100: **CRR-case**



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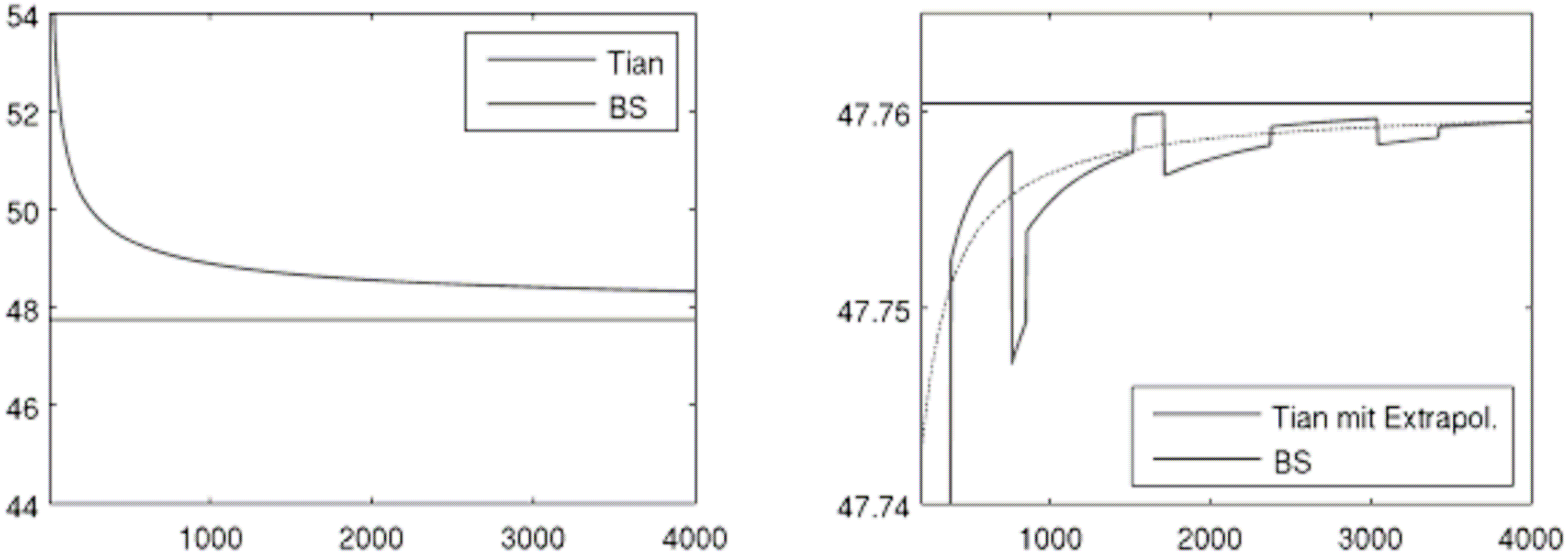
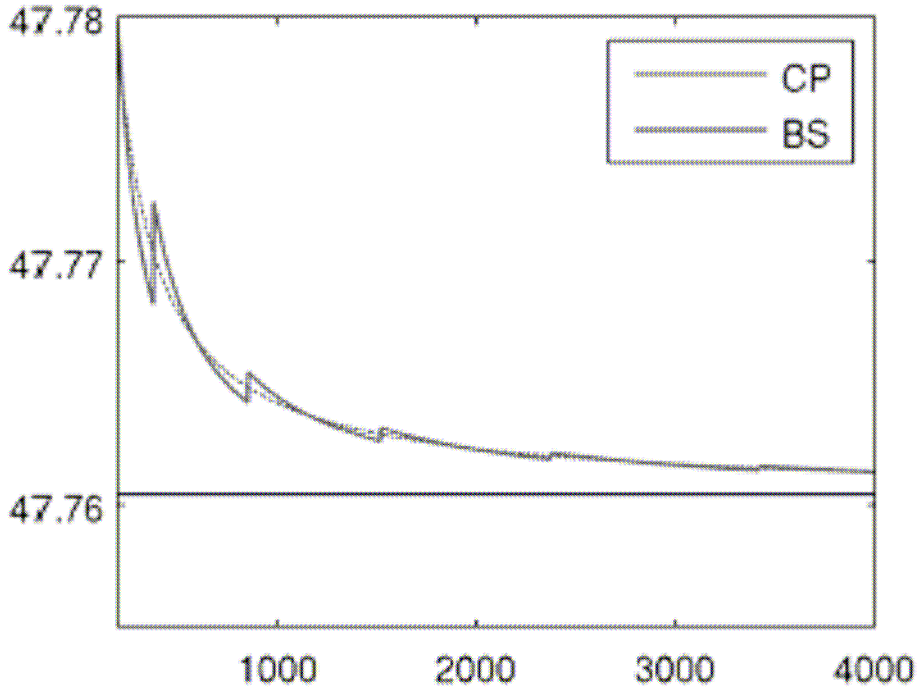
s = 95**,** r = 0.1**,** σ = 0.25**,** T = 1, K = x =100: **Tian-case**



Convergence behaviour for a European digital call with

s = 95, r = 0.1, σ = 0.25, T = 1, K = x =100: **Chang-Palmer-case**



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Idea:

1. Apply the Chang-Palmer idea to a binomial tree with an arbitrary (non-zero) drift parameter α, i.e. choose an additional drift parameter $\alpha(n)$ to obtain $b(n) = 0$.

$$\Rightarrow \quad P^{(n)}\left(S^{(n)}(n) \geq x\right) = Q\left(S(T) \geq x\right) + g(\alpha)\tfrac{1}{n} + o\left(\tfrac{1}{n}\right)$$

as one can show $\alpha(n) = \alpha + o(1)$

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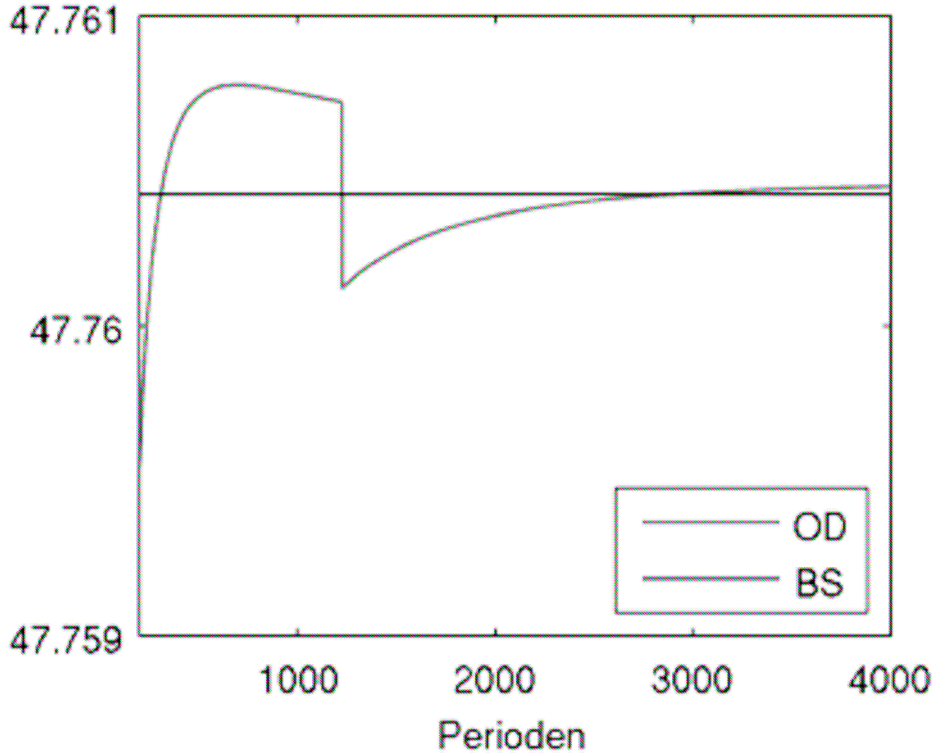
as one can show $\alpha(n) = \alpha + o(1)$

2. Realize that $g(\alpha)$ is a quadratic function.

$\Rightarrow$ If a zero $\tilde{\alpha}$ of $g(\alpha)$ exists then choose it as the “original α “ and obtain an order of convergence of $\boxed{o\left(1/n\right)}$.

$\Rightarrow$ Else choose the minimizing/maximizing α as “original α “ and obtain the same order of convergence as Chang-Palmer, but with an <u>optimal constant</u>.

Convergence behaviour for a European digital call with
s = 95**,** r = 0.1**,** σ = 0.25**,** T = 1, K = x =100: **Optimal drift-case**



Convergence behaviour for a European digital call with

s = 95**,** r = 0.1**,** σ = 0.25**,** T = 1, K = x =100: **The numbers**

N	CRR	RB	Tian	Tian Extrapol.	CP	OD
200	47.5257	47.7912	50.3228	47.6913	47.7798	47.7596
400	46.1524	47.0456	49.5701	47.7529	47.7717	47.7607
1000	47.1805	48.6610	48.9022	47.7555	47.7644	47.7608
2000	47.9034	47.6615	48.5669	47.7574	47.7623	47.760355
3000	47.9178	47.2180	48.4187	47.7596	47.7617	47.760428
5000	47.5104	47.7922	48.2702	47.7594	47.7612	47.760384
10000	47.5869	47.7666	48.1208	47.7601	47.7608	47.760427
15000	47.7984	47.8921	48.0546	47.7602	47.7607	47.760418
BS	47.760425					

More on the optimal drift method:

- Results carry over to puts/calls and to American puts/calls (optimal drift method outperforms Leisen-Reimer (1996) algorithm)
- Optimal drift model makes full use of the free parameters to obtain a better order of convergence
- Theoretical results on superior performance
- Interesting to test it on more advanced options (such as barriers)

4. *Standard multi-dimensional trees*

Define an *m*-dimensional *N*-period tree via

$$S_i^{(N)}(k\Delta t) = S_i^{(N)}((k-1)\Delta t)\, e^{\alpha_i \Delta t + \beta_i \sqrt{\Delta t} Z_{k,i}}, \quad k = 1, \ldots, N, i = 1, \ldots, m$$

with $Z_{k,i}(\omega) \in \{-1, 1\}$.

We choose the constants α_i, β_i and the up-down probabilities such that we obtain:

- $(Z_{k,1}, \ldots, Z_{k,m}), k = 1, \ldots, N$ are i.i.d. for fixed N
- the first two moments of the continuous-time log returns coincide (at least) asymptotically with those in the tree, in particular the covariances satisfy

Disadvantages:

- very tedious
- does not always lead to well-defined probabilities !

Examples:

1. The BEG Tree (Boyle, Evnine, Gibbs (1989), generalized CRR Tree)

$$S_i^{(N)}(k\Delta t) = S_i^{(N)}((k-1)\Delta t)\, e^{\sigma_i \sqrt{\Delta t} Z_{k,i}}, \quad k = 1,\ldots,N,\, i = 1,\ldots,m$$

$$p_{BEG}^{(N)}(\omega) = \frac{1}{2^m}\left(1 + \sum_{j=1}^{m}\sum_{i=1}^{j-1}\delta_{ij}(\omega)\rho_{ij} + \sqrt{\Delta t}\sum_{i=1}^{m}\delta_i(\omega)\frac{r - 0.5\sigma_i^2}{\sigma_i}\right)$$

$$\delta_i(\omega) = \begin{cases} 1 \; if \;\; \omega_i = "up" \\ -1 \; if \;\; \omega_i = "down" \end{cases}, \quad \delta_{ij}(\omega) = \begin{cases} 1 \; if \;\; \omega_i = \omega_j \\ -1 \; if \;\; \omega_i \neq \omega_j \end{cases}$$

Note:

- log-prices in the BEG Tree are **symmetric**
- probabilities depend in a complicated way from the drift and covariance structure
- probabilities are not automatically non-negative !

BEG probabilities are not always well-defined:

Let S_1, S_2, S_3 with $\rho_{12} = -0.8$, $\rho_{23} = -0.6$ and $\rho_{13} = 0.2$.
Then for the tree suggested by Boyle, Evnine and Gibbs (1989),

$$P^{(N)}(\omega_{k1} = \omega_{k2} = \omega_{k3} = -1) = \frac{1}{8}\left(-0.2 - \sqrt{\Delta t}\sum_{i=1}^{3}\frac{r - \frac{1}{2}\sigma_i^2}{\sigma_i}\right),$$

which is negative for all $\Delta t > 0$ if $(r - 1/2\sigma_i^2) > 0$ for all assets i.

Here the problem cannot be fixed by choosing a sufficiently large number of periods!

2. The n-dim. Rendleman-Barrter Tree (Amin (1991), K., Müller (2009))

$$S_i^N(k\Delta t) = S_i^N((k-1)\Delta t)\, e^{\left(r - \frac{1}{2}\sigma_i^2\right)\Delta t + \sigma_i \sqrt{\Delta t} Z_{k,i}}, \quad k = 1, \ldots, N, i = 1, \ldots, m$$

$$p_{RB}^{(N)}(\omega) = \frac{1}{2^m}\left(1 + \sum_{j=1}^{m}\sum_{i=1}^{j-1} \delta_{ij}(\omega)\rho_{ij}\right)$$

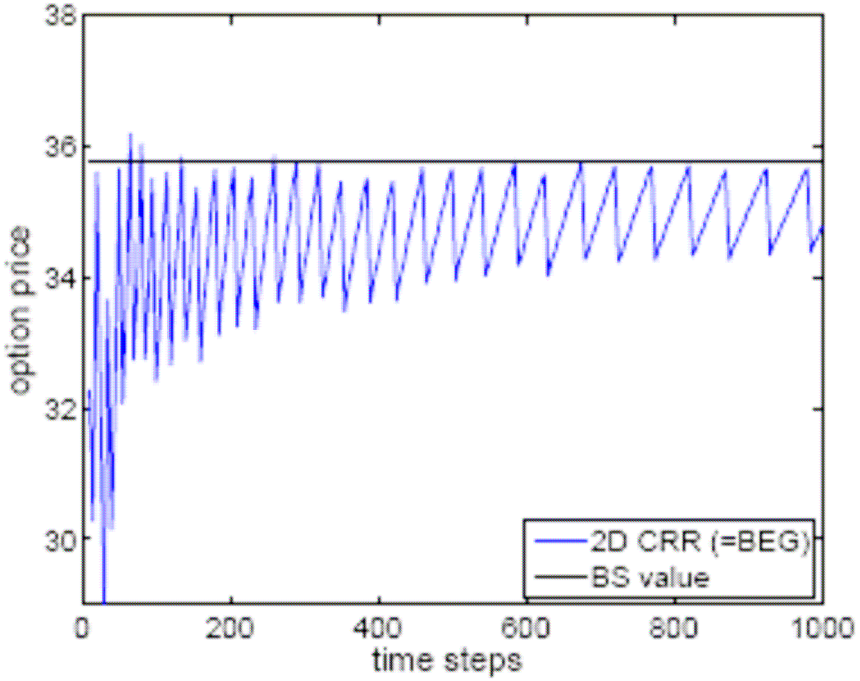
Note:

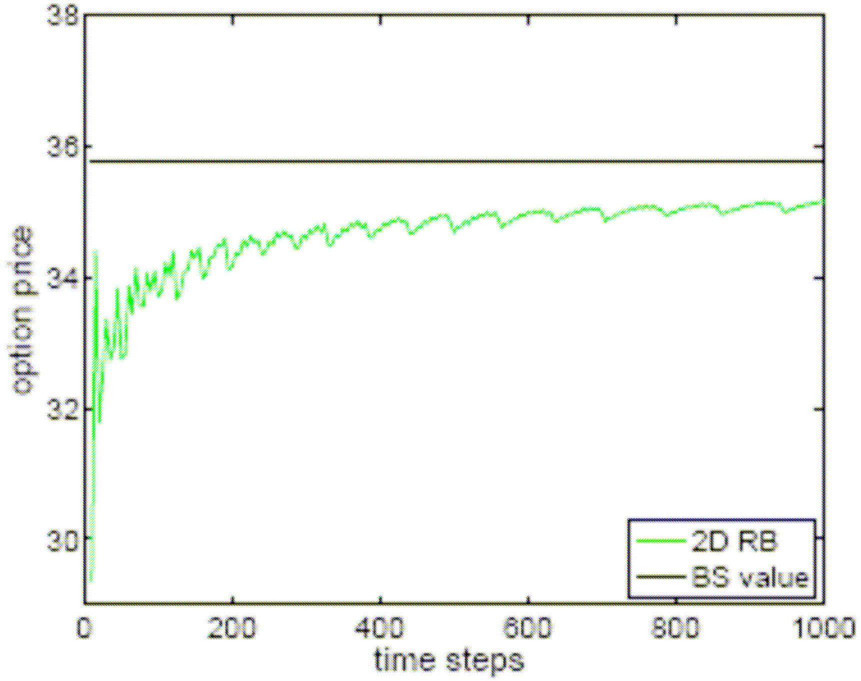
- log-prices in the RB Tree are **non-symmetric**
- probabilities depend only on the covariance structure, not on the fineness of the discretization

An example for multi-asset binomial convergence:

$$B = g\left(S_1(T), S_2(T)\right) = 100 \cdot 1_{\{S_1(t_0) \geq 25 \text{ for some } t_0 \in [0,T], S_2(t) \geq 15 \, \forall t \in [0,T]\}}$$

$$S_1(0) = 20, S_2(0) = 30, \sigma_1 = 0.2, \sigma_2 = 0.3, \rho = 0.5, T = 1, r = 0.1$$





Cash-or-nothing option with up-in-barrier on stock 1 and down-out-barrier on stock 2

5. *A simple universal tree obtained by decoupling*

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A general decoupling rule (K. and Müller (2009))

- Decompose the variance/covariance matrix of the log-returns

(1) $$\Sigma = GDG^T$$

- Set up a new log-return process

(2) $$Y(t) := G^{-1}\left(\ln\left(S_1(t)\right), \ldots, \ln\left(S_m(t)\right)\right)^T$$

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Consequence:

- The components of $Y(t)$ are **independent** and have the dynamics

(3) $$dY_j(t) = \mu_j dt + \sqrt{d_{jj}}\, d\bar{W}_j(t),\ \mu = G^{-1}\left(r\underline{1} - \tfrac{1}{2}\underline{\sigma}^2\right), \underline{\sigma}^2 = \left(\sigma_1^2,\ldots,\sigma_m^2\right)^T$$

- Moment matching can be done **component wise !!!! (only 1D problems !)**
- Always well-defined probabilities
- Combination of different 1D-trees possible !

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$$Y_{k+1}^{(N)} = \begin{pmatrix} Y_{k,1}^{(N)} + \mu_1 \Delta t + Z_{k+1,1}\sqrt{d_{11}}\sqrt{\Delta t} \\ \vdots \\ Y_{k,m}^{(N)} + \mu_m \Delta t + Z_{k+m,1}\sqrt{d_{mm}}\sqrt{\Delta t} \end{pmatrix}$$

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We then transform each node back to obtain the "valuation tree":

- Define (5) $h(\underline{x}) := \left(e^{G_1 \underline{x}}, \ldots, e^{G_m \underline{x}}\right),\ G_i = row\ i\ of\ G$
- Set (6) $S_k^{(N)} := h\left(Y_k^{(N)}\right)$

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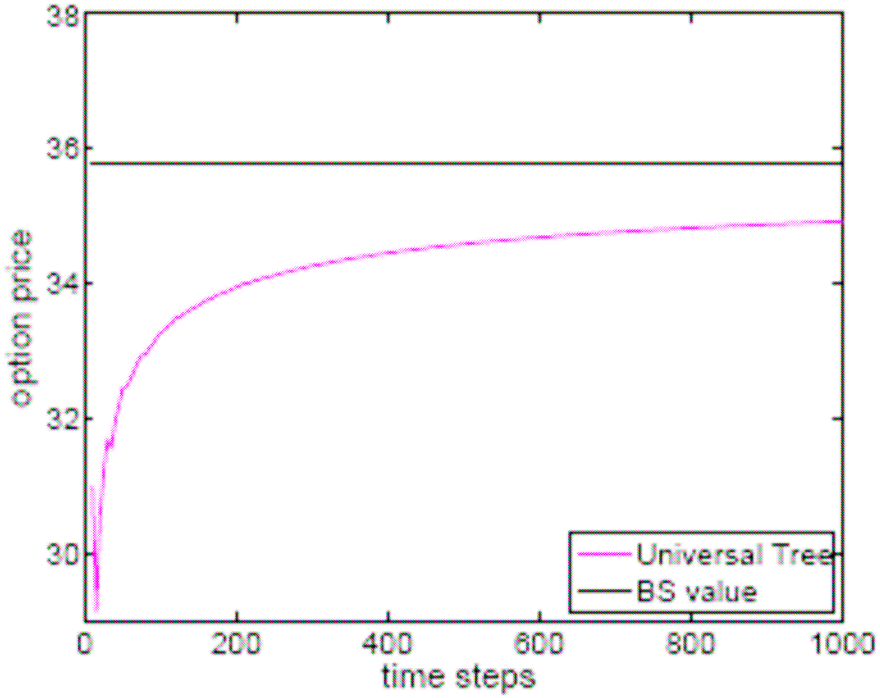
Finally: Calculate the option price via backward induction

<u>Algorithm:</u> Decoupled Tree Option Pricing

Input: payoff function g, model parameters (in part. var.-cov. matrix $\boldsymbol{\Sigma}$), N

1. Decompose the variance-covariance matrix $\Sigma = GDG^T$.
2. Transform the stock price S into $Y(t) := G^{-1}\left(\ln\left(S_1(t)\right),\ldots,\ln\left(S_m(t)\right)\right)^T$ which is component wise a Brownian motion with drift as in (3).
3. Set up an m-dimensional Rendleman-Barrter tree with independent components using the discrete process $Y^{(N)}$ as defined in (4).
4. Apply the transformation (5) to each node of the tree as in (6).
5. Evaluate the payoff functional along the transformed nodes using backward induction. Exploit the fact that all scenarios are equally likely.

An extra gain: *No sawtooth effect !*



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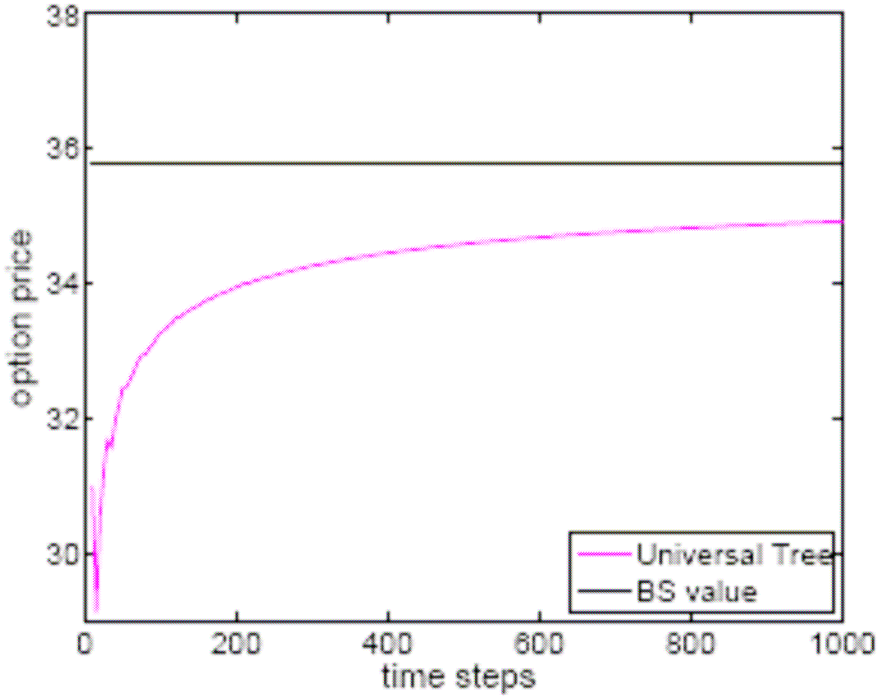

- **How to explain this ?**

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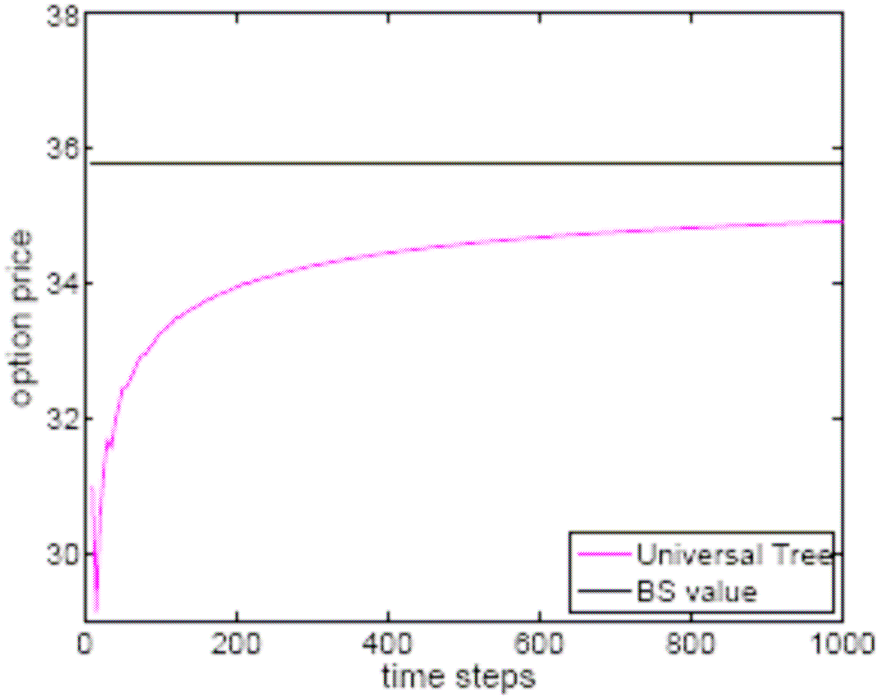
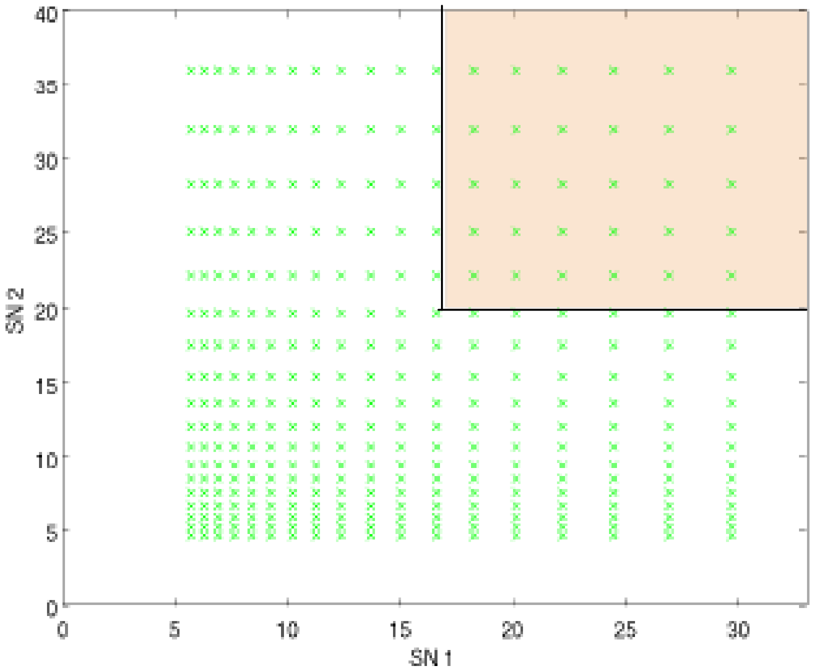


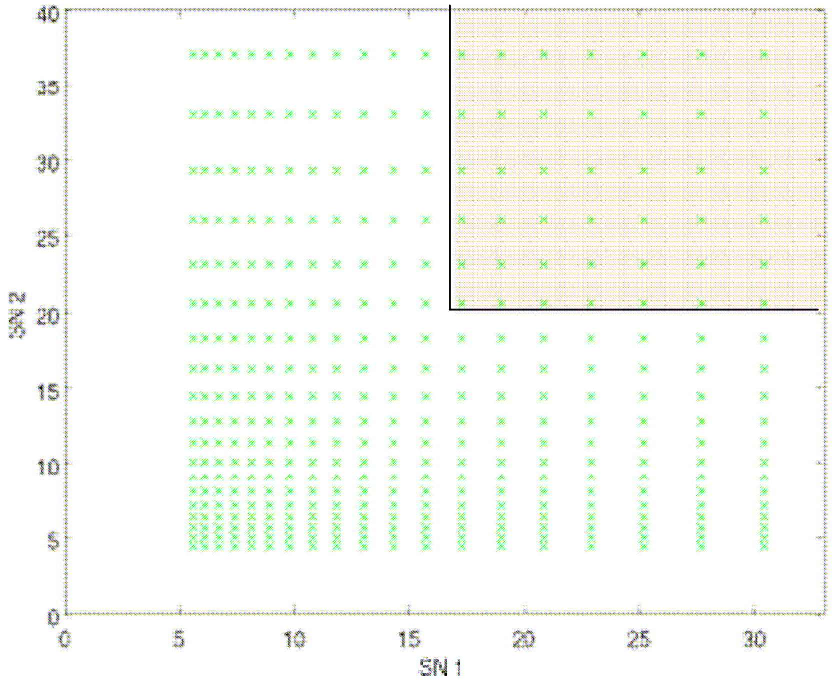
- **How to explain this ?**
- **Is it really a gain ?**

Explanation:

Origin of the sawtooth effect

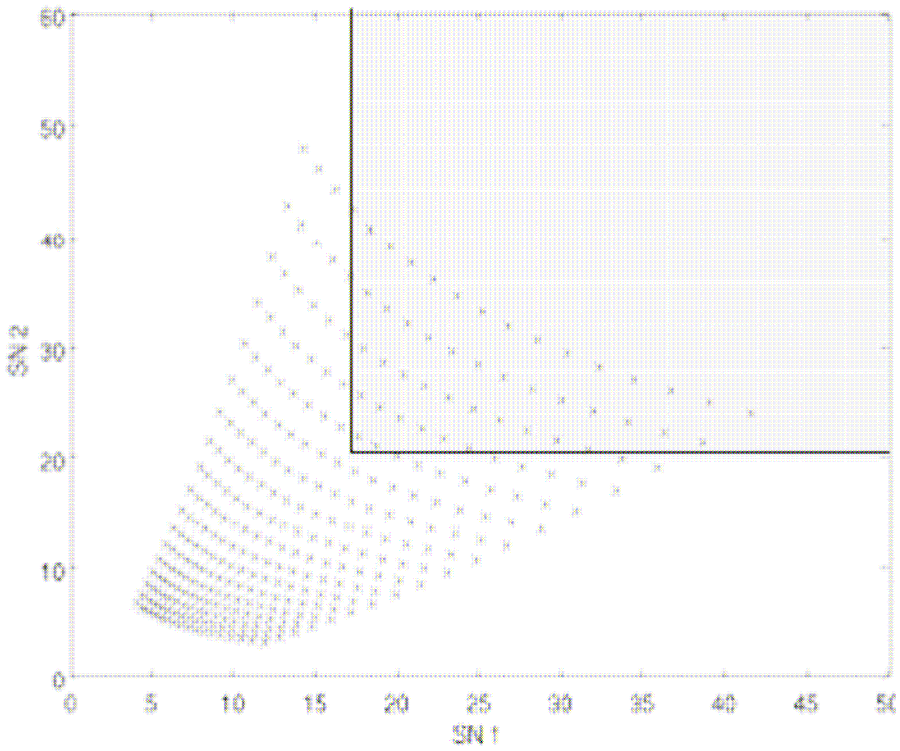
Two discretizations (N=17, N=18) for a Rendleman-Barrter Tree for a cash-or-nothing barrier-option





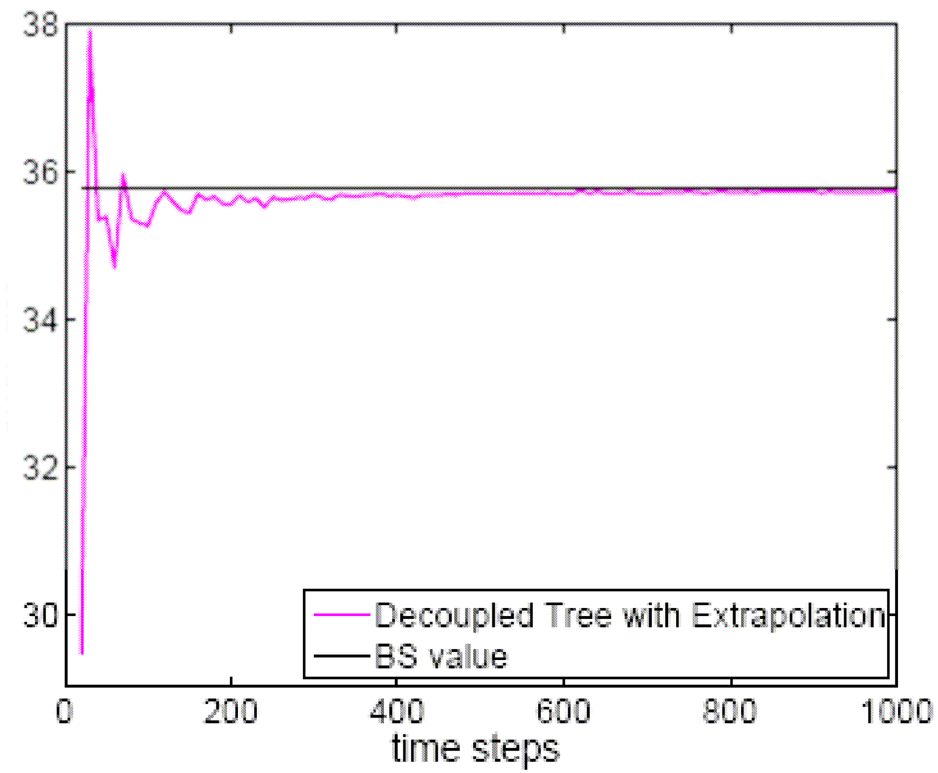
Explanation II:

Discretizations (N=17) for a Rendleman-Barrter orthogonal Tree for a cash-or-nothing barrier-option



Speeding up convergence:

As convergence is approx. monotone we apply *Richardson extrapolation*



In numbers:

N	BEG	$2D\ RB$		$Decoupling$		$Decoupling$ $extrapolation$	
100	32.40	33.71	83 %	33.28	250 %	35.26	283 %
250	34.88	34.55	73 %	34.12	227 %	35.65	253 %
500	35.65	34.69	88 %	34.59	302 %	35.71	316 %
750	35.04	35.08	89 %	34.79	293 %	35.73	311 %
1000	34.79	35.15	88 %	34.92	290 %	35.72	325 %
1250	35.09	35.18	86 %	35.01	287 %	35.74	324 %
1500	34.88	35.27	86 %	35.07	286 %	35.73	321 %
1750	35.41	35.29	88 %	35.12	295 %	35.74	330 %
2000	35.67	35.23	88 %	35.16	299 %	35.75	335 %
2750	35.34	35.38	88 %	35.25	299 %	35.75	335 %
BS	35.76						

Conclusion on decoupled trees

- Decoupled trees are **easy** to implement
- Decoupled trees are **not restricted** in their application by parameter settings
- Decoupled trees **avoid the sawtooth effect** by a non-linear transformation
- Decoupled trees allow for an **efficient implementation** (Richardson extrapolation, component-adapted discretization, model reduction…)
- Decoupled trees need a **higher computing time** for path-dependent options
- Decoupled trees are **not universally best methods**, but can be used universally

Literature:

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Thanks for your attention !